4 Linear Equations, Systems of Linear Equations

Equation: mathematical statement that relates two algebraic expressions involving at least one variable.

- $5x + 3 = 2 - x$
- $x^3 + 3x^2 - 1 = 7 + x - x^2$
- $\frac{3}{x^2 - x + 1} = x + 2$

Properties of equality:

1. if $a = b$ then $a + c = b + c$ — addition
2. if $a = b$ then $a - c = b - c$ — subtraction
3. if $a = b$ then $ca = cb$, $c \neq 0$ — multiplication
4. if $a = b$ then $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$ — division
5. if $a = b$ then they can be used interchangeably — substitution

LINEAR EQUATIONS - $ax + b = 0$

To solve linear equations in one variable we use the properties of equality. Remember, that whatever you do with one side of the equation has to be done with the other side as well.

$7x - 4 = 3$ — add 4 to both sides of equation

$7x - 4 + 4 = 3 + 4$

$7x = 7$ — dived both sides of equation by 7

$\frac{7x}{7} = \frac{7}{7}$

$x = 1$

Example: Solve the following equation and check.

$$6x + 2 = 2x + 14$$
$$6x - 2x + 2 = 14$$
$$4x + 2 = 14$$
$$4x = 14 - 2$$
$$4x = 12$$
$$x = 3$$

Check: we substitute 3 for x in the original equation in order to check that our solution is correct:

$$6x + 2 =^? 2x + 14$$
$$6 \times 3 + 2 =^? 2 \times 3 + 14$$
$$20 =^{\checkmark} 20$$

So indeed, $x = 3$ is a solution to our equation.

Problem: Find 5 consecutive natural numbers such that their sum is 50.

Solution: Let's denote the first number x. Then the four remaining numbers are $x+1, x+2, x+3$ and $x+4$. Their sum is supposed to be equal to 50. So we have the following equation:

$$x + (x + 1) + (x + 2) + (x + 3) + (x + 4) = 50$$
$$5x + 10 = 50$$
$$5x = 40$$
$$x = 8$$

Hence the numbers are 8, 9, 10, 11 and 12.

Problem: Find 4 consecutive odd integers such that the sum of the last two is equal to 2 times the sum of the first two numbers.

Solution: Let's denote the first number x. Then the three remaining numbers are $x+2, x+4$ and $x+6$. Sum of the first two numbers is $x+(x+2)$ and the sum of two last numbers is $(x+4)+(x+6)$. Sum of the last two is 2 times the sum of the first two numbers. Therefore, to have an equality we have to multiply the sum of the first two numbers by 2:

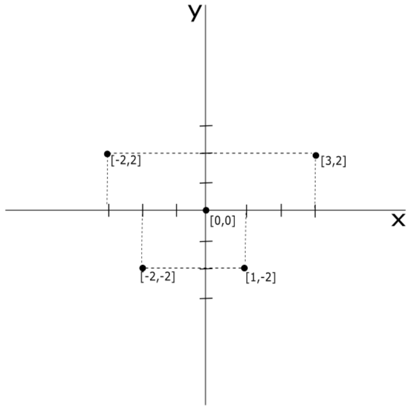
$$2[x + (x + 2)] = (x + 4) + (x + 6)$$
$$4x + 4 = 2x + 10$$
$$2x = 6$$
$$x = 3$$

Hence the numbers are 3, 5, 7 and 9.

Cartesian coordinate system, point, line

Cartesian coordinate system is formed by two real lines, one horizontal and one vertical, which cross through their origins. These two lines are called the horizontal axis and vertical axis.

Point: Every point is represented by two numbers - coordinates. The first number represents the value on axis x and the second number represents the value on axis y.



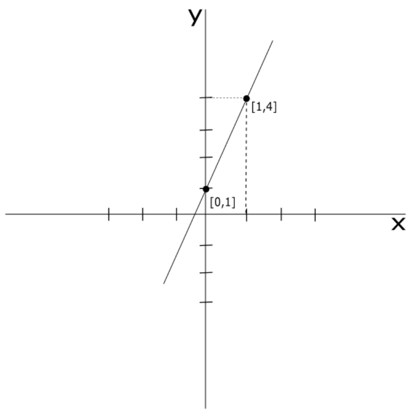
Linear function - Straight line:

Generally, linear function has the following form: $y = ax + b$. This can be graphically represented by a straight line. Any straight line can be represented by two points. If we find two points lying on the line, we can draw the whole line. Coefficient a is called *slope*. The bigger (smaller) a the steeper (flatter) the line.

