Problem 1: Women are denoted A, B, C, D and men are denoted a, b, c, d. The content of the box (a, B), which is (4,1), means that man a ranks woman B as his *fourth* choice and woman B ranks man a as her *first* choice. Find the men most preferred matching and the women most preferred matching for the following preference table. What can you say about the set of stable matchings?

	A	B	C	D
a	(1,1)	(4,1)	(3,1)	(2,1)
b	(4,3)	(3,2)	(2,2)	(1,3)
c	(4,2)	(1,3)	(3,3)	(2,2)
d	(2,4)	(1,4)	(4,4)	(3,4)

Solution: Preferences can be rewritten in the following form:

A:	a $\succ$	c $\succ$	b $\succ$	d	a:	A $\succ$	D $\succ$	C $\succ$	B
B:	a $\succ$	b $\succ$	c $\succ$	d	b:	D $\succ$	C $\succ$	B $\succ$	A
C:	a $\succ$	b $\succ$	c $\succ$	d	c:	B $\succ$	D $\succ$	C $\succ$	A
D:	a $\succ$	c $\succ$	b $\succ$	d	d:	B $\succ$	A $\succ$	D $\succ$	C

To find men most preferred matching we use Gale and Shapley algorithm in which men propose to their best possible choices and women reject all the offers but the best one. In the first step men *a, b, c* and *d* propose to women *A, D, B*, and *B* respectively. Woman B who gets two offers rejects man *d* and keeps man *c* who is preferred. In the second step, man *d* proposes to his second best choice - woman *A* and is rejected, and so on.

	Step 1	Step 2	Step 3	Step 4
a:	**A**			
b:	**D**			
c:	**B**			
d:	B	A	D	**C**

Hence, men most preferred stable matching is ***Aa, Bc, Cd, Db***.

Similarly, we use Gale and Shapley mechanism with women making proposals and men rejecting all but the best offer in order to find women best outcome. In the first step, all women propose to man *a* who rejects all the offers except the best one, which is an offer from woman *A*. In the second step, women *B*, *C* and *D* make new offers to their second choices and so on.

	Step 1	Step 2	Step 3	Step 4	Step 5	Step 6
A:	**a**					
B:	a	b	**c**			
C:	a	b			c	**d**
D:	a	c		**b**		

Women most preferred stable matching is ***Aa, Bc, Cd, Db***.

Note that men best (women worst) outcome and women best (men worst) outcome coincide. This means that there exists a single stable matching in this problem.

Problem 2: Find the set of stable matchings for the following table of preferences:

	A	B	C	D
a	(1,1)	(2,3)	(4,4)	(3,1)
b	(4,3)	(3,2)	(1,3)	(2,2)
c	(4,2)	(2,4)	(3,2)	(1,4)
d	(2,4)	(4,1)	(3,1)	(1,3)

Solution: First, we rewrite preferences into the following form:

A: $a \succ c \succ b \succ d$ a: $A \succ B \succ D \succ C$
B: $d \succ b \succ a \succ c$ b: $C \succ D \succ B \succ A$
C: $d \succ c \succ b \succ a$ c: $D \succ B \succ C \succ A$
D: $a \succ b \succ d \succ c$ d: $D \succ A \succ C \succ B$

As in the previous problem, we first look for men best outcome and women best outcome.

To find men most preferred matching we use Gale and Shapley algorithm in which men propose to their best possible choices and women reject all the offers but the best one.

	Step 1	Step 2
a:	**A**	
b:	**C**	
c:	B	**B**
d:	**D**	

Hence, men most preferred stable matching is ***Aa, Bc, Cb, Dd***.

Similarly, for women best outcome we have:

	Step 1	Step 2	Step 3	Step 4
A:	**a**			
B:	d	b	a	**c**
C:	**d**			
D:	a	**b**		

And therefore, women best outcome is ***Aa, Bc, Cd, Db***.

We see, that in both men best and women best outcome A is matched with a and B is matched with c. Note that due to the polarization result, men best outcome is women worst outcome at the same time. So both in her best and worst outcome, woman A is matched with man a. That means that she is matched with him in all stable matchings. Hence, in all stable matchings, A is matched with a and B is matched with c. But this leaves us with only two possible stable matchings: *Aa, Bc, Cb, Dd* and *Aa, Bc, Cd, Db*. We showed that these two outcomes are stable and there are no more stable matchings.

Problem 3: How many stable matchings are there in the following matching problem?

