Introduction to Game Theory
Lecture 11

Disclaimer: this presentation is only a supporting material and is not sufficient to master the topics covered during the lecture. Study of relevant books is strongly recommended.

Course overview

- static games
  - perfect information: NE
  - imperfect information: (Bayesian) NE
- dynamic games
  - perfect information: SPNE
  - imperfect information: weak perfect Bayesian equilibrium

Preview

- dynamic games
  - perfect information:
  - SPNE (Backward Induction)
  - each player is informed of the actions chosen previously by all players

  - imperfect information:
  - weak perfect Bayesian equilibrium
  - situations in which each player may not be informed of the other players' previous actions
  - imperfect information – information set

Preview

Dynamic Games with Imperfect Information

- in games with perfect information, like chess, the players always know everything that has happened so far in the game
- in other games, players only have a partial information about what has been done so far

- we already know that dynamic (sequential) games can be represented also by normal form game (table)
- similarly, static (simultaneous) games can be represented by extensive form game (decision tree)

Prisoners' Dilemma

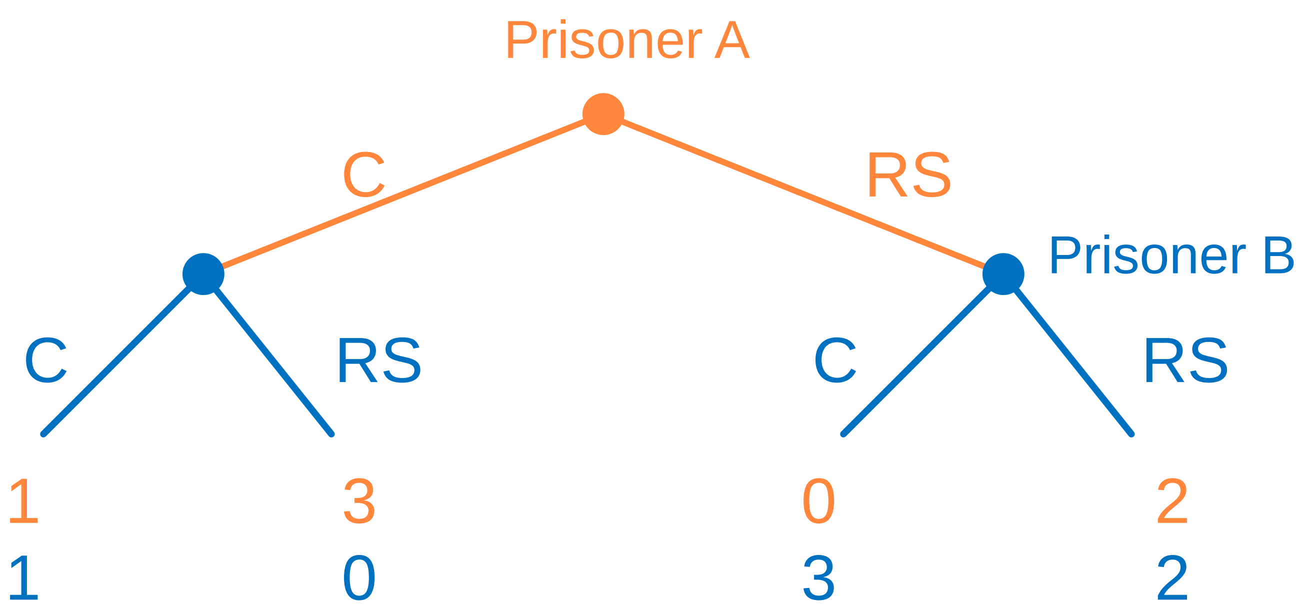
Static version of Prisoners' dilemma game:

- Prisoners' dilemma game:

1 \ 2	Confess	Remain Silent
Confess	1,1	3,0
Remain Silent	0,3	2,2

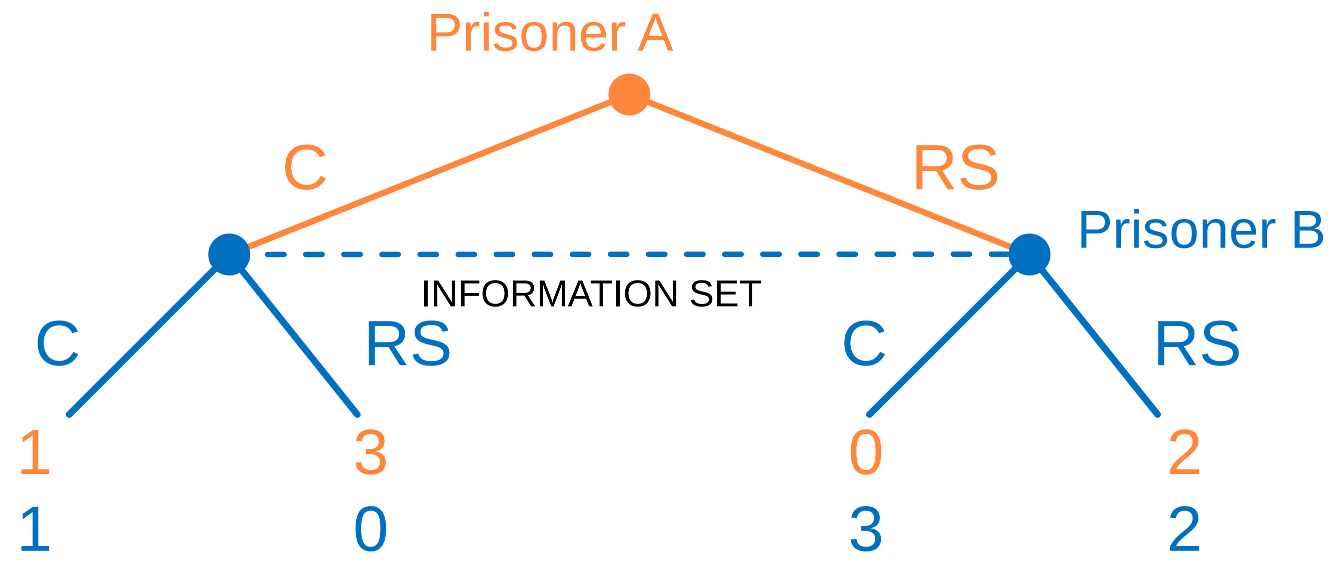
Dynamic Prisoners' Dilemma

Dynamic version of Prisoners' dilemma game:



Dynamic Prisoners' Dilemma

- if second player observes move of the first one, the order of players matter (in general)
- if second player can not observe action of the first one, the game is equivalent to static game



Imperfect Information

Information Set (dashed blue line) – collection of decision nodes such that:

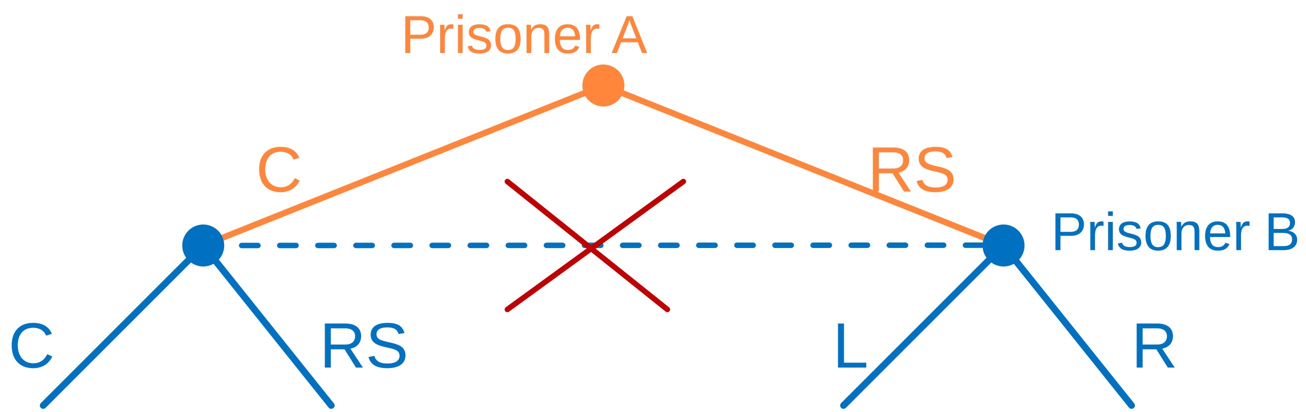
a) when the play reaches a node in the information set, the player with the move does not know which node in the information set has been reached

