Problem 1: A Firm has the following production function: $Q = \sqrt{KL}$. Further we know that $w = 36$ and $r = 9$.

(a) Derive short run cost function when the amount of capital is fixed: $\overline{K} = 9$

(b) Derive long run cost function

Solution:

The difference between short run and long run is that in short run the amount of one of the inputs is fixed - in this case, the amount of capital is fixed at the level of $\overline{K} = 9$. So in short run the firm is optimizing only level of labor L such that the cost is as low as possible.

In short run the production function is given by:

$$Q = \sqrt{KL} = \sqrt{9L} = 3\sqrt{L} \quad \Rightarrow \quad L = \frac{Q^2}{9}$$

So for any level of production, Q, the total cost is given by:

$$TC = wL + rK = 36\frac{Q^2}{9} + 9 * 9 = 81 + 4Q^2$$

Note that in **short run** TC is easy to determine, because we can not vary the level of one input - capital. Hence if the firm wants to produce a certain level of output Q, there is only one way in which it is possible and it is to use $\frac{Q^2}{9}$ units of labor. So we do not really have any optimization problem here.

In **long run** the situation is different because there are many ways (different combinations of L and K) to produce certain level of output and we need to find the "best" one. For that we need to use the optimality condition:

$$TRS = \frac{MP_L}{MP_K} = \frac{w}{r}$$
$$TRS = \frac{\sqrt{K}/2\sqrt{L}}{\sqrt{L}/2\sqrt{K}} = \frac{w}{r} \quad \Rightarrow \quad \frac{K}{L} = \frac{w}{r}$$

Optimality condition and production function represent system of two equations and two unknowns which we can solve:

$$\frac{K}{L} = \frac{w}{r}$$
$$\sqrt{KL} = Q \quad \Rightarrow \quad KL = Q^2 \quad \Rightarrow \quad K = \frac{Q^2}{L}$$

We plug this to the first equation:

$$\frac{K}{L} = \frac{w}{r} \quad \Rightarrow \quad \frac{\frac{Q^2}{L}}{L} = \frac{w}{r} \quad \Rightarrow \quad L = Q\sqrt{\frac{r}{w}}$$
$$K = \frac{Q^2}{L} = \frac{Q^2}{Q\sqrt{\frac{r}{w}}} = Q\sqrt{\frac{w}{r}}$$

Now we know that if the firm wants to produce the level of output Q, the cheapest way to do it is to use $Q\sqrt{\frac{r}{w}}$ units of labor L and $Q\sqrt{\frac{w}{r}}$ units of capital K. So the total cost function (cost of cheapest way to produce Q units of output) is:

$$TC = wL + rK = w * Q\sqrt{\frac{r}{w}} + r * Q\sqrt{\frac{w}{r}} = 36Q\sqrt{\frac{9}{36}} + 9Q\sqrt{\frac{36}{9}} =$$
$$= 18Q + 18Q = 36Q$$

Problem 2: Total cost function of an individual firm facing perfect competition is given by relation:

$$TC(Q) = Q^3 - 20Q^2 + 150Q$$

Find the optimal level of production (supply function) of this firm. For what prices would the firm earn a positive profit? Negative profit? Zero profit?

Solution:

In optimum the firm chooses the level of production such that its marginal cost is equal to the market price:

$$MC = TC' = 3Q^2 - 40Q + 150 = P$$

So the optimal level of production (firm's supply function) is implicitly given by the equation above. For each value of price P we can calculate the quantity supplied Q.

If we knew the exact price and corresponding quantity we would be able to calculate exact profit of this firm. But we do not have this information. So we will use the fact that the profit of the firm is:

- positive if $AC < P$
- zero if $AC = P$
- negative if $AC > P$

So to see if the firm makes positive, zero or negative profit we compute average cost:

$$AC = \frac{TC}{Q} = Q^2 - 20Q + 150$$

Following the argument above we get that the profit of the firm is:

- positive if $Q^2 - 20Q + 150 < P$
- zero if $Q^2 - 20Q + 150 = P$
- negative if $Q^2 - 20Q + 150 > P$

