Five types of matching problems:

- Two-sided market with non transferable utility (marriage market)
- Two-sided market with transferable utility (business partners)
- One-sided market with non transferable utility (roommates)
- One-sided market with transferable utility (roommates with side payments)
- Housing markets and housing allocation problem only with non transferable utility

Problem 1: Two-sided market with non transferable utility. Find the set of stable matchings for the following table of preferences:

A:	b $\succ$	c $\succ$	d $\succ$	a	a:	D $\succ$	B $\succ$	C $\succ$	A
B:	a $\succ$	d $\succ$	c $\succ$	b	b:	C $\succ$	A $\succ$	B $\succ$	D
C:	c $\succ$	d $\succ$	b $\succ$	a	c:	B $\succ$	D $\succ$	A $\succ$	C
D:	a $\succ$	c $\succ$	d $\succ$	b	d:	A $\succ$	D $\succ$	C $\succ$	B

Solution: First we start with men and women most preferred outcome. To find men most preferred matching we use Gale and Shapley algorithm in which men propose to their best possible choices and women reject all the offers but the best one.

	Step 1
a:	**D**
b:	**C**
c:	**B**
d:	**A**

Hence, men most preferred stable matching is ***Ad, Bc, Cb, Da***.

Similarly, for women best outcome we have:

	Step 1	Step 2
A:	**b**	
B:	a	**d**
C:	**c**	
D:	**a**	

And therefore, women best outcome is ***Ab, Bd, Cc, Da***.

We see, that in both men best and women best outcome D is matched with a. Note that due to the polarization result, men best outcome is women worst outcome at the same time. So both in her best and worst outcome, woman D is matched with man a. That means that she is matched with him in all stable matchings.

To find all stable matchings we use an algorithm which consists of three steps.

From the table of preferences eliminate

- all men (women) who are better than the partner in the best possible outcome
- all men (women) who are worse than the partner in the worst possible outcome
- all men (women) who are not mutually interested or affordable for each other

In the table below, those partners who were not eliminated are in bold:

A:	**b** ≻	**c** ≻	**d** ≻	a	a:	**D** ≻	B ≻	C ≻	A
B:	a ≻	**d** ≻	**c** ≻	b	b:	**C** ≻	**A** ≻	B ≻	D
C:	**c** ≻	**d** ≻	**b** ≻	a	c:	**B** ≻	D ≻	**A** ≻	**C**
D:	**a** ≻	c ≻	d ≻	b	d:	**A** ≻	D ≻	**C** ≻	**B**

And the set of all stable matchings is as follows:

{Ac, Bd, Cb, Da}, **{Ab, Bd, Cc, Da}**, **{Ab, Bc, Cd, Da}**, and **{Ad, Bc, Cb, Da}**.

Problem 2: Housing markets and housing allocation problem - Random serial dictatorship. There are three agents i_1, i_2, i_3 and three houses h_1, h_2, h_3. Agent i_1 is a current tenant and he occupies house h_1. Agents i_2, i_3 are new applicants and houses h_2, h_3 are vacant houses.

• Utilities are:

	h_1	h_2	h_3
i_1	3	4	1
i_2	4	3	1
i_3	3	4	1

• Agent i_1 has two options:

1. he can keep house h_1
2. he can give it up and enter the lottery

Which option should he chose?

Solution: The following table summarizes the possible outcomes, in case he enters the lottery:

ordering	i_1	i_2	i_3
$i_1 - i_2 - i_3$	h_2	h_1	h_3
$i_1 - i_3 - i_2$	h_2	h_3	h_1
$i_2 - i_1 - i_3$	h_2	h_1	h_3
$i_2 - i_3 - i_1$	h_3	h_1	h_2
$i_3 - i_1 - i_2$	h_1	h_3	h_2
$i_3 - i_2 - i_1$	h_3	h_1	h_2

Expecting utility from entering the lottery is:

$$\frac{1}{6}u(h_1) + \frac{3}{6}u(h_2) + \frac{2}{6}u(h_3) = \frac{17}{6} < 3$$

Utility from keeping house h_1 is 3, therefore the optimal strategy is to keep house h_1.