b) the player has the same set of choices at each node in the information set

Imperfect Information

Situation below violates b) => no information set

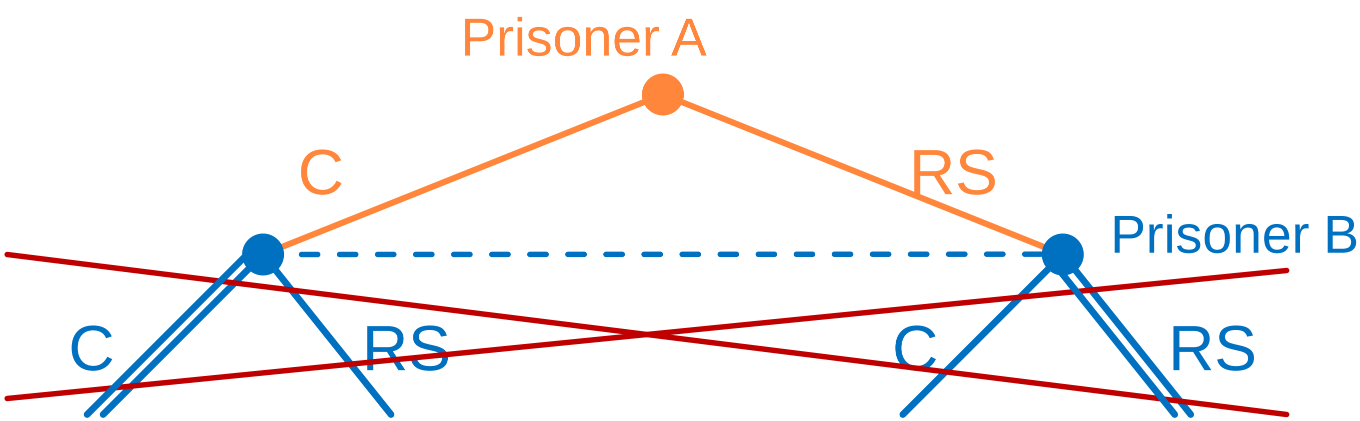
b) the player has the same set of choices at each node in the information set



Imperfect Information

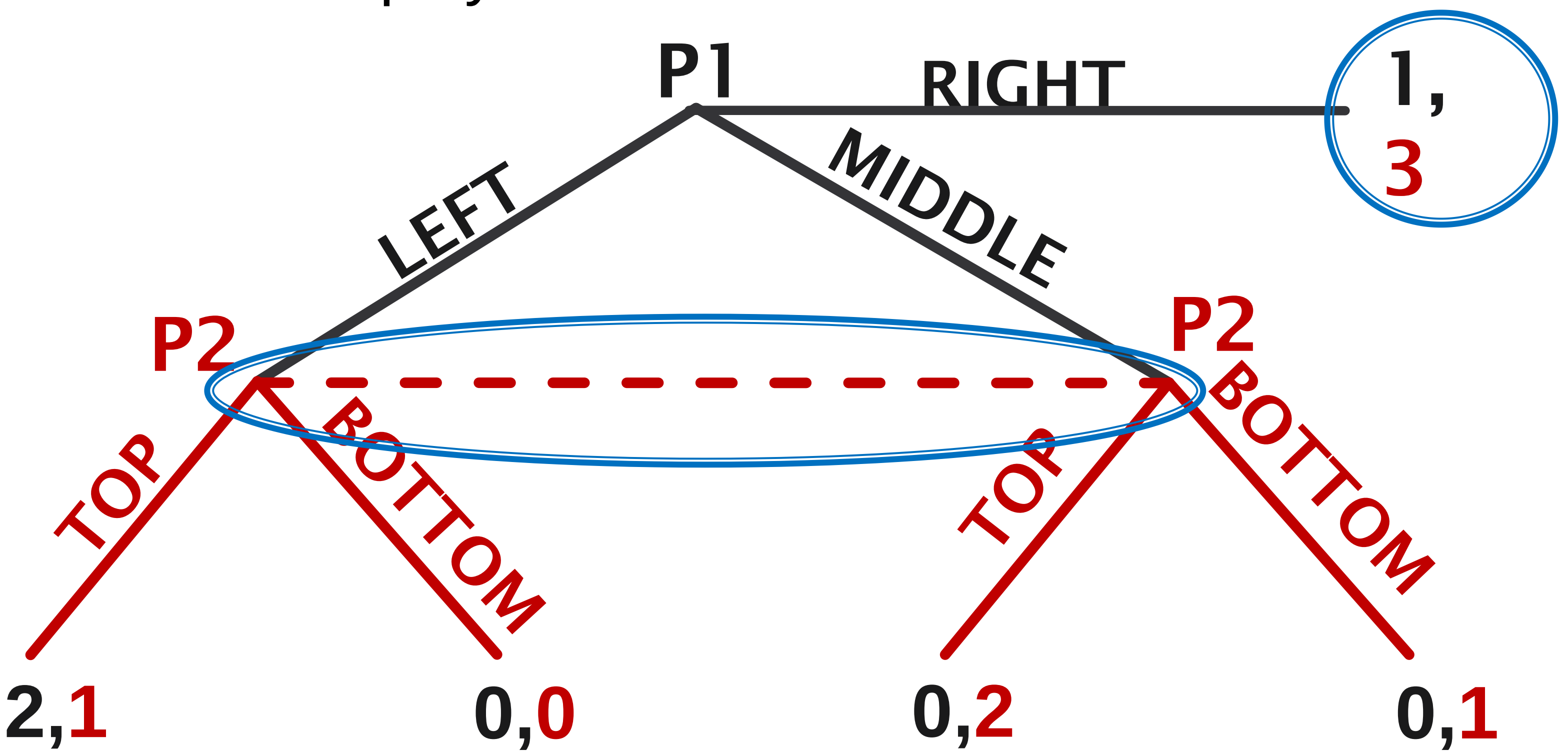
It is not possible to choose different actions within one information set

Prisoner B can not decide to choose C in left node and RS in right node (he does not know in which he is)



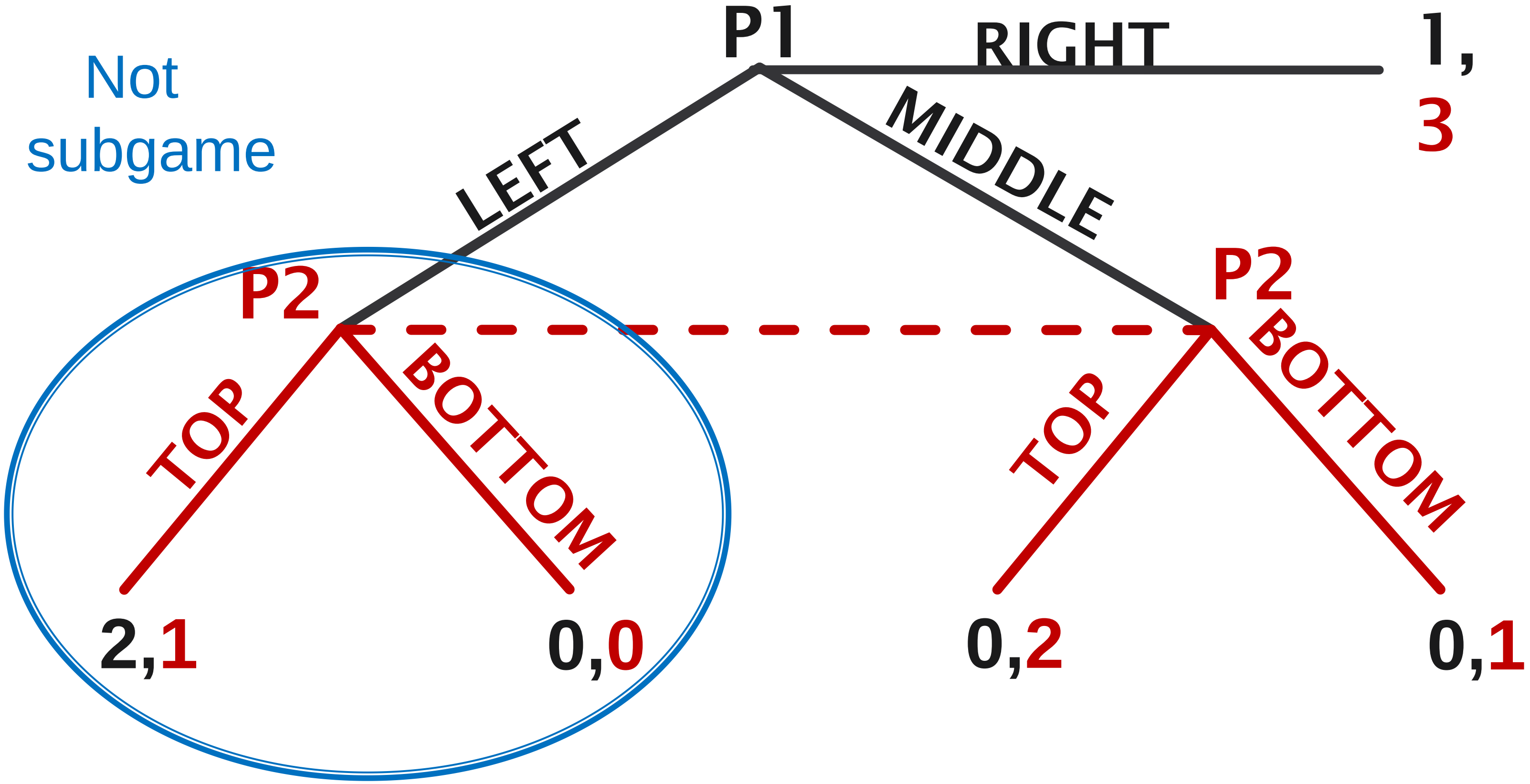
Imperfect Information

- P2 has two information sets:
  - one after P1 plays L or M
  - one after P1 player R



