12 Review Lecture

Equations and Inequalities

Problem 1: Solve the absolute value inequality. Write the solution set using interval notation: $|4x + 7| < 5$

Solution:

$$
\begin{aligned}
&|4x + 7| < 5 \\
&-5 < 4x + 7 < 5 \quad / - 7 \\
&-12 < 4x < -2 \quad / \div 4 \\
&-3 < x < -1/2 \quad \Rightarrow \quad x \in (-3, -1/2)
\end{aligned}
$$

Problem 2: Write as a single interval, using interval notation: $(-\infty, 1) \cap (-10, 5)$

Solution: $(-\infty, 1) \cap (-10, 5) = (-\infty, 5)$

Problem 3: Solve the following inequality for x. Express the solution set using interval notation: $\frac{1}{2}x - 4 < \frac{1}{3}x + 5$

Solution:

$$
\begin{aligned}
&\frac{1}{2}x - 4 < \frac{1}{3}x + 5 \\
&\frac{1}{2}x - \frac{1}{3}x < 5 + 4 \\
&\frac{1}{6}x < 9 \\
&x < 54 \quad \Rightarrow \quad x \in (-\infty, 54)
\end{aligned}
$$

Problem 4: Solve the compound inequality for x. Express the solution set using interval notation: $8 \leq 5 - x$ or $3x - 2 > 10$

Solution:

$$
\begin{aligned}
&8 \le 5 - x \quad \text{or} \quad 3x - 2 > 10 \\
&-3 \le -x \quad \text{or} \quad 3x > 12 \\
&x \le 3 \quad \text{or} \quad x > 4 \quad \Rightarrow \quad x \in (-\infty, 3] \quad \text{or} \quad x \in (4, \infty)
\end{aligned}
$$

Exponents and Logarithms

Problem 5: Solve for x: $\log(3x - 9) = \log(2x - 6)$

Solution:

$$
\begin{aligned}
&\log(3x - 9) = \log(2x - 6) \\
&3x - 9 = 2x - 6 \\
&3x - 2x = -6 + 9 \\
&x = 3
\end{aligned}
$$

Problem 6: Find the exact solution for x: $(1.3)^{2x} = 4$

Solution:

$$
\begin{aligned}
&(1.3)^{2x} = 4 \\
&\ln\left[(1.3)^{2x}\right] = \ln 4 \\
&(2x)\ln\left[(1.3)\right] = \ln 4 \\
&2x = \frac{\ln 4}{\ln\left[(1.3)\right]} \\
&x = \frac{\ln 4}{2\ln\left[(1.3)\right]}
\end{aligned}
$$

Problem 7: Solve the equation for x: $\left(\frac{2}{3}\right)^{x+1} = \frac{8}{27}$

Solution:

$$
\begin{aligned}
&\left(\frac{2}{3}\right)^{x+1} = \frac{8}{27} \\
&\left(\frac{2}{3}\right)^{x+1} = \left(\frac{2}{3}\right)^{3} \\
&x + 1 = 3 \\
&x = 2
\end{aligned}
$$

Matrices and Determinants

Problem 8: When the system of linear equations is solved, what is the value of x and y?
$x - 2y = 4$, $3x - y = -3$

Solution:

$x - 2y = 4 \quad \Rightarrow \quad x = 4 + 2y$ plug in the second equation:
$3(4 + 2y) - y = -3$
$12 + 6y - y = -3$
$5y = -15$
$y = -3$ plug in the first equation to find the value of x
$x = 4 + 2y = 4 + 2(-3) = 4 - 6 = -2$
$\Rightarrow \quad x = -2,\ y = -3$

Problem 9: Find product of the following matrices:

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$$

Solution:

$$AB = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 + 4 \cdot 2 & 1 \cdot (-1) + 4 \cdot 2 \\ -2 \cdot 3 + 0 \cdot 2 & -2 \cdot (-1) + 0 \cdot 2 \end{pmatrix} = \begin{pmatrix} 11 & 7 \\ -6 & 2 \end{pmatrix}$$

Problem 10: Find the following determinants:

