10 Matrices, determinants, Cramer's rule

Properties of determinants:

• **Multiplying a row by a constant:** If each element of any row (or column) of a determinant is multiplied by a constant k, the new determinant is k times the original.

Example:

$$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 1 = 4$$

$$\begin{vmatrix} 2 \times 2 & 2 \times 1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 4 \times 3 - 2 \times 2 = 8 = 2 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

• **Row of zeros:** If every element in a row (or column) is 0, then the value of the determinant is 0.

Example:

$$\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 2 \times 0 - 0 \times 1 = 0$$

• **Interchanging rows:** If two rows (or columns) are interchanged, the new determinant is the negative of the original.

Example:

$$\begin{vmatrix} 1 & 0 & 9 \\ -2 & 1 & 5 \\ 3 & 0 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 9 & 0 \\ -2 & 5 & 1 \\ 3 & 7 & 0 \end{vmatrix}$$

• **Equal rows:** If the corresponding elements are equal in two rows (or columns), the value of the determinant is 0.

• **Addition of rows:** If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.

Cramer's Rule:

Given the system:

$$\begin{matrix} a_{11}x + a_{12}y = k_1 \\ a_{21}x + a_{22}y = k_2 \end{matrix} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

Example: Solve the following system using: 1. matrix method, 2. inverse matrix, 3. Cramer's rule.

$-2x + y = 6$
$x - y = -5$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} -2 & 1 & 6 \\ 1 & -1 & -5 \end{array}\right) \begin{matrix} \nwarrow \\ \swarrow \end{matrix} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ -2 & 1 & 6 \end{array}\right) \begin{matrix} /2 \\ \swarrow \end{matrix} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & -1 & -4 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & 1 & 4 \end{array}\right) \begin{matrix} \nwarrow \\ \nearrow \end{matrix} \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right) \quad \Rightarrow \quad \begin{matrix} x = -1 \\ y = 4 \end{matrix}$$

2. using inverse matrix: First we find the inverse matrix:

$$\left(\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix} \nwarrow \\ \swarrow \end{matrix} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array}\right) \begin{matrix} /2 \\ \swarrow \end{matrix} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right) \begin{matrix} \nwarrow \\ \nearrow \end{matrix} \sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

Now, using the inverse matrix we get x and y:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} x = -1 \\ y = 4 \end{matrix}$$

3. Cramer's rule:

$$x = \frac{\begin{vmatrix} 6 & 1 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-1}{1} = -1$$

$$y = \frac{\begin{vmatrix} -2 & 6 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{4}{1} = 4$$

Example: Solve the following system using: 1. matrix method, 2. inverse matrix, 3. Cramer's rule.

$3x - 2y = 0$
$x + 2y = 8$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} 3 & -2 & 0 \\ 1 & 2 & 8 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -2 & 0 \end{array}\right) \begin{array}{c}/(-3)\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -8 & -24 \end{array}\right) \begin{array}{c}\\ / \div 3\end{array} \sim$$

$$\left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right) \begin{array}{c}\nwarrow\\ /(-2)\end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right) \Rightarrow \begin{array}{c} x = 2 \\ y = 3 \end{array}$$

2. using inverse matrix: First we find the inverse matrix:

$$\left(\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & -2 & 1 & 0 \end{array}\right) \begin{array}{c}/(-3)\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -8 & 1 & -3 \end{array}\right) \begin{array}{c}\\ / \div (-8)\end{array} \sim$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/8 & 3/8 \end{array}\right) \begin{array}{c}\nwarrow\\ /(-2)\end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/4 & 1/4 \\ 0 & 1 & -1/8 & 3/8 \end{array}\right)$$

Now, using the inverse matrix we get x and y:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{c} x = 2 \\ y = 3 \end{array}$$

3. Cramer's rule:

$$x = \frac{\begin{vmatrix} 0 & -2 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{16}{8} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 0 \\ 1 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{24}{8} = 3$$