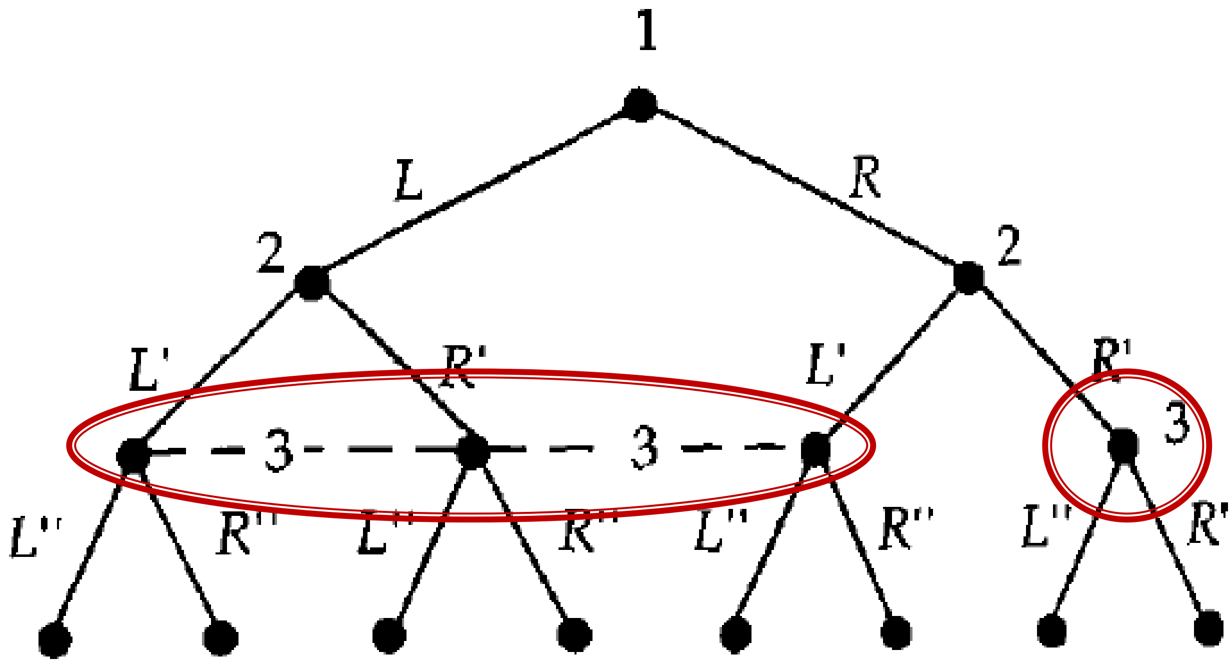
Imperfect Information - Subgames

- game has only one subgame:
  - the whole game
  - we can not cut through information sets



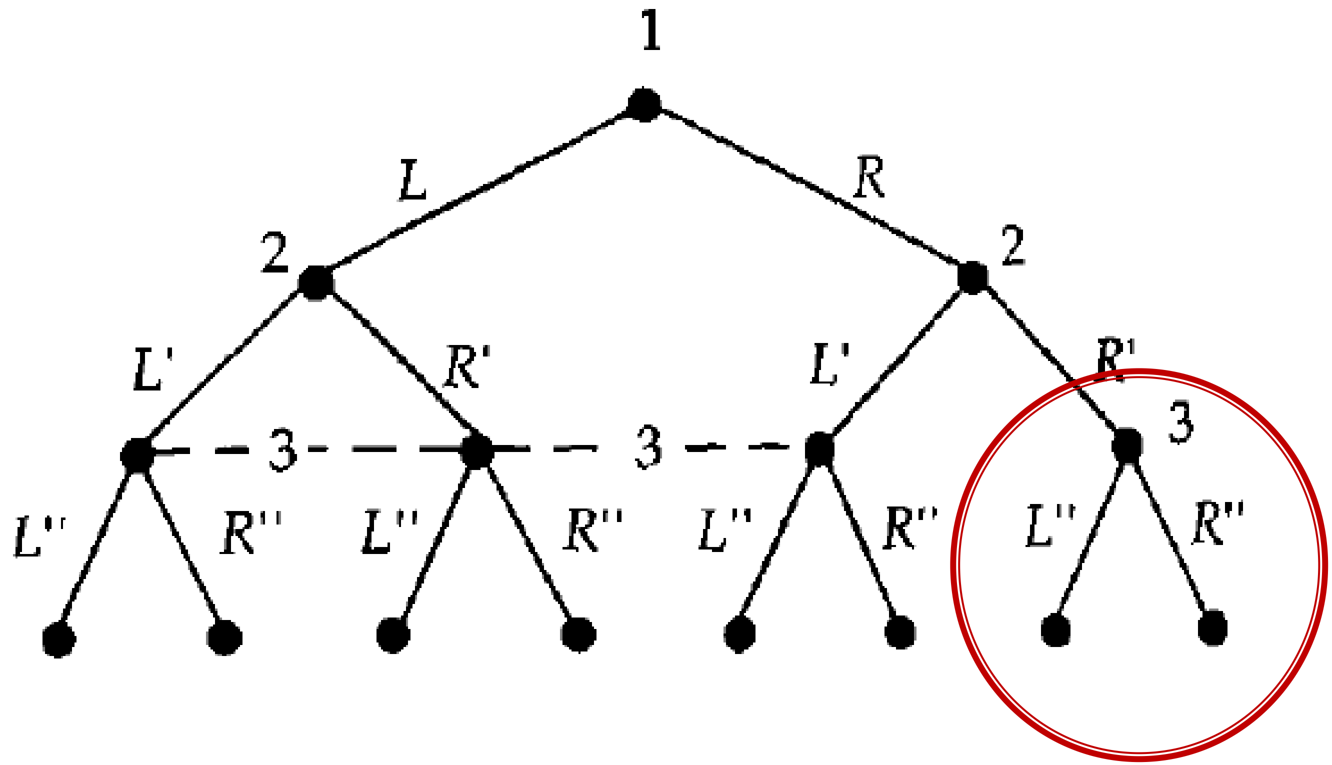
Imperfect Information

- P3 has two information sets:
  - singleton (single node) following 1-R, 2-R’
  - information set that contains 3 nodes



Imperfect Information - Subgames

- game has two subgames:
  - whole game
  - game that follows 1-R, 2-R’