(a) $\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 2 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{vmatrix}$

Solution:

(a) $\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = 1 \cdot 1 - (-2)(-3) = 1 - 6 = -5$

(b) $\begin{vmatrix} 2 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{vmatrix} = 2(-1)^{1+1}[(-1) \cdot 0 - 1 \cdot (-1)] + 2(-1)^{1+2}[3 \cdot 0 - (-1)(-2)] +$

$+3(-1)^{1+3}[3 \cdot 1 - (-1)(-2)] = 2 + 4 + 3 = 9$

Problem 11: Solve the following systems using (i) matrix method, (ii) inverse matrix, and (iii) Cramer's rule:

$x + y = 1,\ \ 3x - 4y = -18$

Solution: Matrix method:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & -4 & -18 \end{array}\right) \begin{array}{c} \times(-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -7 & -21 \end{array}\right) \div(-7) \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-1) \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right) \Rightarrow \begin{array}{c} x = -2 \\ y = 3 \end{array}$$

Inverse matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right) \begin{array}{c} \times(-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -7 & -3 & 1 \end{array}\right) \div(-7) \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/7 & -1/7 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-1) \end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 4/7 & 1/7 \\ 0 & 1 & 3/7 & -1/7 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 4/7 & 1/7 \\ 3/7 & -1/7 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 4/7 & 1/7 \\ 3/7 & -1/7 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4/7 + 1 \cdot 3/7 & 1 \cdot 1/7 + 1 \cdot (-1/7) \\ 3 \cdot 4/7 + (-4) \cdot 3/7 & 3 \cdot 1/7 - 4 \cdot (-1/7) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AX = B \longrightarrow \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -18 \end{pmatrix}$$

$$X = A^{-1}B \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/7 & 1/7 \\ 3/7 & -1/7 \end{pmatrix} \begin{pmatrix} 1 \\ -18 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} \cdot 1 + \frac{1}{7} \cdot (-18) \\ \frac{3}{7} \cdot 1 - \frac{1}{7} \cdot (-18) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{c} x = -2 \\ y = 3 \end{array}$$

Cramer's rule:

$$x = \frac{\begin{vmatrix} 1 & 1 \\ -18 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}} = \frac{14}{-7} = -2$$

$$y = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -18 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}} = \frac{-21}{-7} = 3$$

Financial Mathematics

Problem 12: If \$4500 is deposited in a bank account paying 8% compounded quarterly, then how much interest will be earned at the end of 6 years? Hint: $A = P\left(1 + \frac{i}{t}\right)^{nt}$

Solution:

$$A = P\left(1 + \frac{i}{t}\right)^{nt}$$
$$A = 4500\left(1 + \frac{0.08}{4}\right)^{4\cdot 6} = 4500 \cdot 1.02^{24} = 7237.97$$

The amount of money in a bank account will be \$7237.97 what means that the interest earned is \$7237.97 − \$4500 = \$2737.97

Problem 13: Celia has invested \$2500 at 11% yearly interest. How much must she invest at 12% so that the interest from both investments totals \$695 after a year?

Solution: The interest from the first investment is:

$$A = P(1+i)^n$$
$$A = 2500(1+0.11)^1 = 2500 \cdot 1.11 = 2775$$

Hence the interest from the first investment is \$2775 − \$2500 = \$275.

The total investment is supposed to be \$695 what means that the interest from the second investment has to be \$695 − \$275 = \$420.

$$P + 420 = P(1+i)^n$$
$$P + 420 = P(1+0.12)^n$$
$$P + 420 = P \cdot 1.12^1$$
$$420 = P \cdot 1.12 - P = 0.12P$$
$$P = \frac{420}{0.12} = 3500$$

Problem 14: After a 7% increase in salary, Laurie makes \$1016.50 per month. How much did she earn per month before the increase?

Solution: Let's denote the original salary S. The according to the setup we have that:

$$
\begin{aligned}
&S + 0.07S = 1016.50 \\
&1.07S = 1016.50 \\
&S = \frac{1016.50}{1.07} = 950
\end{aligned}
$$