Working with these quadratic equation and inequalities would be a little complicated, but we can use the fact that the profit of the firm is zero if and only if the price equals minimum of average cost (or the price is where average cost and marginal cost curves cross). In this case the optimality condition $P = MC$ leads to optimal quantity for which price equals average cost and hence the profit is zero. Then whenever the price is higher (lower) than this value the firm makes positive (negative) profit. So let's find the value of price P such that $MC = AC$, or $P = \min AC$:

$$MC = AC \quad \Rightarrow \quad 3Q^2 - 40Q + 150 = Q^2 - 20Q + 150 \quad \Rightarrow \quad 2Q^2 - 20Q = 0$$

This only holds for $Q = 0$ and $Q = 10$. If $Q = 10$ then the corresponding price P is:

$$P = MC \quad \Rightarrow \quad P = 3Q^2 - 40Q + 150 = 3 * 10^2 - 40 * 10 + 150 = 50 \quad \Rightarrow \quad P = 50$$

Let's check if our result is correct - for price $P = 50$ we get that the optimal (cost minimizing, profit maximizing) level of production is such that:

$$P = MC \quad \Rightarrow \quad 50 = 3Q^2 - 40Q + 150 \quad \Rightarrow \quad 3Q^2 - 40Q + 100 = 0$$
$$Q_{1,2} = \frac{40 \pm \sqrt{40^2 - 4 * 3 * 100}}{6} = \frac{40 \pm \sqrt{400}}{6} = \frac{40 \pm 20}{6} = \frac{20}{6}, \ 10$$

Remember, that it is **never optimal** to produce at the decreasing part of MC curve (why?) so the firm will choose to produce $Q = 10$. Then the profit is:

$$\pi = P * Q - TC(Q) = P * Q - (Q^3 - 20Q^2 + 150Q) =$$
$$= 50 * 10 - 10^3 + 20 * 10^2 - 150 * 10 = 0$$

So we showed that if price $P = 50$ and firm is producing profit maximizing level of output $Q = 10$ then it makes zero profit. From this result it directly follows that if $P > 50$ the firm makes positive profit and if $P < 50$ the firm makes negative profit.

Problem 3: The demand for oranges is $q = 120 - 4p$ and the supply is $q = 2p - 30$, where p is the price measured in dollars per hundred pounds and q is the quantity measured in hundred pound units.

(a) On one graph, draw the demand curve and the supply curve for oranges.

(b) Write down the equation that you would solve to find the equilibrium price.

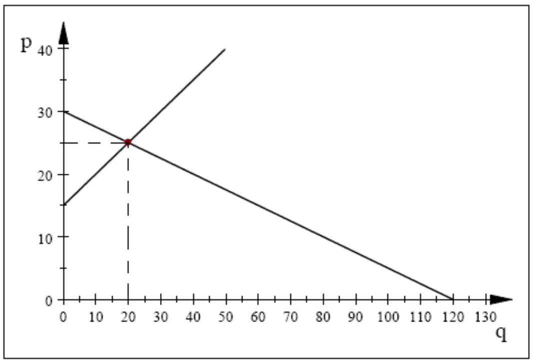
(c) What is the equilibrium price of oranges? What is the equilibrium quantity? Show the equilibrium price and quantity on the graph and label them p_1 and q_1.

(d) A terrible drought strikes California, traditional homeland of oranges. The supply schedule shifts to $2p - 60$. The demand schedule remains as before. Draw the new supply schedule. Write down the equation that you would solve to find the new equilibrium price of oranges.

(e) What is the new equilibrium price of oranges? What is the new equilibrium quantity? Show the equilibrium price and quantity on the graph and label them p_2 and q_2.

Solution:

(a) Graph with demand and supply curve:



(b)

$$D(p^*) = 120 - 4p^* = 2p^* - 30 = S(p^*)$$

(c) Solving the equation in (b):

$$120 - 4p_1 = 2p_1 - 30$$
$$150 = 6p_1 \quad \Rightarrow p_1 = \frac{150}{6} = 25$$

Evaluating the demand (or supply) equation at this price:

$$q_1 = 120 - 4p_1 = 20$$

(d) The equation is:

$$D(p^*) = 120 - 4p^* = 2p^* - 60 = S(p^*)$$

(e) Solving the equation in (b):

$$p_2 = \frac{180}{6} = 30$$

Evaluating the demand (or supply) equation at this price:

$$q_2 = 120 - 4p_2 = 0$$