Imperfect Information

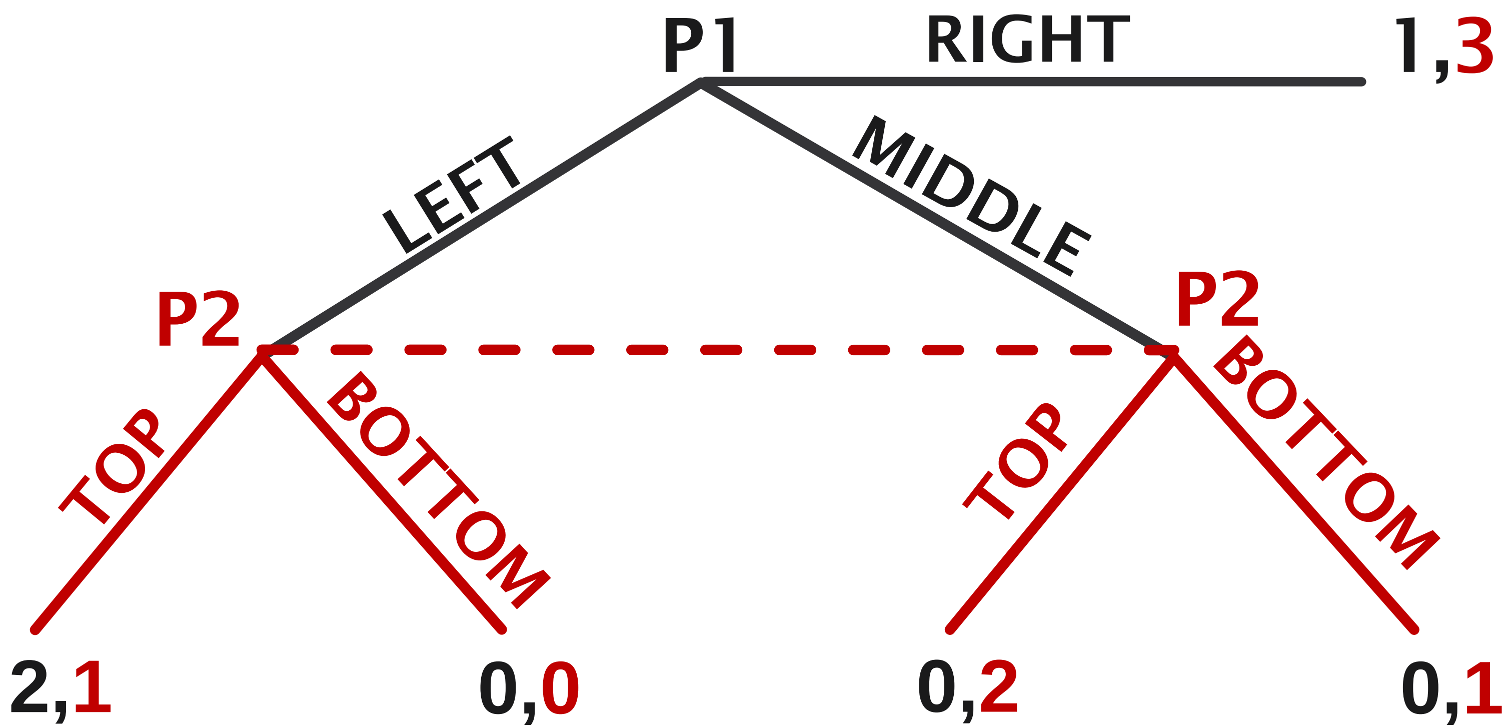
- Game is with perfect information if:
  - player who is about to make a move knows full history of the play of the game so far

  Alternatively:
  - all information sets are singletons

- Game is with imperfect information if:
  - players do not know full history of the game
  - at least one information set is not singleton

Weak Perfect Bayesian Equilibrium

- Dynamic game with imperfect information:



NE in Dynamic Game

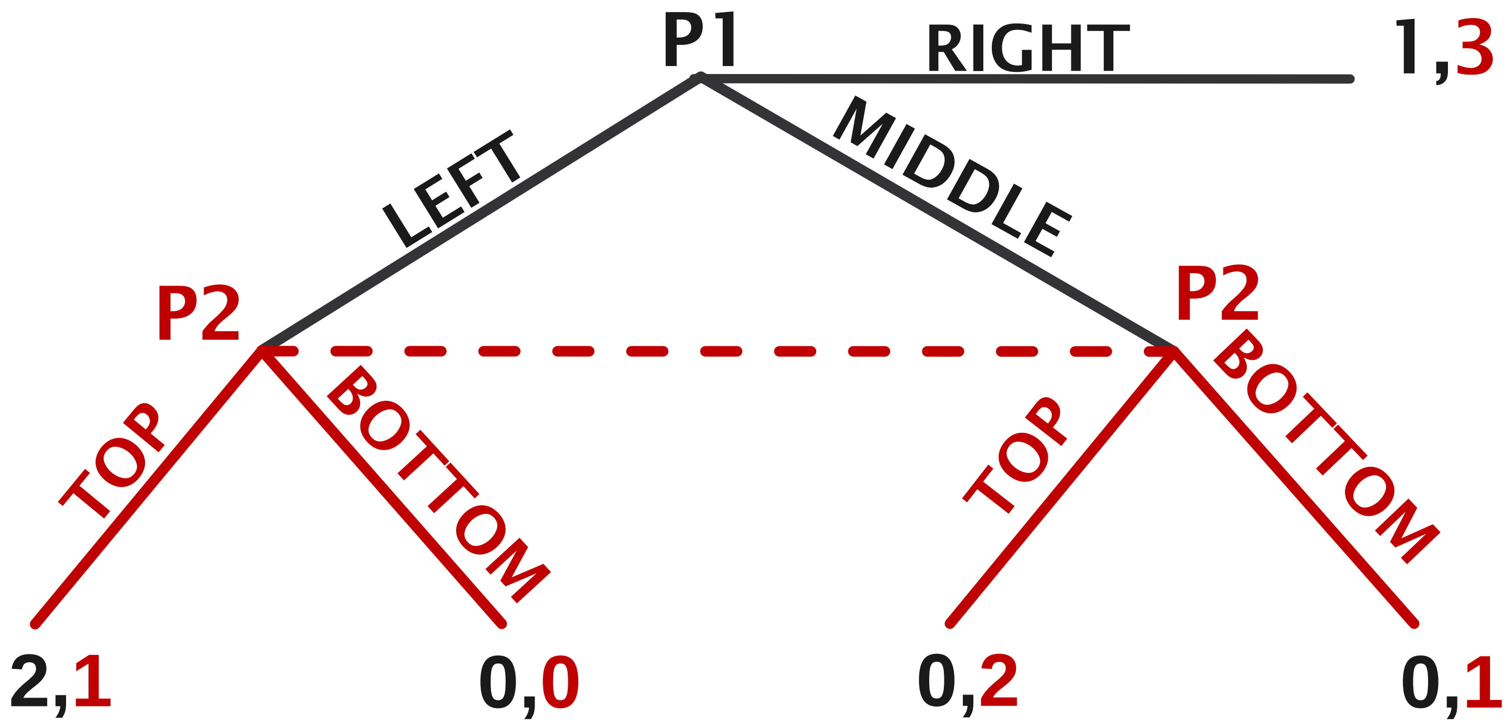
- Dynamic game with imperfect information:

1 \ 2	T	B
L	2,1	0,0
M	0,2	0,1
R	1,3	1,3

- Two NE: (L,T) and (R,B)
- Are these subgame perfect as well?
  - only one subgame, whole game (subgames can not cut through information sets) => yes, they are SPNE as well

NE in Dynamic Game

- But: (R,B) is empty threat and we want to eliminate this, and keep only (L,T) -> wPBE



NE in Dynamic Game

- But: (R,B) is empty threat and we want to eliminate this, and keep only (L,T) -> wPBE

- In dynamic games with perfect information
  - SPNE eliminates non-credible threats

- In dynamic games with imperfect information
  - SPNE is not strong enough, we need a new concept – weak perfect Bayesian equilibrium

Weak Perfect Bayesian Equilibrium

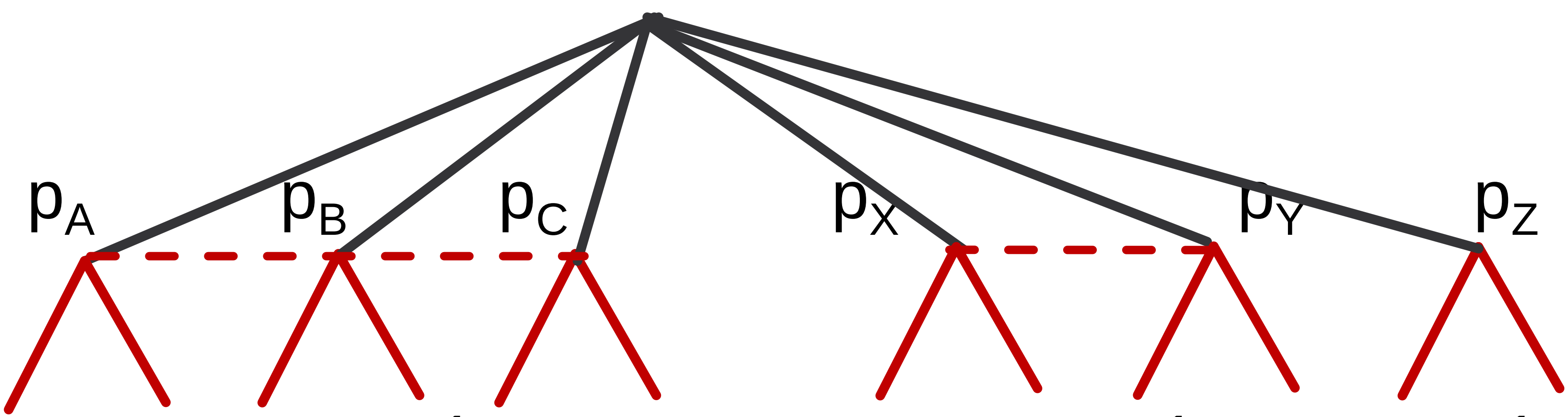
- weak perfect Bayesian equilibrium consists of equilibrium strategies and beliefs

