7 Exponential equations and logarithmic equations

Properties of logarithmic functions: If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then:

$$\begin{aligned} \log_b 1 &= 0 & \log_b MN &= \log_b M + \log_b N \\ \log_b b &= 1 & \log_b \frac{M}{N} &= \log_b M - \log_b N \\ \log_b b^x &= x & \log_b M^p &= p \log_b M \\ b^{\log_b x} &= x, x > 0 & \log_b M &= \log_b N \text{ iff } M = N \end{aligned}$$

There are two basic methods of solving exponential equations:

First: if it is possible to transform the equation to one with the same base on both sides of the equation, we just compare the exponents.

Second: if it is not possible to transform the equation to one with the same base on both sides of the equation, we use logarithm.

1. Transformation to the same base on both size:

Example: Solve $4^{x-3} = 16$

$$\begin{aligned} &(2^2)^{x-3} = 16 \\ &2^{2(x-3)} = 2^4 \\ &2x - 6 = 4 \\ &2x = 10 \\ &x = 5 \end{aligned}$$

Example: Solve $27^{x+1} = 9$

$$\begin{aligned} &(3^3)^{x+1} = 3^2 \\ &3^{3(x+1)} = 3^2 \\ &3x + 3 = 2 \\ &3x = -1 \\ &x = -1/3 \end{aligned}$$

Example: Solve $7^{x^2} = 7^{2x+3}$

$$7^{x^2} = 7^{2x+3}$$
$$x^2 = 2x + 3$$
$$x^2 - 2x - 3 = 0$$
$$D = b^2 - 4ac = 4 - 4 \times 1 \times (-3) = 16$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 4}{2} = -1, 3$$

Example: Solve $4^{5x-x^2} = 4^{-6}$

$$4^{5x-x^2} = 4^{-6}$$
$$-x^2 + 5x + 6 = 0$$
$$D = b^2 - 4ac = 25 - 4 \times (-1) \times 6 = 49$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-5 \pm 7}{-2} = -1, 6$$

Example: Find y: $y = \log_3 27$

$$y = \log_3 27 \Leftrightarrow 3^y = 27$$
$$y = 3$$

Example: Find y: $y = \log_9 27$

$$y = \log_9 27 \Leftrightarrow 9^y = 27$$
$$(3^2)^y = 3^3$$
$$3^{(2y)} = 3^3$$
$$2y = 3$$
$$y = 3/2$$

Example: Find x: $\log_2 x = -3$

$$\log_2 x = -3 \Leftrightarrow x = 2^{(-3)}$$
$$x = \frac{1}{2^3}$$
$$x = 1/8$$

Example: Find b: $\log_b 100 = 2$

$$\log_b 100 = 2 \Leftrightarrow b^2 = 100$$
$$b = \sqrt{100}$$
$$b = 10$$

Examples:

$$\log_e 1 = 0 \qquad \log_{10} 10 = 1$$
$$10^{\log_{10} 7} = 7 \qquad \log_e e^{2x+1} = 2x + 1$$
$$e^{\log_e x^2} = x^2$$

- If you know that $\log_e 3 = 1.1$ and $\log_e 7 = 1.95$, find $\log_e(\frac{7}{3})$ and $\log_e \sqrt[3]{21}$.

$$\log_e \left(\frac{7}{3}\right) = \log_e 7 - \log_e 3 = 1.95 - 1.1 = 0.85$$
$$\log_e \sqrt[3]{21} = \log_e 21^{1/3} = 1/3 \log_e 21 = 1/3 \log_e(3 \times 7) = 1/3[\log_e 3 + \log_e 7] =$$
$$= 1/3[1.1 + 1.95] = 1/3 \times 3.05 \approx 1.02$$

- Find x: $\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$

$$\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$$
$$\log_b x = \log_b 27^{2/3} + \log_b 2^2 - \log_b 3$$
$$\log_b x = \log_b [27^{2/3} \times 2^2/3]$$
$$\log_b x = \log_b [9 \times 4/3]$$
$$x = \frac{9 \times 4}{3} = 12$$

- $2 \log_5 x = \log_5(x^2 - 6x + 2)$

$$2 \log_5 x = \log_5(x^2 - 6x + 2)$$
$$\log_5 x^2 = \log_5(x^2 - 6x + 2)$$
$$x^2 = x^2 - 6x + 2$$
$$6x = 2$$
$$x = 1/3$$

