Problem 1: Robinson-Crusoe economy. Consider Robinson-Crusoe sitting on his island who is trying to survive by either gathering bananas (x denotes the number of bananas) or going fishing (y is the number of fish). The technologies which are available for these two production processes are given by:

$$x = \sqrt{L_x}$$
$$y = 0.5\sqrt{L_y}$$

where L_x is the amount of labor (time) used for gathering bananas and L_y the time devoted to fishing. Notice that there are diminishing returns in both activities. Total time (L) is constrained to 15 hours per day. So $L_x + L_y = L = 15$.

Further we assume that Robinson Crusoe's preferences are described by a Cobb-Douglas utility function:

$$U(x, y) = \sqrt{x}\sqrt{y}$$

(a) Construct the production possibility frontier (PPF).

(b) Sketch several indifference curves and find marginal rate of substitution (MRS).

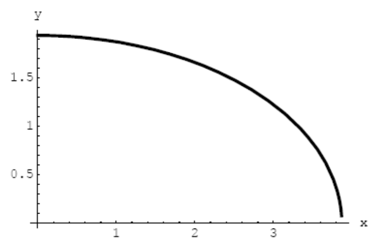
(c) Find the optimal combination of two goods and optimal allocation of time.

Solution:

(a) An implicit form for the PPF is given by:

$$L = x^2 + 4y^2$$
$$\text{or:} \quad y = \frac{\sqrt{L - x^2}}{2}$$

PPF is depicted on the picture below:



The marginal rate of transformation (MRT) is the slope of the PPF.

$$y = \frac{\sqrt{L - x^2}}{2} \quad \Rightarrow \quad MRT = y' = -\frac{x}{2\sqrt{L - x^2}} = -\frac{x}{2\sqrt{15 - x^2}}$$

in absolute terms: $MRT = \dfrac{x}{2\sqrt{15 - x^2}}$

Interpretation: Assume that Robinson-Crusoe produces 2 units of x initially. Evaluating the MRT at $x = 2$ gives:

$$MRT = \frac{2}{2\sqrt{15 - 4}} = 0.3$$

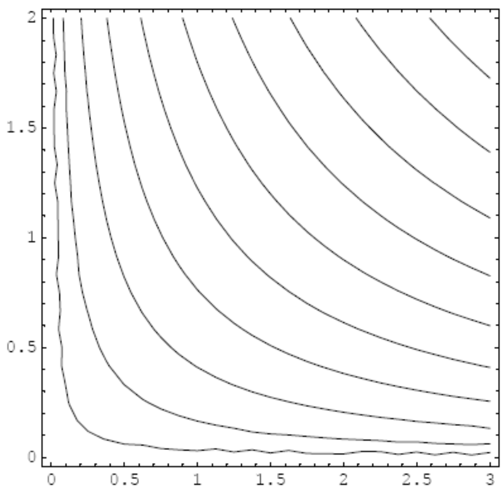
If Robinson Crusoe wants to produce one more unit of x, then he is forced to reduce the number of y by 0.3.

(b) MRS or the slope of an indifference curve is:

$$MRS = \frac{\partial U/\partial x}{\partial U/\partial y} = \frac{\frac{\sqrt{y}}{2\sqrt{x}}}{\frac{\sqrt{x}}{2\sqrt{y}}} = \frac{y}{x}$$

Interpretation: Assume Robinson Crusoe consumes initially 1 unit of y and 2 units of x; hence $MRS = 0.5$. Reducing the consumption of y by 0.1 units requires to increase the consumption of x by 0.2 units to keep utility constant. The MRS hence gives Robinson's marginal valuation of good y in terms of good x (and vice versa).

To illustrate, we can plot a number of indifference curves for alternative utility levels:



(c) Utility is maximized if MRT = MRS. We employ this efficiency condition to solve the underlying problem.

$$MRT = \frac{x}{2\sqrt{15 - x^2}}$$
$$MRS = \frac{y}{x}$$
$$MRT = MRS \quad \Rightarrow \frac{x}{2\sqrt{15 - x^2}} = \frac{y}{x}$$

In addition we know that the optimum lies on PPF. So we have:

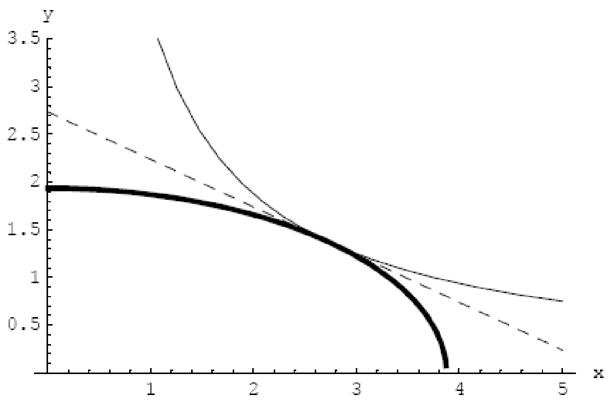
$$MRT = MRS \quad \Rightarrow \frac{x}{2\sqrt{15 - x^2}} = \frac{y}{x}$$
$$y = \frac{\sqrt{L - x^2}}{2} = \frac{\sqrt{15 - x^2}}{2}$$