Definition: A wPBE equilibrium consists of behavioral strategies and beliefs systems satisfying following conditions:

1. Sequential rationality - Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies
2. Consistency of beliefs with strategies – Each player's belief is consistent with strategy profile (behavioral strategies of all players)

Information Set - Beliefs

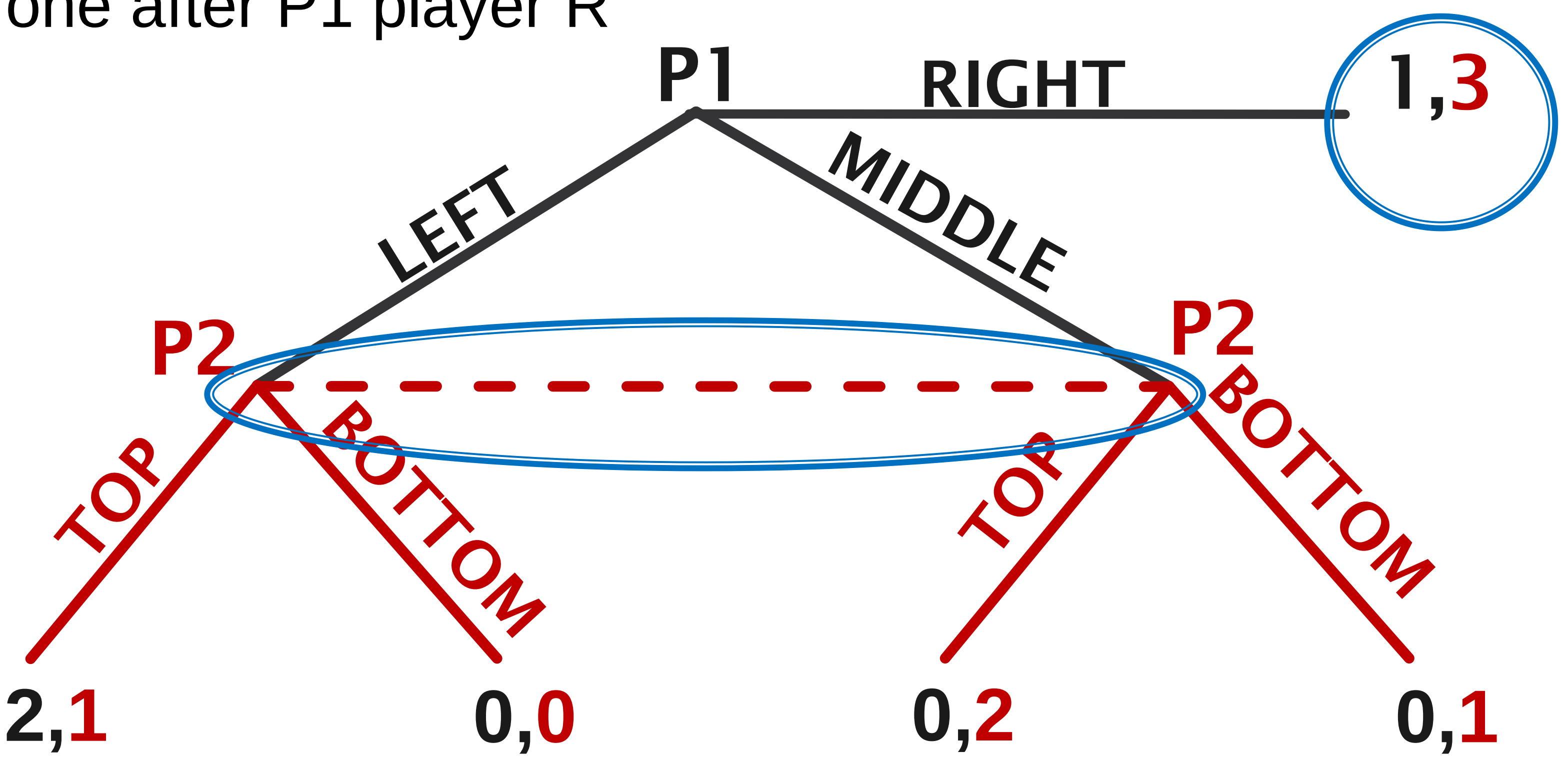
- weak perfect Bayesian equilibrium consists of equilibrium strategies and beliefs
- Belief system assigns to each information set a probability distribution over the decision nodes in that information set



$p_A + p_B + p_C = 1$; $p_X + p_Y = 1$; $p_Z = 1$

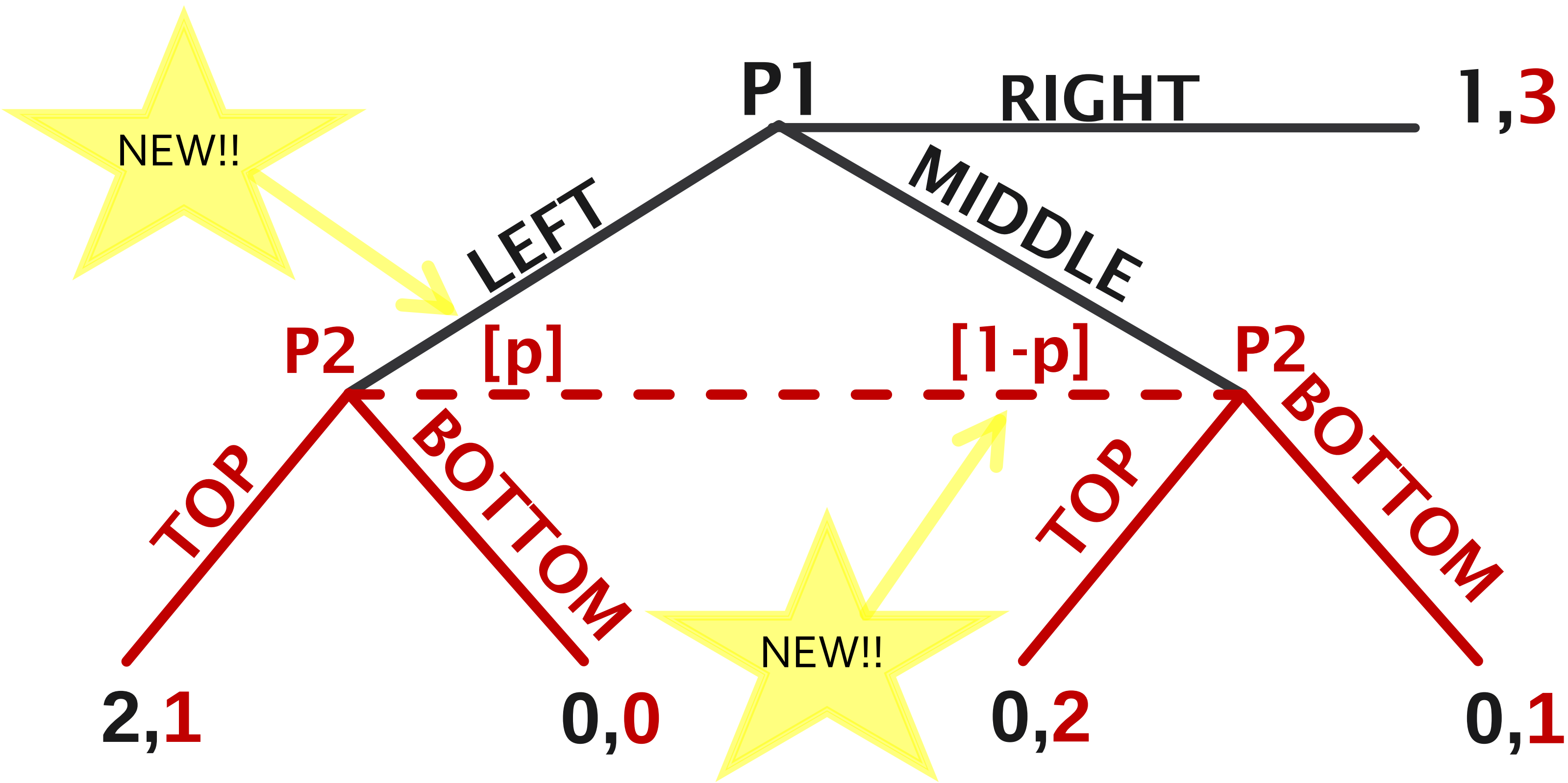
Imperfect Information

- P2 has two information sets:
  - one after P1 plays L or M
  - one after P1 player R



