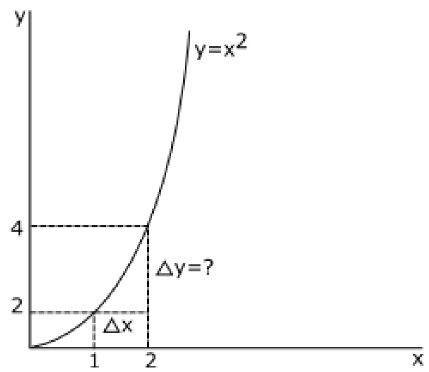
Derivatives, Extremes and Optimization

The derivative is a measurement of how a function changes when its input changes. Loosely speaking, a derivative can be thought of as how much a quantity is changing at some given point. The derivative of a function at a chosen input value describes the best linear approximation of the function near that input value. For a real-valued function of a single real variable, the derivative at a point equals the slope of the tangent line to the graph of the function at that point.

Differentiation is a method to compute the rate at which a dependent output y, changes with respect to the change in the independent input x. This rate of change is called the derivative of y with respect to x. In more precise language, the dependency of y on x means that y is a function of x. If x and y are real numbers, and if the graph of y is plotted against x, the derivative measures the slope of this graph at each point. This functional relationship is often denoted y = f(x), where f denotes the function.

In economics we start in some equilibrium point and want to analyze what happens in case of some exogenous changes. For example in a market model the initial equilibrium is represented by equilibrium price P^* and quantity Q^*. If some exogenous change occur the initial equilibrium will change. Derivatives measure the direction and speed of this change.

Example: $y = f(x) = x^2$. We refer to y and x as dependent and independent variable respectively. Suppose that x changes by Δx. The question is, how much will y change. Even more importantly, we are interested in rate of change: $\frac{\Delta y}{\Delta x}$. As Δy approaches zero we talk about the derivative of f - $f'(x)$. In case of function $y = f(x) = x^2$, the derivative is $f'(x) = 2x$.



Examples:

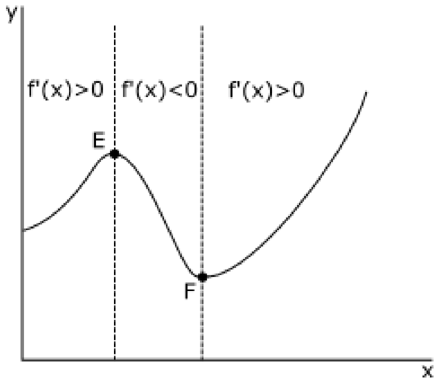
$$(const)' = 0 \qquad (10)' = 0$$
$$(ax)' = a \qquad (2x)' = 2$$
$$(x^n)' = nx^{n-1} \qquad (x^3)' = 3x^2$$
$$(\ln x)' = \frac{1}{x} \qquad (2\ln x)' = \frac{2}{x}$$

Problem 1: Total cost function has the following form: $C = Q^3 - 4Q^2 + 10Q + 75$. Find the marginal-cost function.

Solution: $MC = C' = \frac{dC}{dQ} = 3Q^2 - 8Q + 10$.

Extremes:

Extreme denotes point of maximum or minimum. Finding extremes is used in all optimization problems (profit/utility maximization, cost minimization). Point E on the picture below is relative (local) maximum, point F is absolute (global) minimum.



Note:

- If $f'(x) > 0$ then a function $f(x)$ increases
- If $f'(x) < 0$ then a function $f(x)$ decreases

If a relative extreme occurs at some point A then the derivative of a function at point A is equal to zero.