Problem 3: Two-sided market with non transferable utility. In this problem we describe the mechanisms that we study in the context of constrained school choice: the Gale-Shapley Student-Optimal Stable mechanism, the Boston mechanism, and the Top Trading Cycles mechanism. We define a school choice problem by a set of schools and a set of students, each of which has to be assigned a seat at not more than one of the schools. Each student is assumed to have strict preferences over the schools and the option of remaining unassigned. Each school is endowed with a strict priority ordering over the students and a fixed capacity of seats.

Let $I = \{i_1, i_2, i_3, i_4\}$ be the set of students, $S = \{s_1, s_2, s_3\}$ be the set of schools, and $q = (1, 2, 1)$ be the capacity vector. The students' preferences P and the priority structure f are given in the table below. So, for instance, $P_{i_1} = s_2, s_1$ and $f_{s_1}(i_1) < f_{s_1}(i_2) < f_{s_1}(i_3) < f_{s_1}(i_4)$.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	f_{s_1}	f_{s_2}	f_{s_3}
s_2	s_1	s_1	s_2	i_1	i_3	i_4
s_1	s_2	s_2	s_3	i_2	i_4	i_1
	s_3		s_1	i_3	i_1	i_2
				i_4	i_2	i_3

If the students truthfully report their preferences, find outcomes of Gale and Shapley algorithm, Boston matching algorithm, and Top trading cycles algorithm.

Solution:

If the students truthfully report their preference lists, then the mechanisms yield the following matchings.

The Gale and Shapley student-optimal stable mechanism.

Step 1. Each student proposes to his most preferred school. So, school s_1 receives a proposal from i_2 and i_3. Student i_2 has a higher priority, so i_3's proposal is rejected. School s_2 receives a proposal from i_1 and i_4. Since school s_2 has 2 seats it does not reject any of the two students. The tentative matching is $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$.

Step 2. Student i_3 proposes to school s_2. So, now school s_2 has two (tentatively) accepted students, i_1 and i_4, and one new proposal, from i_3. Since school s_2 has 2 seats it rejects i_1, the student with the lowest priority. The tentative matching becomes $\{\{s_1, i_2\}, \{s_2, i_3, i_4\}, \{s_3\}, \{i_1\}\}$.

Step 3. Student i_1 proposes to school s_1. The unique seat of school s_1 is tentatively occupied by i_2. Since i_1 has a higher priority than student i_2, the latter is rejected. The tentative matching becomes $\{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3\}, \{i_2\}\}$.

Step 4. Student i_2 proposes to school s_2. Now two seats in school s_2 are tentatively occupied by i_3 and i_4. School s_2 rejects student i_2 as the least preferrer from the three students that proposed and the tentative matching stays $\{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3, i_2\}\}$.

Step 5. Student i_2 proposes to school s_3. Since school s_3's unique seat is available, student i_2 is accepted. No student has been rejected in this step, so the tentative matching is the final matching and is given by $\{\{s_1, i_1\}, \{s_2, i_3, i_4\}, \{s_3, i_2\}\}$.

The Boston mechanism.

Step 1. Each student proposes to his most preferred school. So, school s_1 receives a proposal from i_2 and i_3. Student i_2 has a higher priority, so i_3's proposal is rejected and i_2 is assigned the unique seat at s_1. School s_2 receives a proposal from i_1 and i_4. Since school s_2 has 2 seats each of the students i_1 and i_4 is assigned a seat at s_2. Schools s_1 and s_2 have filled all their seats and hence are removed. The tentative matching is $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$.

Step 2. Student i_3 cannot propose to his next preferred school, s_2. Since he finds school s_3 unacceptable he is removed and remains unassigned. So, the final matching is given by $\{\{s_1, i_2\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_3\}\}$.

The top trading cycles mechanism.

Step 1. Each student points to his most preferred school, and each school points to the student with highest priority. There is a unique cycle that is given by (i_1, s_2, i_3, s_1). So, students i_1 and i_3 are assigned a seat at schools s_2 and s_1, respectively. Students i_1 and i_3 are removed. Since school s1 had only 1 available seat it is also removed. School s_2 still has an available seat and is therefore not removed. The tentative matching is $\{\{s_1, i_3\}, \{s_2, i_1\}, \{s_3\}, \{i_2\}, \{i_4\}\}$.

Step 2. There is a unique cycle given by (i_4, s_2). So, student i_4 is assigned the remaining seat at school s_2. Both student i_4 and school s_2 are removed. The tentative matching is $\{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3\}, \{i_2\}\}$.