Weak Perfect Bayesian Equilibrium

- We need to specify equilibrium beliefs for information sets of Player 2



Weak Perfect Bayesian Equilibrium

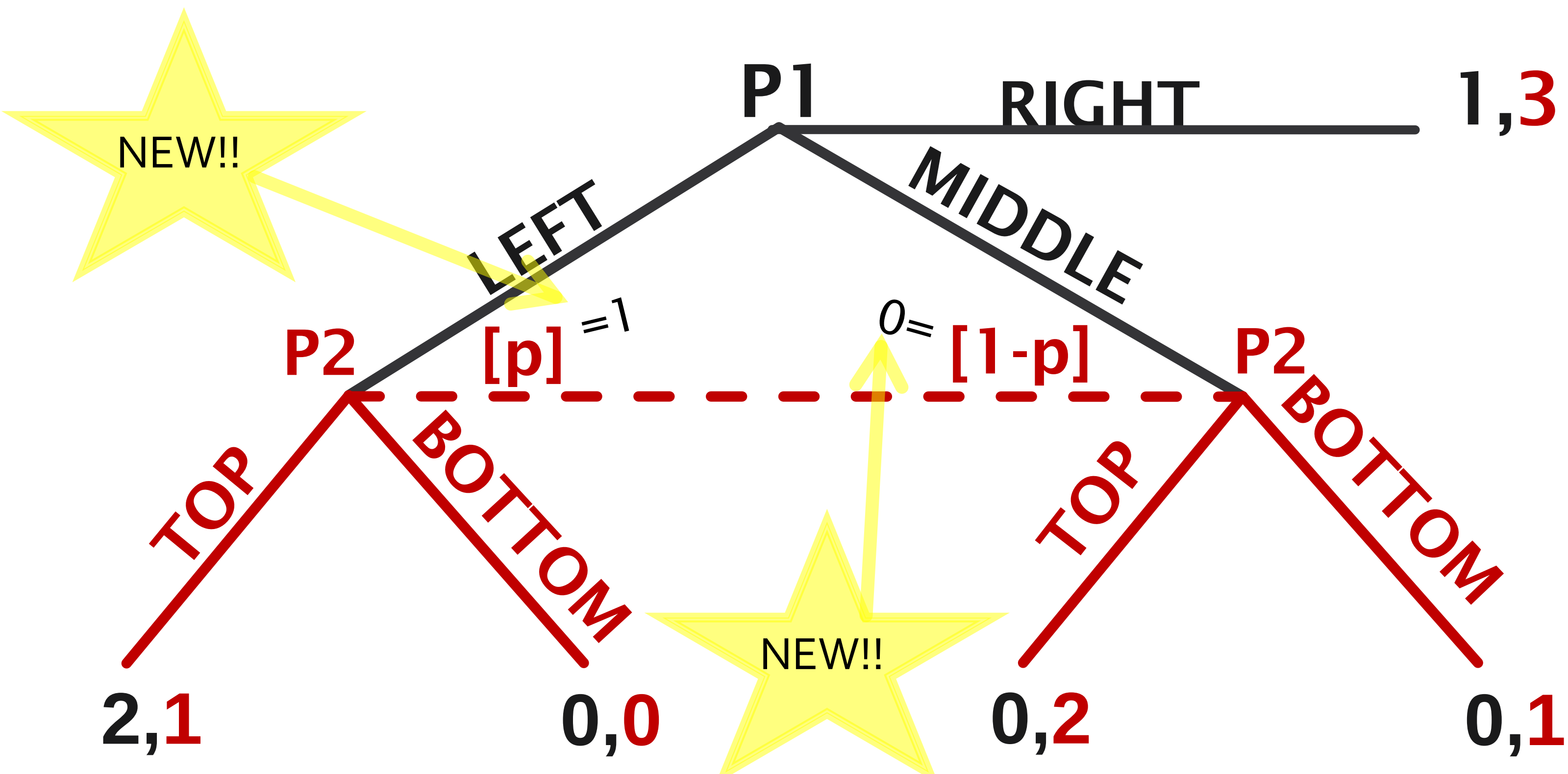
For all p:

- EP(T) = 1p + 2(1-p) = 2-p
- EP(B) = 0p + 1(1-p) = 1-p

- T is always better than B irrespective of the beliefs, because 2-p is always more than 1-p
- In other words, there are no such beliefs which would make P2 play B
- Sequential rationality requires that P2 chooses T => (R,B) is not wPBE and is eliminated

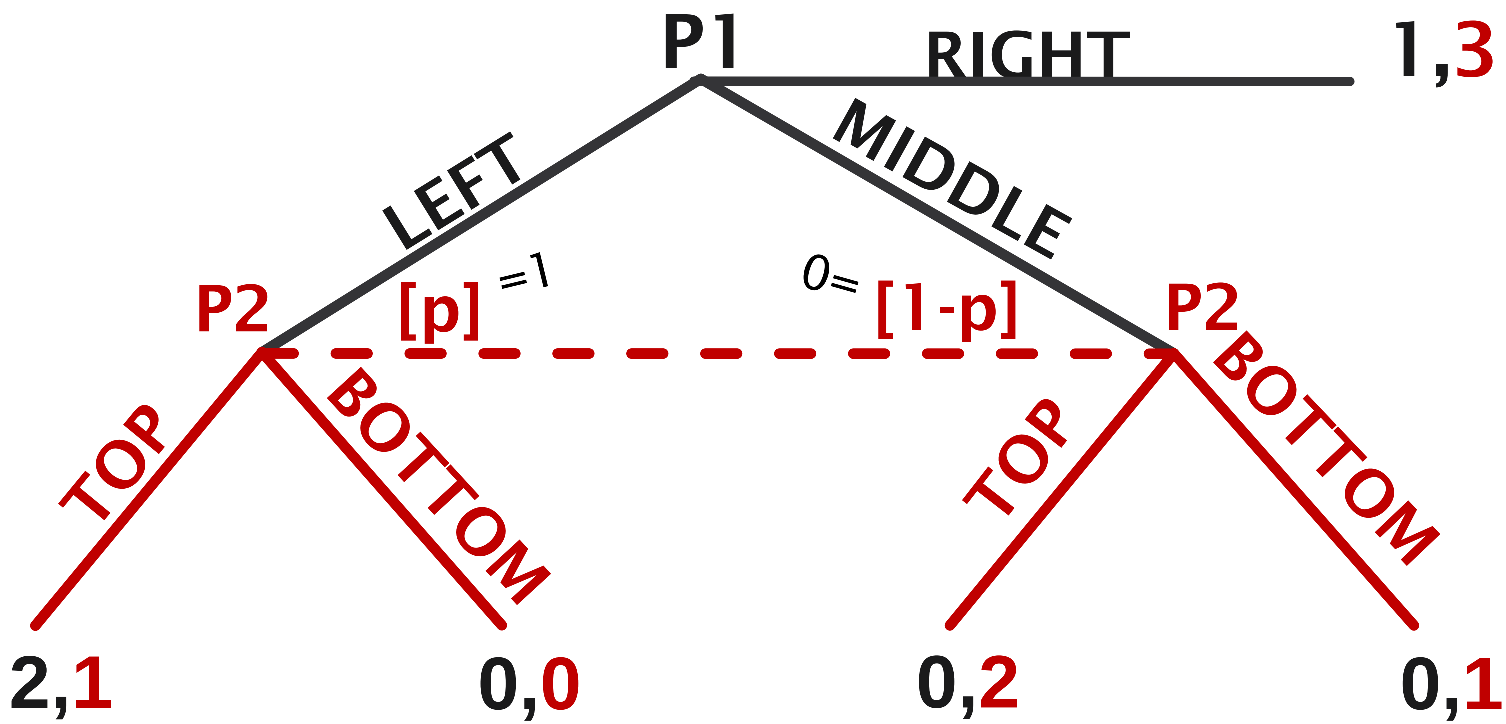
Weak Perfect Bayesian Equilibrium

- P1 knows that P2 always plays T => for P1, LEFT is better than MIDDLE
- consistency of beliefs requires that p = 1, 1-p = 0



Weak Perfect Bayesian Equilibrium

- wPBE: (L,T) and P2 beliefs that if he is in his information set, then P1 chose LEFT with probability 1 and Middle with probability 0



