Problem 1: Suppose the agent has a utility of wealth w and effort a given by $u(w,a) = \sqrt{w} - a/2$. The reservation utility is given by 0. The effort a is chosen from the set $A = [0, \infty)$, and conditional on this effort, the distribution of profits is governed by the exponential distribution with parameter $1/a$, whose density on $[0, \infty)$ is given by

$$f(\pi|a) = \frac{e^{-\pi/a}}{a}.$$

Note that under this distribution, $E(\pi|a) = a$, and $Var(\pi|a) = a^2$. Assume throughout the problem that the first-order approach is valid. Your calculations may also be simplified by the observation that $f_a/f = \partial \ln(f)/\partial a$.

(a) Write out the principal's problem when the agent's effort is *observable*. Clearly state the objective, principal's choice variables, and any constraints and their meaning.

(b) Solve for the optimal compensation scheme when the agent's effort is *observable*. What is the welfare of the principal and the agent in this case?

(c) Write out, in two stages, the principal's problem when the agent's effort is *unobservable*. In the first stage, compensation cost is minimized for a fixed effort level. Write down both the original constraints, and their modification using the first-order approach. In the second stage, optimal effort level is chosen. In each stage, clearly state the objective, principal's choice variables, and any constraints and their meaning.

(d) Using the first-stage program, characterize the optimal compensation scheme for a fixed effort level a when the agent's effort is *unobservable* in terms of the two multipliers. Don't pin down the multipliers at this stage. What is the sign of the multiplier on the participation constraint? Why? Assuming that it is not negative, what is the sign of the multiplier on the incentive constraint? Does it depend on a? Why? What is the shape of the compensation schedule?

(e) Using the participation and the incentive constraint, solve for the wage cost minimizing compensation schedule for a fixed effort level a. That is, pin down the two multipliers. How does the compensation schedule depend on a? Derive the expected cost function for inducing an effort level a.

(f) Using the second-stage program, pin down the optimal level of effort and the profit-maximizing compensation schedule. Assume that is the manager is indifferent among multiple levels of effort, he will pick whatever level is intended by the principal. How do the effort level and the welfare differ from the first-best? Why?

(g) Now suppose that there is another signal π_2 of the manager's effort that does not on its own contribute to the principal's payoff. It is distributed according to the same exponential distribution shown above. In addition, conditional on effort, the two profit levels are independent. Effort is unobservable. What is the compensation schedule a function of in this case? Using the two-stage approach, solve for the optimal compensation schedule for a fixed effort level a, the expected cost of effort, and then pin down the optimal level of effort and the final form of the compensation schedule. Does total compensation only depend on the sum of $\pi_1 + \pi_2$? How do the effort level and the welfare differ from the case with one signal only? Why?

Solution:

(a) Principal solves

$$\max_{a,w(\cdot)} \int_0^\infty [\pi - w(\pi)]\, f(\pi|a)d\pi = \max_{a,w(\cdot)} \left\{a - \int_0^\infty w(\pi) f(\pi|a)d\pi\right\}$$
$$\text{s.t. PC } \int_0^\infty \sqrt{w(\pi)} f(\pi|a)d\pi - a/2 \geq 0.$$

(b) When effort is observable, it can be stipulated by a forcing contract that imposes huge penalties on the manager for deviating from the prescribed level of effort. At the same time, as long as this effort level is provided, efficient risk-sharing implies that the wage will be fixed at some level w irrespective of the realization of π. Then the participation constraint implies that $\sqrt{w} - a/2 = 0$, or $w = a^2/4$. Being in full control of the effort, the principal solves

$$\max_{a\in[0.\infty)} a - a^2/4,$$

giving the optimal effort level

$$a^* = 2.$$

As a result, the manager is paid a flat wage $w = 1$. Under this effort level, the manager's utility is equal to the reservation utility 0, while the principal's expected payoff is equal to 1.