Step 3. Only student i_2 and school s_3 remain. Since i_2 finds school s_3 acceptable, he points to the school. Since i_2 is the only remaining student, school s_3 points to i_2. This creates a cycle and hence i_2 is assigned a seat at school s_3. So, the final matching is $\{\{s_1, i_3\}, \{s_2, i_1, i_4\}, \{s_3, i_2\}\}$.

Note that for the school choice problem above the three mechanisms generate different matchings. Also, the obtained matchings illustrate directly some of the problems of the mechanisms. For instance, Boston mechanism leads to a Pareto-efficient outcome but not stable because student i_3 would like to go to school s_2 and s_2 prefers i_3 to both current students i_1 and i_4. We also see that Boston mechanism does not ensure stating true preferences. (Had student i_3 have announced the list that only contains school s_2 he would have guaranteed a seat at this school.) Similarly, one easily verifies that the outcome of Gale and Shapley algorithm is stable but not Pareto-efficient (i_1 and i_3 could switch their places) and that outcome of TTC is Pareto-effcient and it is a dominant strategy for each player to reveal their true preferences but the outcome is not stable (i_2 and s_1 prefer to brake their current matches and be matched together).

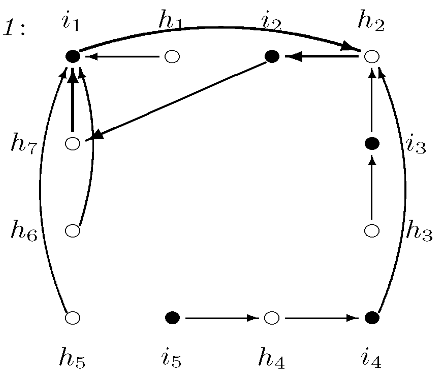
Problem 4: Let $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, and $H_V = \{h_5, h_6, h_7\}$. Let the ordering f order the agents as $i_1 - i_2 - i_3 - i_4 - i_5$ and the preferences (from best to worst) as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
h_2	h_7	h_2	h_2	h_4
h_6	h_1	h_1	h_4	h_3
h_5	h_6	h_4	h_3	h_7
h_1	h_5	h_7	h_6	h_1
h_4	h_4	h_3	h_1	h_2
h_3	h_3	h_6	h_7	h_5
h_7	h_2	h_5	h_5	h_6
h_0	h_0	h_0	h_0	h_0

Use top trading cycles mechanism to find the outcome of this matching problem.

Solution:

Step 1:

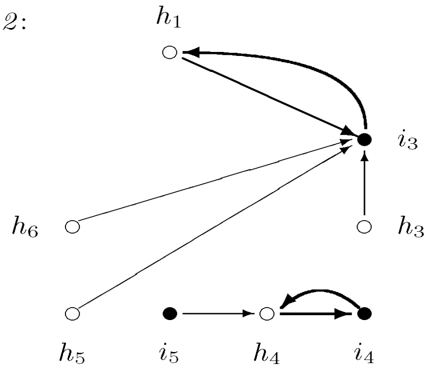


• The set of available houses in Step 1 is $H_V = \{h_5, h_6, h_7\}$.

• The only cycle that is formed in this step is (i_1, h_2, i_2, h_7). Therefore i_1 is assigned h_2 and i_2 is assigned h_7.

• Since i_1 leaves the market, his house h_1 becomes available for the next step. Therefore the set of available houses for Step 2 is $\{h_1, h_5, h_6\}$.

Step 2:

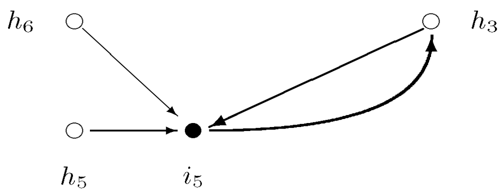


• There are two cycles (i_3, h_1) and (i_4, h_4) in Step 2.

• Therefore i_3 is assigned h_1 and i_4 is assigned h_4.

• Since i_3 leaves the market his house h_3 becomes available for the next step. Therefore the set of available houses for Step 3 is $\{h_3, h_5, h_6\}$.

Step 3:



• There is one cycle (i_5, h_3) in Step 3.

• Therefore i_5 is assigned h_3.

• There are no remaining agents so the algorithm terminates and the matching it induces is: $\{\{i_1, h_2\}, \{i_2, h_7\}, \{i_3, h_1\}, \{i_4, h_4\}, \{i_5, h_3\}\}$