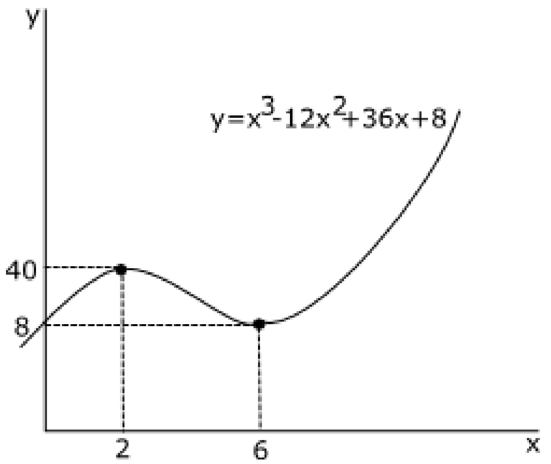
The opposite implication (if derivative is zero in point A then a relative extreme occurs in that point) does not hold. For example $y = f(x) = x^3$. First derivative has to change the sign around point A.

Problem 2: Find the relative extrema of the function: $y = f(x) = x^3 - 12x^2 + 36x + 8$.

Solution: First, we find the derivative:

$$f'(x) = 3x^2 - 24x + 36 = 0$$

Applying quadratic formula we get that the last equation holds for $x = 6$ and $x = 2$. Since f'(6)=f'(2)=0, these two values of x are the critical values we desire. Now, we need to decide what type of extreme (minimum/maximum) we found. In the immediate neighborhood of $x = 6$, we have $f'(x) < 0$ for $x < 6$ and $f'(x) > 0$ for $x > 6$; thus the value of the function f(6)=8 is a relative minimum. Similarly, in the immediate neighborhood of $x = 2$, we have $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$; thus the value of the function f(2)=40 is a relative maximum.



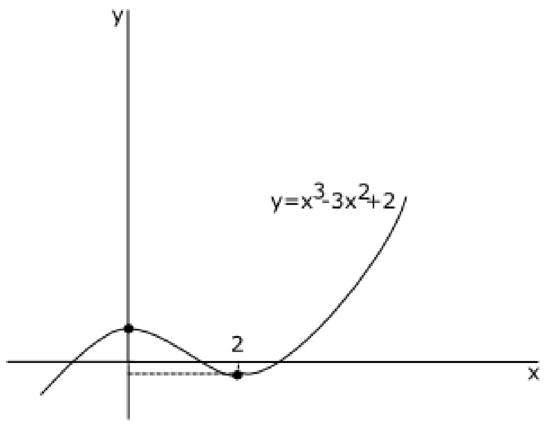
To decide if a relative extreme point is minimum or maximum we also can use the second derivative of the function at a given point. If it is positive (negative) then we have the case of minimum (maximum).

Problem 3: Find relative extreme of the function $y = f(x) = 4x^2 - x$

Solution: We find first derivative of the function: $y' = f'(x) = (4x^2 - x)' = 8x - 1$. We put this derivative equal to zero and get a point of extreme: $f'(x) = 8x - 1 = 0 \;\Rightarrow\; x = \frac{1}{8}$. We use second derivative to decide if this point of extreme is point of minimum or maximum: $f"(x) = (8x - 1)' = 8 > 0$. Second derivative is positive and hence $x = \frac{1}{8}$ is point of minimum.

Problem 4: Find relative extreme of the function $y = f(x) = x^3 - 3x^2 + 2$

Solution: We find first derivative of the function: $y' = f'(x) = (x^3 - 3x^2 + 2)' = 3x^2 - 6x$. We put this derivative equal to zero and get a point of extreme: $f'(x) = 3x^2 - 6x = 0 \Rightarrow x = 0$ or $x = 2$. We use second derivative to decide if these points of extreme are points of minimum or maximum: $f"(x) = (3x^2 - 6x)' = 6x - 6$. For $x = 0$ the value of second derivative is negative $f"(x) = 6x - 6 = 60 - 6 = -6 < 0$ and thus $x = 0$ is a point of relative maximum. Similarly, for $x = 2$ the value of second derivative is positive $f"(x) = 6x - 6 = 62 - 6 = 6 > 0$ and thus $x = 2$ is a point of relative minimum.



Summary:

1. Find first derivative of the function
2. Put the derivative equal to 0 and find potential point(s) of extreme
3. Find second derivative of the function to decide if point(s) from 2. is minimum/maximum/neither of them
4. If constrained optimization: check for corner solutions, i.e. check if the end points from the constraint do not give higher (lower) value that value in maximum (minimum)