Solving this system we get:

$$x^2 = 2y\sqrt{15 - x^2} = 15 - x^2 \quad \Rightarrow \quad x = \sqrt{\frac{15}{2}}$$
$$2y = \sqrt{15 - x^2} \quad \Rightarrow \quad y = \frac{\sqrt{15 - x^2}}{2} = \frac{\sqrt{\frac{15}{2}}}{2}$$

And finally:

$$x = \sqrt{L_x} \quad \Rightarrow \quad L_x = x^2 = \frac{15}{2}$$
$$y = 0.5\sqrt{L_y} \quad \Rightarrow \quad L_y = 4y^2 = \frac{15}{2}$$



Problem 2: Two countries, Brazil and Vietnam, can each produce two goods: coffee and tea. In one year, Brazil can produce 40 units of coffee (millions of 60kg-bags) or 200 units (tons) of tea or a mix of the two. In one year, Vietnam can produce 15 units of coffee or 150 units of tea or a mix of the two. For both countries, the marginal returns are decreasing. This means that using more and more resources to produce coffee brings less and less additional output of tea and vice versa for both countries.

(a) What are Brazil's opportunity cost of producing coffee? How about Vietnam's opportunity cost of tea production?

(b) Suppose that each country prefers to consume the commodity which is produced with relatively higher costs. Which of the commodities is it (for both Brazil and Vietnam)?

(c) Sketch the optimal points of consumption on a graph with tea on horizontal and coffee on vertical axis.

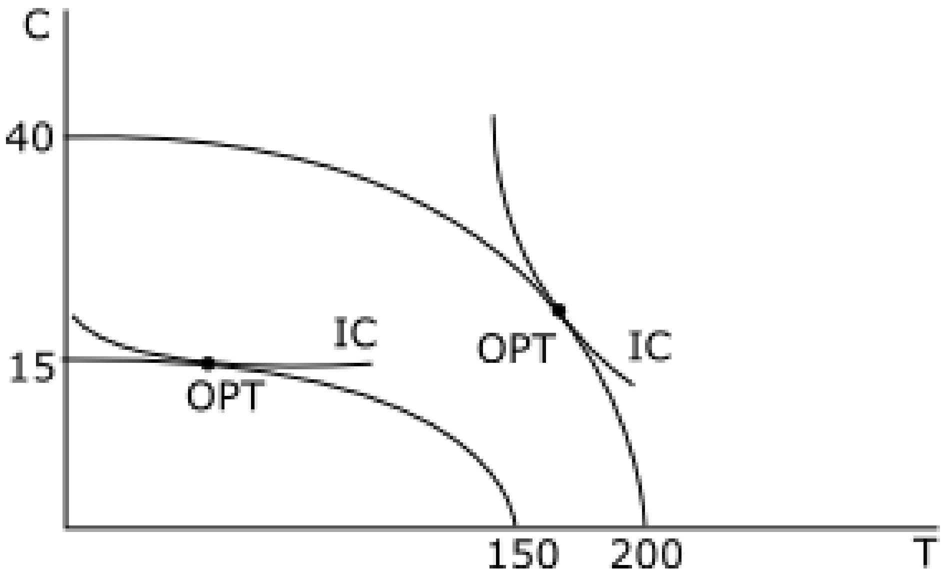
(d) Show graphically, that there exist prices (coffee for tea), that both countries would accept (they will both get to higher IC).

Solution:

(a) For each unit of coffee, Brazil has to sacrifice $200/40 = 5$ units of tea. This is the opportunity cost of coffee production. For each produced unit of tea, Vietnam has to forgo $15/150 = 1/10$ units of coffee which is the Vietnam's opportunity cost of tea production.

(b) Similarly, Brazil's opportunity cost of tea production is $1/5$ units of coffee and Vietnam's opportunity cost of coffee production is 10 units of tea. Hence coffee production is relatively cheaper in Brazil (5 versus 10 in terms of opportunity cost) and tea production is relatively cheaper in Vietnam ($1/10$ versus $1/5$ in terms of opportunity cost). So Brazil prefers to consume tea and Vietnam prefers to consume coffee.

(c) The optimal consumption is depicted on the picture below.



(d) The ratio of prices (slope of a black dashed line) has to be such that it is higher than MRS of Vietnam and lower than MRS of Brazil (slopes of blue lines). This way both countries will have incentive to specialize more in production of a commodity that they can produce at relatively lower cost (point where the line with the slope equal to the ratio of prices touches production possibility frontier) and exchange it for consumption of a commodity that they prefer more (point where the highest possible indifference curve touches the line with the slope equal to the ratio of prices).