Example: $y = 3x + 1$.

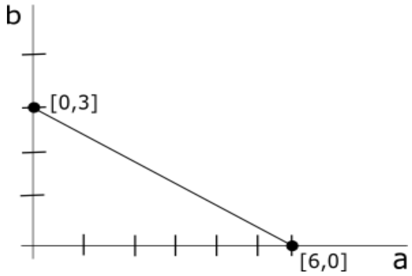
To find two points lying on this line we use 0 and 1 for x and find corresponding values of y from the equation:

x	0	1
y	$3 \times 0 + 1 = 1$	$3 \times 1 + 1 = 4$



In economics, we often deal with the budget constraint. We can draw the budget line or alternatively budget set in the following way:

Example: Assume that there are only two goods: apples and bananas. The price of apples is \$2 and the price of bananas is \$4. You have \$12. If you spend all the money on apples, you can afford to buy 6 of them. If you spend all the money on bananas, you can buy 3. So the budget line goes through points [6,0] and [0,3]. The budget line can be represented by the following equation $2a + 4b = 12$ and graphically:



Budget line represents all combinations of apples and bananas that we can buy spending *exactly* \$12.

The budget set represents all combinations of apples and bananas that we can afford, i.e. that we can buy spending *at most* \$12. This can be represented by inequality $2a + 4b \leq 12$ or graphically it is the triangle below the budget line.

Forms for linear equations:

1. **General form:** $Ax + By + C = 0$, where A and B are not both equal to zero. If A is nonzero, then the x-intercept, that is the x-coordinate of the point where the graph crosses the x-axis (y is zero), is $-C/A$. If B is nonzero, then the y-intercept, that is the y-coordinate of the point where the graph crosses the y-axis (x is zero), is $-C/B$, and the slope of the line is $-A/B$.

2. **Slope-intercept form:** $y = mx + c$, where m is the slope of the line and c is the y-intercept, which is the y-coordinate of the point where the line crosses the y axis. This can be seen by letting x = 0, which immediately gives that y = c.

Special cases:

1. **y=b:** This is a special case of the general form where $A = 0$ and $B = 1$, or of the slope-intercept form where the slope $m = 0$. The graph is a horizontal line with y-intercept equal to b. There is no x-intercept, unless $b = 0$, in which case the graph of the line is the x-axis, and so every real number is an x-intercept.

2. **x=c:** This is a special case of the standard form where $A = 1$ and $B = 0$. The graph is a vertical line with x-intercept equal to c. The slope is undefined. There is no y-intercept, unless $c = 0$, in which case the graph of the line is the y-axis, and so every real number is a y-intercept.

SYSTEM OF TWO EQUATIONS IN TWO VARIABLES

$$3x + 2y = 12$$
$$4x - y = 5$$

SOLVING BY SUBSTITUTION:

Eliminate one of the variables by replacement when solving a system of equations. Think of it as "grabbing" what one variable equals from one equation and "plugging" it into the other equation.

Solution:

$$3x + 2y = 12$$
$$4x - y = 5 \qquad \Rightarrow y = 4x - 5$$

Now, plug $4x - 5$ for y in the first equation:

$$3x + 2(4x - 5) = 12$$
$$3x + 8x - 10 = 12$$
$$11x = 22$$
$$x = 2$$

Now we get back to $y = 4x - 5$ and therefore $y = 4 \times 2 - 5 = 3$.

Problem: Solve the system of linear equations given below using substitution.

Suppose there is a piggybank that contains 57 coins, which are only quarters and dimes. The total number of coins in the bank is 57, and the total value of these coins is \$9.45. This information can be represented by the following system of equations:

$$D + Q = 57$$
$$00.10D + 0.25Q = 9.45$$

Determine how many of the coins are quarters and how many are dimes.

