Problem 1: Most of the time we are interested in the following properties of matchings:

- Individual rationality
- Pareto efficiency
- Incentive compatibility

What properties does the Random serial dictatorship with squatting rights exhibit?

Solution: Random serial dictatorship with squatting rights is a mechanism for housing problem where some individuals are endowed with houses. These individuals have a choice whether to keep their houses or give it up and participate in the mechanism. If they decide to participate they can get a better house but also they may end up in a worse house. Hence, some individuals will choose not to participate and this leads to inefficiencies in the market (in Pareto sense). For a particular problem see ES #2, problem #2. The outcome of SRDwSR is incentive compatible, not individually rational nor Pareto efficient.

Problem 2: Let $I_E = \{i_1, i_2, i_3\}$, $I_N = \emptyset$, $H_O = \{h_1, h_2, h_3\}$, and $H_V = \{h_4\}$. Here the existing tenant i_k occupies the house h_k for $k = 1, 2, 3$. Let the ordering f order the agents as $i_1 - i_2 - i_3 - i_4$ and let the preferences (from best to worst) be as follows:

P_{i_1}	P_{i_2}	P_{i_3}
h_2	h_3	h_1
h_3	h_1	h_4
h_1	h_2	h_3
h_4	h_4	h_2
h_0	h_0	h_0

Find the outcome of Random serial dictatorship with waiting list. Is it Pareto efficient?

Solution: The random serial dictatorship with waiting list a direct mechanism in which agents announce their preferences over all houses. For a given ordering f of agents, the outcome is obtained as follows: Define the set of available houses for to be the set of vacant houses. Define the set of acceptable houses for agent i to be the set of all houses in case agent i is a new applicant, and the set of all houses better than his or her current house h_i in case he or she is an existing tenant.

The agent with the highest priority among those who have at least one acceptable available house is assigned his or her top available house and removed from the process. His or her assignment is deleted from the set of available houses for the next step. In case he or she is an existing tenant, his or her current house becomes available for the next step. If there is at least one remaining agent and one available house that is acceptable to at least one of them, then we go to the next step.

Step 1. The only available house at Step 1 is house h_4. It is acceptable to only agent i_3. So agent i_3 is assigned house h_4.

Step 2. The only available house at Step 2 is house h_3. It is acceptable to both agent i_1 and agent i_2. Since agent i_1 has the higher priority, agent i_1 is assigned house h_3.

Step 3. The only available house at Step 3 is house h_1. It is acceptable to agent i_2. So agent i_2 is assigned house h_1.

Since there are no remaining agents at the end of Step 3, the process terminates and the final matching is $\{(i_1, h_3), (i_2, h_1), (i_3, h_4)\}$. This outcome, however, is Pareto dominated by $\{(i_1, h_2), (i_2, h_3), (i_3, h_1)\}$. So the outcome of RSDwWL is not Pareto efficient. Moreover, this mechanism is not incentive compatible either (agent i_1 has an incentive to state his preferences as $h_3 - h_2 - h_1 - h_4 - h_0$ in which case this agent gets his top choice h_2 instead of h_3).

Problem 3: Let $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, and $H_V = \{h_5\}$. Here the existing tenant i_k occupies the house h_k for $k = 1, 2, 3, 4$. Let the ordering f order the agents as $i_1 - i_2 - i_3 - i_4 - i_5$ and let the preferences be as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
h_3	h_4	h_5	h_3	h_4
h_4	h_5	h_3	h_5	h_5
h_5	h_2	h_4	h_4	h_3
h_1	h_3	h_2	h_2	h_1
h_2	h_1	h_1	h_1	h_2
h_0	h_0	h_0	h_0	h_0

Find the outcome of MIT-NH4 mechanism. Is it Pareto efficient?

Solution:

Step 1. First agent i_1 is tentatively assigned h_3, next agent i_2 is tentatively assigned h_4, next agent i_3 is tentatively assigned h_5, and next its agent i_4's turn and a squatting conflict occurs. The conflicting agent is agent i_2 who was tentatively assigned h_4. Agent i_2's tentative assignment, as well as that of agent i_3, is erased. Agent i_4 is assigned his or her current house h_4 and removed from the process. This resolves the squatting conflict.