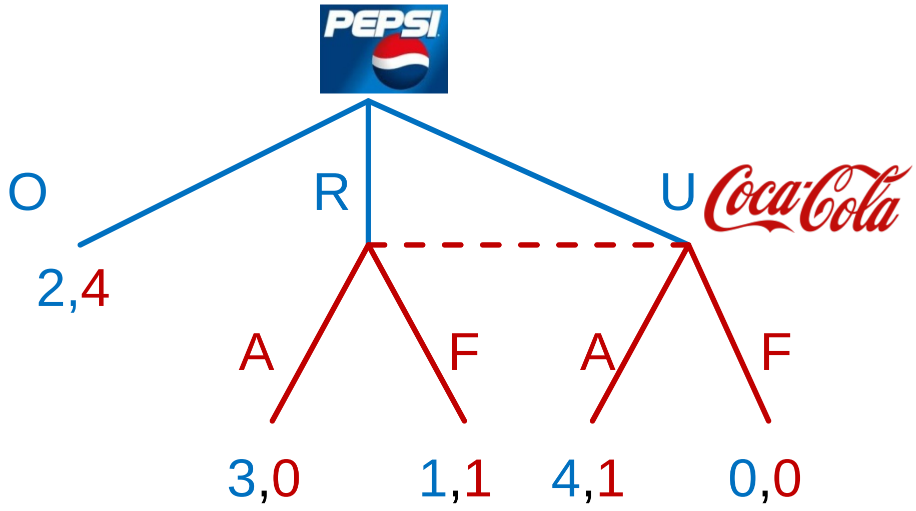
Weak Perfect Bayesian Equilibrium

Definition: A wPBE equilibrium consists of behavioral strategies and beliefs systems satisfying following conditions:

1. Sequential rationality - Each players' strategy is optimal whenever she has to move, given her belief and the other players' strategies
   P1->L, P2->T is optimal choice given P2's beliefs
2. Consistency of beliefs with strategies – Each player's belief is consistent with strategy profile (behavioral strategies of all players)
   P2 has beliefs consistent with action of P1 (he is certain to be in the left node)

WPBE

Pepsi is considering a new marketing war against Coca Cola. Pepsi can decide to stay out of the market (O). If Pepsi decides to enter the market Coca Cola knows that there is a new increased competition but does not observe whether Pepsi is ready (R) or unready (U). Coca Cola decides to accept (A) the new competitor or to fight (F).



How to Find WPBE

The way to find Weak perfect Bayesian equilibrium:

1. Write down normal form of the game
2. Find Nash equilibria (optimal actions)
3. Check NE one by one to see if consistent beliefs can be found (if yes, we have WPBE; if no, no WPBE)

This works because WPBE is subset of NE

P \ C	A	F
O	2,4	2,4
R	3,0	1,1
U	4,1	0,0

There are two NE: (0,F) and (U,A)

WPBE 1

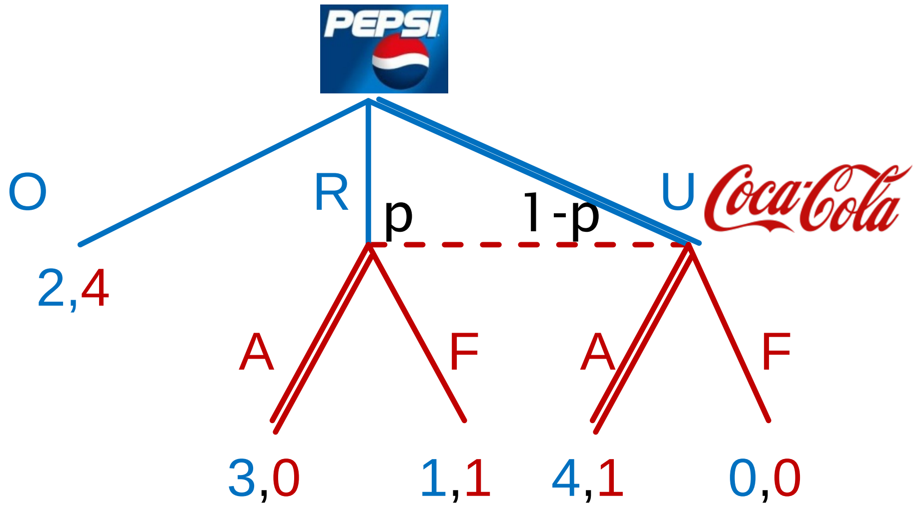
1. (U,A) is NE. Is it WPBE?

   Option 1: Information set is reached

   beliefs have to be consistent with actions =>

   p = 0 and 1-p = 1

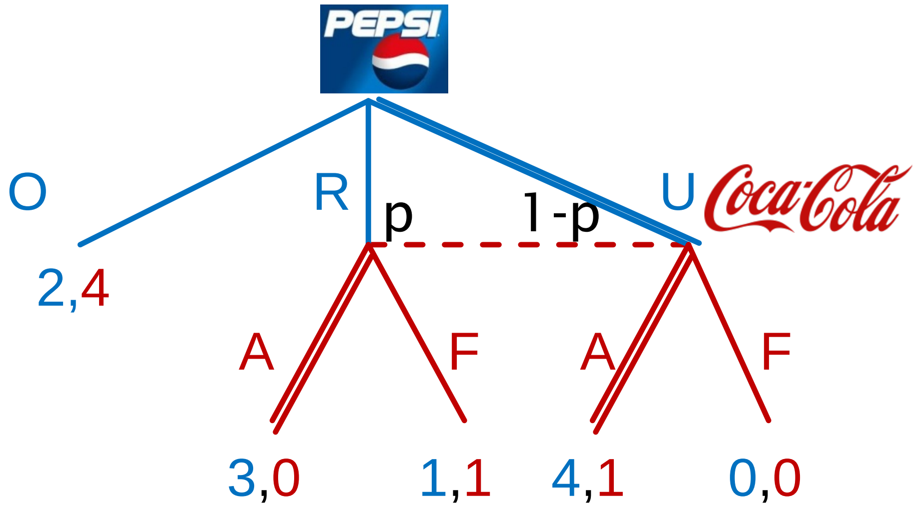
   WPBE 1 is (U,A) and p=0, 1-p=1



WPBE 1

WPBE 1 is (U,A) and p=0, 1-p=1

- P1 behaves optimally given action of P2
- P2 behaves optimally given his beliefs and P1's action
- P2's beliefs are consistent with P1's actions



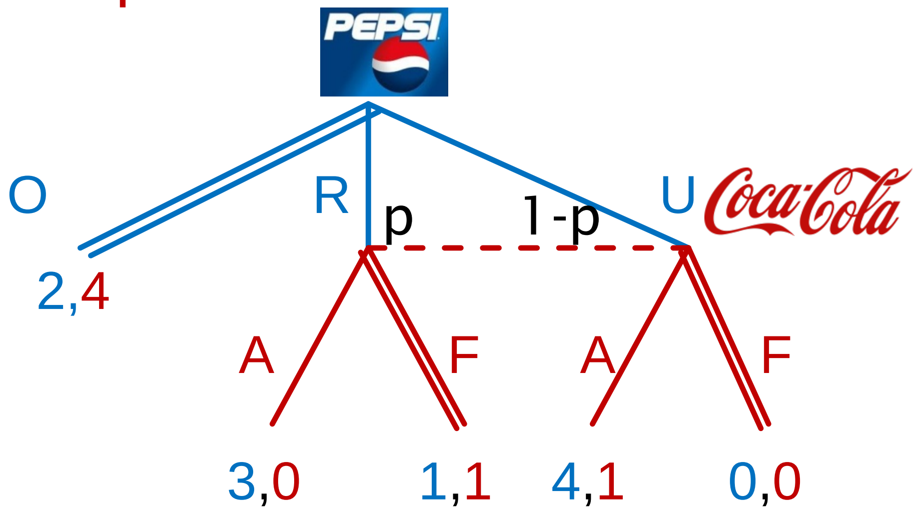
WPBE 2

2. (O,F) is NE. Is it WPBE?