Solution:

$$D + Q = 57 \qquad \Rightarrow D = 57 - Q$$
$$00.10D + 0.25Q = 9.45$$

Plug $57 - Q$ for D in the second equation

$$\begin{aligned} 00.10(57 - Q) + 0.25Q &= 9.45 \\ 5.7 - 0.1Q + 0.25Q &= 9.45 \\ 0.15Q &= 3.75 \\ Q &= 25 \qquad D = 57 - Q = 57 - 25 = 32 \end{aligned}$$

SOLVING BY ADDITION (ELIMINATION) METHOD:

The addition method says we can just add everything on the left hand side and add everything on the right hand side and keep the equal sign in between.

$$3x + y = 14$$
$$4x - y = 14$$

Solution: Add the two equations; i.e sum left hand sides, sum right hand sides and keep equal sign in between. This way, we eliminate variable y and get only one equation in one variable x:

$$3x + 4x + y - y = 14 + 14$$
$$7x = 28$$
$$x = 4$$

Now we plug 4 for x and use any of two equations to determine y:

$$3x + y = 14$$
$$3 \times 4 + y = 14$$
$$y = 2$$

Check:

$$3x + y = 14 \ldots 3 \times 4 + 2 \stackrel{?}{=} 14 \ldots 14 \stackrel{\checkmark}{=} 14$$
$$4x - y = 14 \ldots 4 \times 4 - 2 \stackrel{?}{=} 14 \ldots 14 \stackrel{\checkmark}{=} 14$$

Problem:

$$2x + 2y = 12$$
$$3x - y = 14$$

Solution: First multiply the second equation by 2 so that we can use the addition method.

$$2x + 2y = 12$$
$$6x - 2y = 28$$

Adding the two equations we get:

$$8x = 40$$
$$x = 5$$

Now we plug 5 for x and use any of two equations to determine y:

$$2x + 2y = 12$$
$$2 \times 5 + 2y = 12$$
$$2y = 2$$
$$y = 1$$

Check:

$$2x + 2y = 12 \ldots 2 \times 5 + 2 \times 1 =^{?} 12 \ldots 12 =^{\checkmark} 12$$
$$3x - y = 14 \ldots 3 \times 5 - 1 =^{?} 14 \ldots 14 =^{\checkmark} 14$$

Problem: Find the equilibrium price of apple and equilibrium quantity consumed if demand and supply equations are as follows:

$$p = -q + 20 \quad \text{Demand equation (consumer)}$$
$$p = 4q - 55 \quad \text{Supply equation (supplier)}$$

Solution:

$$p = -q + 20$$
$$p = 4q - 55 \quad \Rightarrow \quad -q + 20 = 4q - 55 \quad \Rightarrow \quad 5q = 75 \quad \Rightarrow \quad q = 15$$
$$p = -q + 20 = -15 + 20 = 5$$

We know already that an equation represents a straight line. Intuitively, the system of equations represents the system of lines. Solving system of equation means looking for the intercept of lines. See the following example:

Example: Solve the following system numerically and graphically:

$x + y = 5$
$2x - y = 1$

Numerical solution to this system is $x = 2$ and $y = 3$.

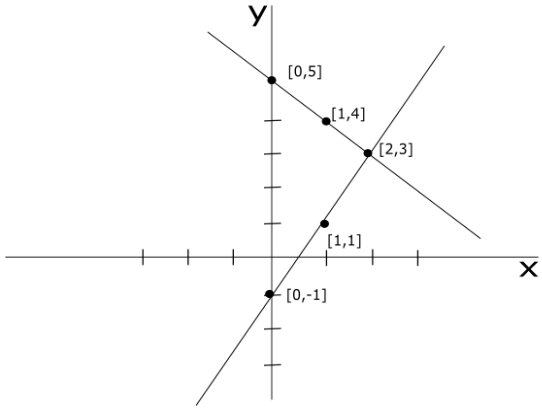
To find graphical solution we first need to draw both lines:

$x + y = 5$ or alternatively $y = 5 - x$

x	0	1
y	5	4

$2x - y = 1$ or alternatively $y = 2x - 1$

x	0	1
y	-1	1



The two lines intercept in point [2,3].

Generally, the system of two equations and two variables can have no solution, exactly one solution (see the example above) or infinitely many solutions.