Step 2. The process starts over with the conflicting agent i_2. Agent i_2 is tentatively assigned h_5 and next it is agent i_3's turn and another squatting conflict occurs. The conflicting agent is agent i_1 who was tentatively assigned h_3. His tentative assignment, as well as that of agent i_2 are erased. Agent i_3 is assigned his current house h3 and removed from the process. This resolves the second squatting conflict.

Step 3. The process starts over with the conflicting agent i_1. He is tentatively assigned h_5, next agent i_2 is tentatively assigned h_2 and finally agent i_5 is tentatively assigned h_1. At this point all tentative assignments are finalized.

The resulting outcome is $\{(i_1, h_5), (i_2, h_2), (i_3, h_3), (i_4, h_4), (i_5, h_1)\}$. This outcome is Pareto dominated by e.g. $\{(i_1, h_3), (i_2, h_2), (i_3, h_5), (i_4, h_4), (i_5, h_1)\}$. The outcome of MIT-NH4 mechanism is incentive compatible and individually rational but not Pareto efficient.

Problem 4: Let $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, and $H_V = \{h_5, h_6, h_7\}$. Let the ordering f order the agents as $i_1 - i_2 - i_3 - i_4 - i_5$ and the preferences (from best to worst) as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
h_2	h_7	h_2	h_2	h_4
h_6	h_1	h_1	h_4	h_3
h_5	h_6	h_4	h_3	h_7
h_1	h_5	h_7	h_6	h_1
h_4	h_4	h_3	h_1	h_2
h_3	h_3	h_6	h_7	h_5
h_7	h_2	h_5	h_5	h_6
h_0	h_0	h_0	h_0	h_0

Use YRMH-IGYT (You request my house-I get your turn) to find the outcome of this matching problem.

Solution: We see that none of mechanisms above is Pareto efficient, incentive compatible and individually rational at the same time. However, TTC mechanism has all these properties. YRMH-IGYT is an alternative way to find the outcome of TTC mechanism.

We can find the outcome of the top trading cycles mechanism using the following you request my house - I get your turn (or in short YRMH-IGYT) algorithm: For any given ordering f, assign the first agent his or her top choice, the second agent his or her top choice among the remaining houses, and so on, until someone demands the house of an existing tenant. If at that point the existing tenant whose house is demanded is already assigned a house, then do not disturb the procedure. Otherwise modify the remainder of the ordering by inserting him to the top and proceed. Similarly, insert any existing tenant who is not already served at the top of the line once his or her house is demanded. If at any point a loop forms, it is formed by exclusively existing tenants and each of them demands the house of the tenant next in the loop. (A loop is an ordered list of agents $(i_1, i_2, \ldots, i_k)$ where agent i_1 demands the house of agent i_2, agent i_2 demands the house of agent i_3, $\ldots$, agent i_k demands the house of agent i_1.) In such cases remove all agents in the loop by assigning them the houses they demand and proceed.

We start with the ordering $1-2-3-4-5$. i_1 asks for his top choice which is h_2. But the house h_2 is occupied by agent i_2 and thus the new ordering is $2-1-3-4-5$.

$i_2 \longrightarrow h_7$

$i_1 \longrightarrow h_2$

$i_3 \longrightarrow h_1$

$i_4 \longrightarrow h_4$

$i_5 \longrightarrow h_3$

So the resulting outcome is $\{\{i_1, h_2\}, \{i_2, h_7\}, \{i_3, h_1\}, \{i_4, h_4\}, \{i_5, h_3\}\}$. Note that this result coincides with the result of problem #5 in ES # 2 where we used TTC mechanism for the same problem. Also note that these two mechanisms (TTC and YRMH-IGYT) give the same outcomes only if we use the same ordering of agents. If we use for example ordering $5-4-3-2-1$ in YRMH-IGYT, resulting outcome is $\{\{i_1, h_6\}, \{i_2, h_7\}, \{i_3, h_1\}, \{i_4, h_2\}, \{i_5, h_4\}\}$.