- $\log_e(x + 8) - \log_e x = 3 \log_e 2$

$$\log_e(x + 8) - \log_e x = 3 \log_e 2$$
$$\log_e \frac{x + 8}{x} = \log_e 2^3$$
$$x + 8 = 8x$$
$$7x = 8$$
$$x = 8/7$$

• $(\ln x)^2 = \ln x^2$, where ln is a short notation for $\log_e$

$$\begin{aligned}
&(\ln x)^2 = \ln x^2 \\
&(\ln x)^2 = 2\ln x \\
&(\ln x)^2 - 2\ln x = 0 \\
&\ln x(\ln x - 2) = 0 \\
&\ln x = 0 \qquad \text{OR} \quad \ln x - 2 = 0 \\
&x = e^0 = 1 \quad \text{OR} \quad x = e^2
\end{aligned}$$

2. Using logarithm:

• $2^{3x-2} = 5$ This equation can not be transformed to get the equation with the same base on each side, therefore we will use logarithm to solve it.

$$\begin{aligned}
&2^{3x-2} = 5 / \log_{10} \\
&\log_{10} 2^{3x-2} = \log_{10} 5 \\
&(3x-2)\log_{10} 2 = \log_{10} 5 \\
&(3x-2) = \frac{\log_{10} 5}{\log_{10} 2} \\
&x = \frac{1}{3}\left(2 + \frac{\log_{10} 5}{\log_{10} 2}\right)
\end{aligned}$$

• You have \$10000 and the annual interest rate is 10%. Imagine that you have to options. You can deposit the money for 4 years and interest rate being compounded annually or you can deposit the money for 2 years and interest rate being compounded semiannually. Decide which option is better (in terms of money).

First option: $A = P(1+r)^t = 10000 \times (1 + 0.10)^4$

Second option: $A = P(1 + r/n)^n t = 10000 \times (1 + 0.10/2)^4 = 10000 \times (1 + 0.05)^4$

Hence, the first option gives more money that the second one.

Example: You deposit 20000 CZK on a bank account with an interest rate of 10%. How many years does it take to get 29282 CZK back?

Solution:

$$\begin{aligned}
&20000(1+0.1)^t = 29282 \\
&1.1^t = \frac{29282}{20000} = 1.4641 \\
&\log 1.1^t = \log 1.4641 \\
&t \log 1.1 = \log 1.4641 \\
&t = \frac{\log 1.4641}{\log 1.1} = \frac{0.16557}{0.0414} \approx 4
\end{aligned}$$

It takes 4 years to get 29282 back.

Example: There are two options for investing money. If you invest 10000 Bank A offers to give you back 13382 after 5 years. Bank B offers 8% interest rate compounded annually if you leave your money there for 5 years. Which option should you choose?

Solution: We have different types of information about these two options. To be able to compare them we need to know either what is the interest rate in Bank A or what will be the amount of money that we get in Bank B. We will find both.

$$
\begin{aligned}
&10000(1+r)^5 = 13382 \\
&(1+r)^5 = \frac{13382}{10000} = 1.3382 \\
&5\log(1+r) = \log 1.3382 \\
&\log(1+r) = \frac{1}{5}\log 1.3382 \\
&\log(1+r) = \log 1.3382^{\frac{1}{5}} \\
&r = 1.3382^{\frac{1}{5}} - 1 = 0.06 = 6\%
\end{aligned}
$$

This means that Bank A offers lower interest rate and hence we should choose Bank B.

If we deposit 10000 in Bank B after 5 years we will get:

$$10000(1+r)^5 = 10000(1+0.08)^5 = 14693$$

This means that in Bank B we get more money back compared to Bank A and hence again we should choose bank B.

Example: You are told that if you invest one million into stocks you will get back 1.225.043 in three years. What is the annual interest rate?

Solution:

$$
\begin{aligned}
&1000000(1+r)^3 = 1225043 \\
&100^3(1+r)^3 = 107^3 \\
&(1+r)^3 = \frac{107^3}{100^3} = \left(\frac{107}{100}\right)^3 = 1.07^3 \\
&1+r = 1.07 \quad \Rightarrow \quad r = 0.07
\end{aligned}
$$