Example: Solve the following system numerically and graphically:

$3x - y = 2$
$-\, 9x + 3y = -4$

Solution:

$$3x - y = 2 \qquad \Rightarrow \quad y = 3x - 2$$
$$-9x + 3y = -4$$

$$-9x + 3(3x - 2) = -4$$
$$-9x + 9x - 6 = -4$$
$$-6 = -4$$

The last equality does not hold for any values of x and y. This means that this system does not have any solution.

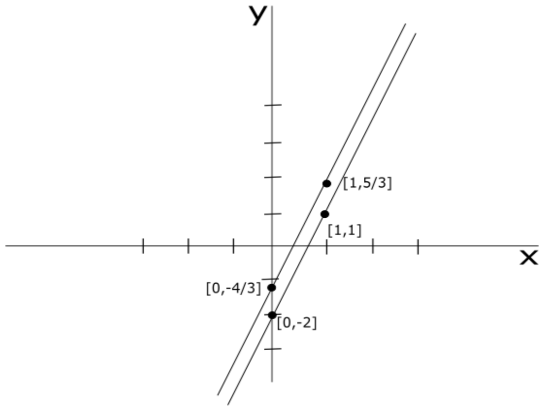
Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -4$ or alternatively $y = \frac{1}{3}(9x - 4)$

x	0	1
y	$-4/3$	$5/3$



From the picture we see that the two lines are parallel, i.e. they do not intercept in any point. That is the reason why the system does not have any solution.

Example: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -6$

Solution:

$$3x - y = 2 \qquad \Rightarrow \quad y = 3x - 2$$
$$-9x + 3y = -6$$

$$-9x + 3(3x - 2) = -6$$
$$-9x + 9x - 6 = -6$$
$$-6 = -6$$

The last equality holds for all values of x and y (-6 = -6 no matter what are the values of x and y). This means that this system has infinitely many solutions.

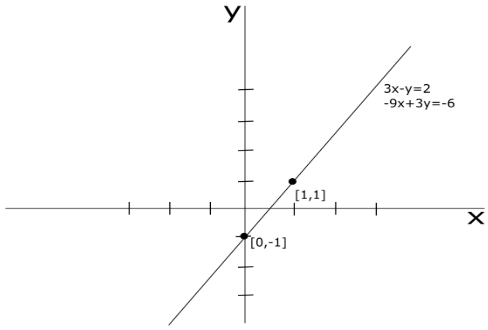
Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -6$ or alternatively $y = \frac{1}{3}(9x - 6) = 3x - 2$

Note that both lines are represented by the same equation. This means that the two lines coincide and therefore there are infinitely mane points where these two lines intercept and hence the system has infinitely many solutions.



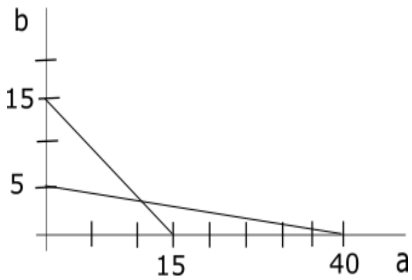
Example: Assume that there are only two goods apples and bananas. Some company produces apple-banana juice. The budget of the company is \$200. The price of apples is \$5 and the price of bananas is \$40. Further, the company has a limited capacity and can only store 15 pieces of fruit at the time. Sketch the budget set, the production possibilities set and find on the graph all combinations of apples and bananas which are feasible in terms of money and capacity.

Solution:

Budget set: The budget set is defined by the following inequality: $10a + 20b \leq 200$. If the company buys only apples, it can buy 40 kilograms. If the company spends all the money on bananas only, it can afford 10 kilograms. Therefore, the budget line goes through points [40,0] and [0,5].

Production set: The production set is defined by the inequality: $a + b \leq 15$. If the company buys only apples, it can buy 15 kilograms of apples. Similarly for bananas.

The budget set and production set are depicted on the following figure. Two lines correspond to budget line and production line. Budget (production) set is the area below the budget (production) line.



Combinations of apples and bananas which are feasible in terms of money and capacity are combinations which belong to both sets at the same time. In other words, we find the intercept of two triangles. This intercept is represented be the shaded area on the picture below.