(c) In the first stage, principal solves

$$
\begin{aligned}
c(a) \equiv \min_{w(\cdot)} & \int_0^\infty w(\pi) f(\pi|a) d\pi \\
\text{s.t. PC} \quad & \int_0^\infty \sqrt{w(\pi)} f(\pi|a) d\pi - a/2 \geq 0 \\
\text{s.t. IC} \quad a \in \arg\max_{\tilde{a}} & \int_0^\infty \sqrt{w(\pi)} f(\pi|\tilde{a}) d\pi - \tilde{a}/2
\end{aligned}
$$

Using the first-order approach, the IC constraint is replaced by

$$\int_0^\infty \sqrt{w(\pi)} f_a(\pi|a) d\pi = 1/2.$$

In the second stage, the principal solves

$$\max_a a - c(a).$$

(d) In the unobservable effort case, for any fixed effort level a, using the first-order approach, the optimal compensation schedule is characterized by (recalling the formula from the class or the handout)

$$2\sqrt{w(\pi)} = \lambda + \mu \frac{f_a(\pi|a)}{f(\pi|a)},$$

where λ is the multiplier on PC, and μ is the multiplier on IC. PC must bind, hence $\lambda > 0$, since otherwise the principal could reduce the compensation cost. Iff $\mu = 0$, $w(\pi)$ does not depend on π, which happens if and only if $a = 0$. Hence iff $a > 0$, $\mu > 0$. With the exponential distribution,

$$
\begin{aligned}
\frac{f_a(\pi|a)}{f(\pi|a)} &= \tfrac{\partial}{\partial a} \ln f(\pi|a) \\
&= \tfrac{\partial}{\partial a} [-\ln a - \pi/a] \\
&= -\tfrac{1}{a} + \tfrac{\pi}{a^2}.
\end{aligned}
$$

Therefore the optimal compensation schedule is characterized by

$$\sqrt{w(\pi)} = \left(\frac{\lambda}{2} - \frac{\mu}{2a}\right) + \frac{\mu}{2a^2}\pi, \tag{1}$$

and hence

$$w(\pi) = \left[\left(\frac{\lambda}{2} - \frac{\mu}{2a}\right) + \frac{\mu}{2a^2}\pi\right]^2. \tag{2}$$

As a result, the optimal compensation schedule is quadratic in realized profit.

(e) By the PC,

$$\int_0^\infty \sqrt{w(\pi)} f(\pi|a) d\pi = a/2.$$

Using (1), this becomes

$$\left(\frac{\lambda}{2} - \frac{\mu}{2a}\right) + \frac{\mu}{2a^2} a = a/2,$$

or

$$\lambda = a.$$

By the IC,

$$\int_0^\infty \sqrt{w(\pi)} f_a(\pi|a) d\pi = 1/2.$$

Using (1) and the fact that $f_a(\pi|a) = f(\pi|a)\left(-\frac{1}{a} + \frac{\pi}{a^2}\right)$, this becomes

$$\frac{\mu}{2a^2} = 1/2,$$

or

$$\mu = a^2.$$

As a result, the optimal compensation scheme implementing an effort level a is given by

$$w(\pi) = \pi^2/4.$$

Then the expected cost function is

$$c(a) = \frac{1}{4} E(\pi^2) = \frac{1}{4}\left[Var(\pi) + E(\pi)\right] = a^2/2.$$

(f) The principal solves

$$\max_a a - c(a) = \max_a a - a^2/2,$$

giving the optimal effort level $a^* = 1$, which is less than the optimal level of effort in the first-best. The distortion is due to the insurance motive in the compensation schedule, which reduces incentives below the first-best. Manager is still held to his reservation utility 0, while the principal's payoff is 1/2, which is less than in the first best. The reduction in the principal's payoff measures the welfare cost of asymmetric information.

(g) Now the principal observes a *pair* (π_1, π_2) of realized profit levels, which, by independence, have a density $f(\pi_1|a)f(\pi_2|a)$, where $f(\cdot)$ is as defined above. Then following the same steps as above, we obtain

$$\sqrt{w(\pi_1, \pi_2)} = \left(\frac{\lambda}{2} - \frac{\mu}{a}\right) + \frac{\mu}{2a^2}\left(\pi_1 + \pi_2\right),$$