A:	b $\succ$	e $\succ$	c $\succ$	d $\succ$	a	a:	D $\succ$	B $\succ$	C $\succ$	E $\succ$	A
B:	d $\succ$	b $\succ$	e $\succ$	c $\succ$	a	b:	E $\succ$	D $\succ$	A $\succ$	B $\succ$	C
C:	e $\succ$	a $\succ$	d $\succ$	c $\succ$	b	c:	C $\succ$	A $\succ$	D $\succ$	E $\succ$	B
D:	c $\succ$	d $\succ$	b $\succ$	e $\succ$	a	d:	E $\succ$	B $\succ$	D $\succ$	C $\succ$	A
E:	c $\succ$	a $\succ$	d $\succ$	b $\succ$	e	e:	B $\succ$	A $\succ$	C $\succ$	E $\succ$	D

First, we find men best outcome:

	Step 1	Step 2	Step 3	Step 4	Step 5
a:	D		B	**C**	
b:	E	**D**			
c:	C				**A**
d:	**E**				
e:	**B**				

Men best outcome is ***Ac, Be, Ca, Db, Ed.***

Now, we find women best outcome:

	Step 1	Step 2
A:	**b**	
B:	**d**	
C:	**e**	
D:	**c**	
E:	c	**a**

Women best outcome is ***Ab, Bd, Ce, Dc, Ea.***

We found two stable matchings. How about others? How many of them are there? We could find out by checking every single matching for stability but obviously there must be a better way. To find all stable matchings we use an algorithm which consists of three steps.

From the table of preferences eliminate

- all men (women) who are better than the partner in the best possible outcome
- all men (women) who are worse than the partner in the worst possible outcome
- all men (women) who are not mutually interested or affordable for each other

In the table below, those partners who were not eliminated are in bold:

A:	**b** $\succ$	**e** $\succ$	**c** $\succ$	d $\succ$	a	a:	D $\succ$	B $\succ$	**C** $\succ$	**E** $\succ$	A
B:	**d** $\succ$	b $\succ$	**e** $\succ$	c $\succ$	a	b:	E $\succ$	**D** $\succ$	**A** $\succ$	B $\succ$	C
C:	**e** $\succ$	**a** $\succ$	d $\succ$	c $\succ$	b	c:	C $\succ$	**A** $\succ$	**D** $\succ$	E $\succ$	B
D:	**c** $\succ$	d $\succ$	**b** $\succ$	e $\succ$	a	d:	**E** $\succ$	**B** $\succ$	D $\succ$	C $\succ$	A
E:	c $\succ$	**a** $\succ$	**d** $\succ$	b $\succ$	e	e:	**B** $\succ$	**A** $\succ$	**C** $\succ$	E $\succ$	D

And the set of all stable matchings is as follows:

Ac, Bd, Ce, Db, Ea

Ab, Bd, Ce, Dc, Ea

Ab, Be, Ca, Dc, Ed and

Ac, Be, Ca, Db, Ed.

Roth (1982) - Theorem 5: In the matching procedure which always yields the optimal stable outcome for a given one of the two sets of agents (i.e. for Men or for Women), truthful revelation is a dominant strategy for all the agents in that set.

Notation: P - set of true preferences; *P'* - set of preferences where everybody reveals their preferences truthfully except m_i; x - outcome of G(P) - Gale & Shapley algorithm with respect to preferences P; y - outcome of G(P'). The theorem says that there does not exist P' such that $y(m_i) > x(m_i)$.

Proof: proceeds in two steps: first, we show that in G(P') no man is made worse off. Second, we show that no man is made better off either.

No man is made worse off in G(P'). Proof by contradiction: assume that there is a man m_j such that $y(m_j) < x(m_j)$. Then m_j proposed to woman $x(m_j)$ and was rejected. Let's assume that this is the first man to whom this happened. Since $x(m_j)$ rejected m_j, she got a better offer in G(P') from a man m_k which she did not get in G(P). Since m_k did not propose to $x(m_j)$ in G(P) where he was matched with $x(m_k)$ it must be the case that in G(P') m_k got rejected by $x(m_k)$. This, however, is in contradiction with assumption that m_j was the first such man.

No man is made better off in G(P'). Let's assume that G(P) stops after period t. Proof by mathematical induction. First we show that man who makes a match (i.e. proposes to his ultimate partner) in period t is not made better off. Second we show that if men who make a match at periods $r+1$ through t are not made better off than man who makes a match in period r is not better off either. Man m_t who makes a match in period t is the only man who proposes to woman $x(m_t)$ (otherwise she would have more offers and algorithm would continue). Those men who did not propose to $x(m_t)$ in G(P) will not propose to her in G(P') either otherwise they would be worse off. Since no other man except m_t proposed to $x(m_t)$ in G(P), no other man proposes to her in G(P'). This means that m_t ends up with the same partner and hence $y(m_t) = x(m_t)$.

Now, let manipulator m_i make a match in period k and m_q in period r such that $k < r < t$. We can divide all men into three groups: those who did not propose to $x(m_q)$ in G(P), m_q, and those rejected by $x(m_q)$ - where the most preferred one was m_u. No man from the first group can propose to woman $x(m_q)$ in G(P'). Further, man m_u is not a manipulator and was rejected by $x(m_q)$ in period r because m_q came. Hence m_u makes a match in period $r+1$ or later and by assumption of mathematical induction it must hold that $y(m_u) = x(m_u)$. Therefore m_u proposes to $x(m_q)$ gets rejected and ends up with the same partner as in G(P). Since $x(m_q)$ rejects m_u in G(P') she must have a better option. But only more preferred man who proposes to her is m_q. Therefore, $y(m_q) = x(m_q)$ and this completes the proof.