3 Polynomials and their factorization

Polynomial functions

Polynomial function has the following general form:

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

in which each term contains a coefficient as well as a nonnegative-integer power of the variable x. (We can write $x^1 = x$ and $x^0 = 1$, thus the first two terms may be taken to be $a_0 x^0$ and $a_1 x^1$ respectively.)

Depending on the value of the integer n (which specifies the highest power of x), we have several subclasses of polynomial function:

Case of $n = 0$: $y = a_0$ *constant* function
Case of $n = 1$: $y = a_0 + a_1 x$ *linear* function
Case of $n = 2$: $y = a_0 + a_1 x + a_2 x^2$ *quadratic* function
Case of $n = 3$: $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ *cubic* function

Factoring: Factor of an algebraic expression is one of two or more algebraic expressions whose product is the given algebraic expression; e.g. $x^3 - x^2 - 4x + 4 = (x-1)(x+2)(x-2)$. $(x-1)$, $(x+2)$ and $(x-2)$ are factors.

In general, quadratic function $y = a_0 + a_1 x + a_2 x^2$ can be written as $(x-r_1)(x-r_2)$; cubic function $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ can be written as $(x-r_1)(x-r_2)(x-r_3)$ and so on. Numbers r_1, r_2, r_3 are called *roots*. Note that if all roots are integers then all roots must be divisors of a_0 (Why?).

Example: Find all integer roots of the (polynomial) cubic function $x^3 - 2x^2 - 9x + 18$. All the roots have to be solutions of equation $x^3 - 2x^2 - 9x + 18 = 0$.

First of all, note that there will be three roots in this example. All possible roots have to be divisors of 18, so it can be: $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9$. We have to check all possible number to check if they are solution of above mentioned equation.

-1: $(-1)^3 - 2(-1)^2 - 9(-1) + 18 = 22 \neq 0$

1: $1^3 - 2 * 1^2 - 91 + 18 = 10 \neq 0$

-2: $(-2)^3 - 2(-2)^2 - 9(-2) + 18 = 20 \neq 0$

2: $2^3 - 2 * 2^2 - 92 + 18 = 0$

-3: $(-3)^3 - 2(-3)^2 - 9(-3) + 18 = 0$

3: $3^3 - 2 * 3^2 - 93 + 18 = 0$

We already found three roots, so all the integer roots are: $2, -3$, and 3.

Factoring quadratic polynomials:

Quadratic polynomial can sometimes be factored into two binomials with simple integer coefficients without the need to use the quadratic formula. In a quadratic equation, this will expose its two roots. The formula

$$ax^2 + bx + c$$

would be factored into:

$$(mx + p)(nx + q)$$

where

$$mn = a$$
$$pq = c$$
$$pn + mq = b$$

and m, n, p, and q are integers.

Example: Consider $x^2 - 5x + 6$. In this case: $a = 1$, $b = -5$ and $c = 6$.

This polynomial would be factored into:

$$(mx + p)(nx + q)$$

where

$$mn = 1 \quad \Rightarrow \quad m = n = 1$$
$$pq = 6$$
$$pn + mq = -5 \quad \Rightarrow \quad p + q = -5$$

This only holds for p and q being equal to -2 and -3. Hence, polynomial $x^2 - 5x + 6$ can be written as $(x - 2)(x - 3)$ and therefore has two roots: 2, and 3; and two factors $(x - 2)$ and $(x - 3)$.

Example: Now, take for example $2x^2 - 5x + 2$. Because $a = 2$ and $mn = a$, $mn = 2$, which means that of m and n, one is 1 and the other is 2. Now we have $(2x + p)(x + q)$. Because $c = 2$ and $pq = c$, $pq = 2$, which means that of p and q, one is 1 and the other is 2 or one is -1 and the other is -2. A guess and check of substituting the 1 and 2, and -1 and -2, into p and q (while applying $pn + mq = b$) tells us that $2x^2 - 5x + 2 = (2x - 1)(x - 2)$, giving us the roots $x = \{0.5, 2\}$

Factoring by grouping

Here, the general idea, as the name of the method suggests, is to group in some sense similar terms together. The method is illustrated on the following examples:

Example: $x^3 + x^2 + x + 1 = x^2(x + 1) + (x + 1) = (x + 1)(x^2 + 1)$

Example: $x^3 - x^2 - 4x + 4 = x^2(x - 1) - 4(x - 1) = (x - 1)(x^2 - 4) = (x - 1)(x - 2)(x + 2)$

Domain, Image, Range

In general we use term *function* to describe relationship between independent variable x and dependent variable y and we write $y = f(x)$.

Domain: the set of numbers that are permitted to replace the independent variable x (no "0" in the denominator, no negative number under the square root).

Image: value of y into which an x value is mapped.

Range: set of all values that the y variable can take.

When no specification is given, it is to be understood that the domain will only include number for which a function makes an economic sense.

Example: The total cost C of a firm per day is a function of its daily output Q: $C = 150 + 7Q$. The firm has a capacity limit of 100 units of output per day. What are the domain and the range of the cost function?

Q can vary only between 0 and 100, hence the domain is:

$$\{Q|0 \leq Q \leq 100\}$$

The minimum value of C is 150 (for $Q = 0$) and maximum value of C is 850 (for $Q = 100$). Thus the range is

$$\{C|150 \leq C \leq 850\}$$

Note: extreme values of the range may not always occur where the extreme values of the domain are attained. It is so in this example because of the linearity of relationship between Q and C.