then $\lambda = a$, $\mu = a^2/2$, and

$$w(\pi_1, \pi_2) = \left(\pi_1 + \pi_2\right)^2 / 16.$$

Then the expected cost of effort is

$$c(a) = \frac{1}{16}E((\pi_1 + \pi_2)^2) = \frac{3}{8}a^2.$$

In the first-stage, the principal solves

$$\max_a a - c(a) = \max_a a - \frac{3}{8}a^2,$$

giving the optimal effort level $a^* = 4/3$, which is less than the optimal level of effort in the first-best, but more than the effort level with one signal only. This is because two signals make effort inference more precise and therefore allow the compensation contract to be flatter in each signal while preserving incentives, hence inducing less risk on the manager. As a result, the marginal cost of effort inducement goes down, and hence the equilibrium level of effort goes up. Manager is still held to his reservation utility 0, while the principal's payoff is 2/3, which is more than in the one signal case, reflecting a reduction in informational asymmetry.

Problem 2: Based on *Aggarwal and Samwick: Executive Compensation, Strategic Competition, and Relative Performance Evaluation: Theory and Evidence (1999).* The main idea is to find optimal compensation scheme for managers in a model of duopoly and Bertrand competition (competition in prices). First, principal (owner of the company) chooses a compensation scheme. Then the agent (manager of the company) chooses the price to be charged.

As a benchmark model we use standard example of duopoly and the compensation of manager is an increasing function of firm's profit. In this case, the manager has an incentive to earn as high profit as possible, so he will choose a profit maximizing price. The demand is given by $D_i(p_i, p_j) = A - bp_i + ap_j$ and profit is given by $\pi_i = (p_i - c)(A - bp_i + ap_j) + \varepsilon$, where ε is some noise with distribution $N(0, \sigma^2)$, so it is not possible to determine prices from observed profit. In our example, $A = 2$, $c = 1$, $a = 1$ and $b = 2$.

Bertrand competition - standard contract

$$\begin{aligned}
\max_{p_i} \pi_i &= (p_i - 1)(2 - 2p_i + p_j) + \varepsilon \\
FOC: \quad & 2 - 2p_i + p_j + p_i(-2) + 2 = 0 \\
& 4 - 4p_i + p_j = 0
\end{aligned}$$

Manager of firm j solves the analogical problem and symmetry implies that $p_i^* = p_j^*$. Therefore, $p_i^* = p_j^* = 4/3$ and the corresponding expected profit is $E^*(\pi) = (4/3-1)(2-4/3) = 2/9$.

The main question now is whether the standard contract can be improved upon. Suppose that the compensation of a manager depends on the own firm's profit but also on profit of the other firm i.e. $w_i = k_i + \alpha_i \pi_i + \beta_i \pi_i$. If this contract is revealed to both managers and then they choose prices can profits be higher?

Bertrand competition - compensation depends on both firms' profit

$$\max_{p_i} \alpha_i[(p_i - 1)(2 - 2p_i + p_j) + \varepsilon] + \beta_i[(p_j - 1)(2 - 2p_j + p_i) + \varepsilon]$$

Here, the idea is to find the optimal price chosen by a manager as a function of α and β. Then the owner of the company, knowing this, chooses optimal values of α and β to maximize his own profit.

We skip the algebra here, because of its level of difficulty, it can be found in the paper. Given α and β, manager chooses the price:

$$p^{**} = \frac{\beta c - \alpha(2c + A)}{\beta - 3\alpha} = \frac{\beta - 4\alpha}{\beta - 3\alpha}$$

The owner of the company plugs this value of p^{**} into his profit function and chooses values of α and β such that the profit is maximized. Using results from the paper we get, that $\alpha^{**} = 3k$ and $\beta^{**} = k$. With this choice of parameters α and β, the price chosen by the manager is $p^{**} = \frac{11}{8}$ and the corresponding expected profit is $E^{**}(\pi) = (p^{**} - 1)(2 - p^{**}) = (\frac{11}{8} - 1)(2 - \frac{11}{8}) = 15/64$. We see, that with this compensation scheme, the price is higher and the expected profit is higher, so the standard contract can be improved upon.

Note: in case of Cournot competition (competition in quantities) the optimal quantity increases under the new compensation scheme and the expected profit decreases. So it is optimal to choose the new compensation scheme only in case of Bertrand duopoly.