Option 2: Information set is not reached

beliefs have to be consistent with actions =>
beliefs have nothing to be consistent with

set p in any way you like; such that actions of all players are still optimal



WPBE 2

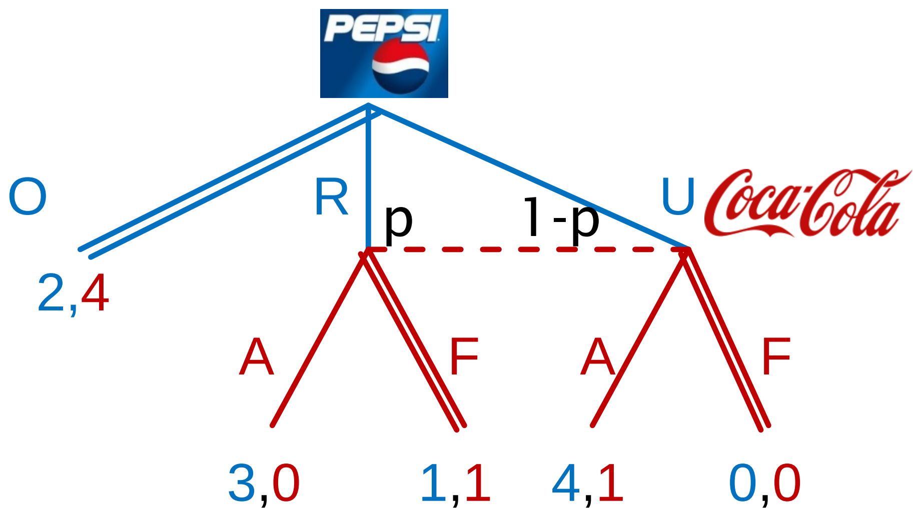
2. (O,F) is NE. Is it WPBE?

set p in any way you like; such that actions of all players are still optimal => F has to be optimal for CocaCola

<=> EP(F) ≥ EP(A)

$$p*1+(1-p)*0 \geq p*0+(1-p)*1$$

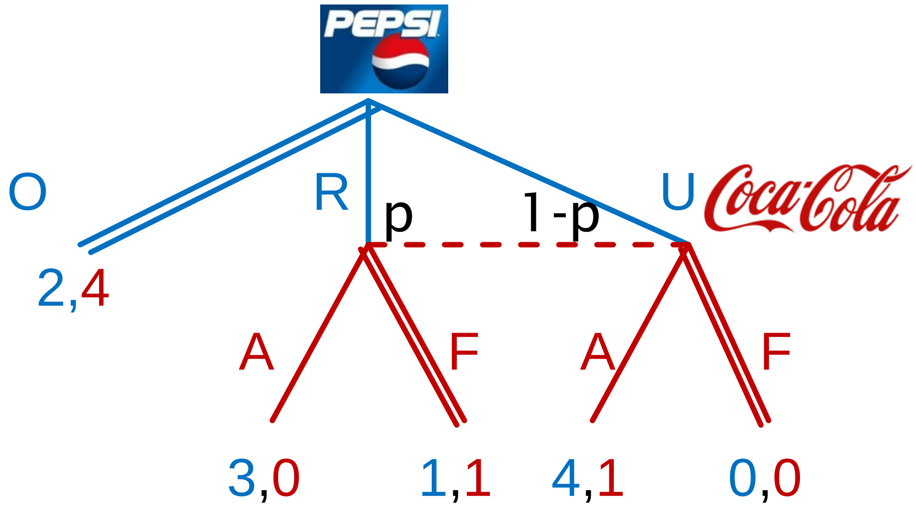
$$p \geq 1/2$$



WPBE 2

WPBE 2 is (O,F) and any p > ½ (F is better than A)

- P1 behaves optimally given action of P2
- P2 behaves optimally given his beliefs and P1's action
- P2's beliefs are consistent with P1's actions



WPBE 3

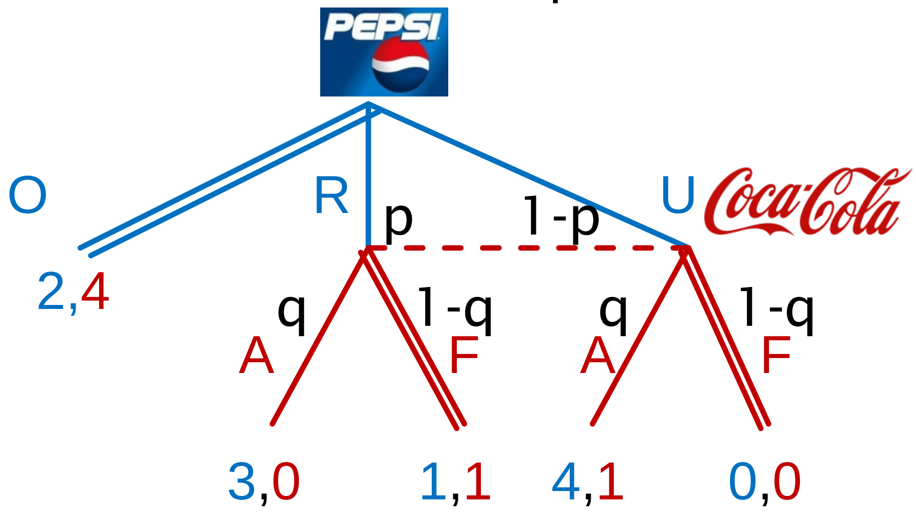
WPBE 3 is (O,F) and p = ½ (F and A are equally good)

- If P2 mixes, it has to be such that O is still optimal for P1:

  EP(O) > EP(R); EP(O) > EP(U)

  2 > 3*q + 1*(1-q) and 2 > 4*q + 0*(1-q)

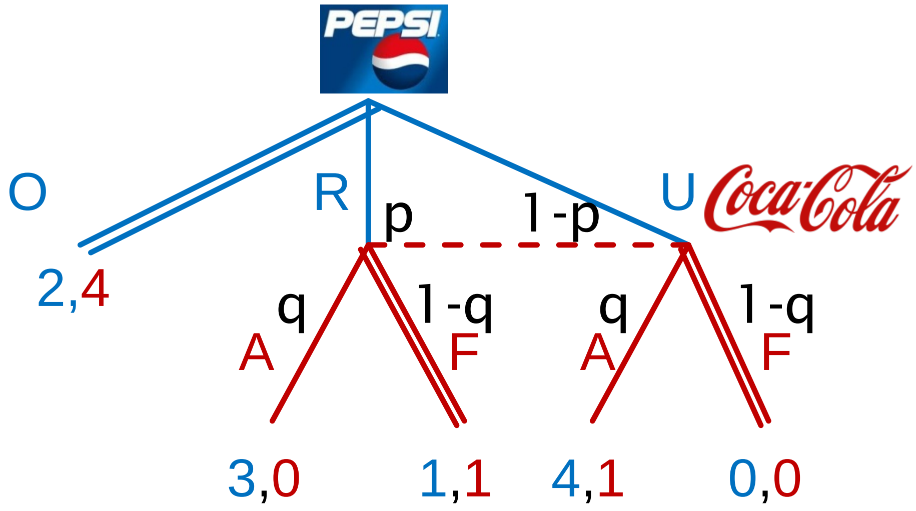
  q < 1/2 and q < 1/2



WPBE 3

WPBE 3 is (O,F) and p = ½; q < 1/2

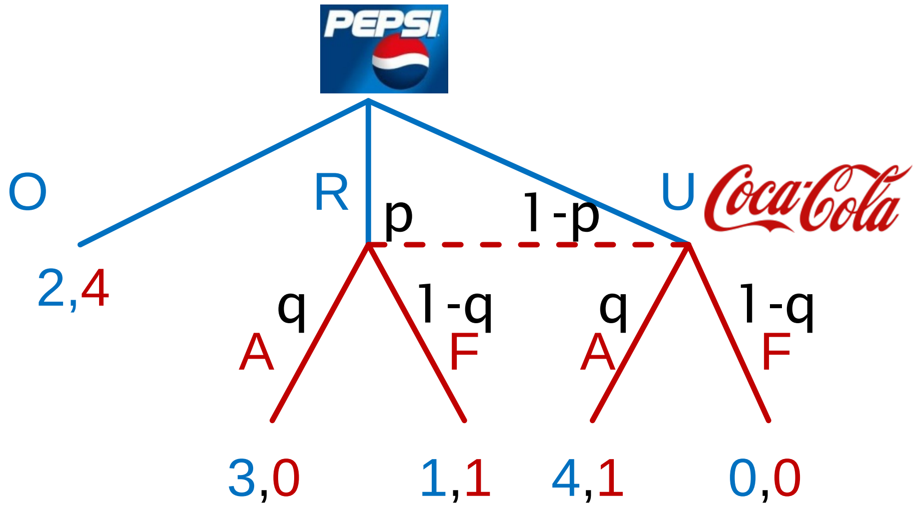
- P1 behaves optimally given action of P2
- P2 behaves optimally given his beliefs and P1's action
- P2's beliefs are consistent with P1's actions



WPBE

This game has three WPBE

- WPBE 1 is (U,A) and p=0, 1-p=1
- WPBE 2 is (O,F) and any p > ½
- WPBE 3 is (O,F) and p = ½; q < 1/2



Summary

- static games
  - perfect information: NE
  - imperfect information: (Bayesian) NE
- dynamic games
  - perfect information: SPNE
  - imperfect information: weak perfect Bayesian equilibrium
- weak perfect Bayesian equilibrium
  - sequential rationality
  - consistency of beliefs

Summary

Weak Perfect Bayesian Equilibrium:

- in Osborne – weak sequential equilibrium
- concept for dynamic games with imperfect info
- eliminates NE – non-credible threats

- find normal-form game
- find all NE
- for each NE find corresponding beliefs
- check for Sequential rationality and Consistency of beliefs