AAU - Business Mathematics I

Lecture #9, November 5, 2009

# 11 Matrices, Determinants, Cramer's Rule

## 11.1 Matrices

**Matrices:** In mathematics, a matrix (plural matrices) is a rectangular table of elements (or entries), which may be numbers or, more generally, any abstract quantities that can be added and multiplied. Matrices are mostly used to describe linear equations and solve systems of equations in a more efficient way. Matrices can be added, multiplied, and decomposed in various ways, making them a key concept in linear algebra and matrix theory.

The horizontal lines in a matrix are called rows and the vertical lines are called columns. A matrix with $m$ rows and $n$ columns is called an $m-by-n$ matrix (written $m\times n$) and $m$ and $n$ are called its dimensions. The dimensions of a matrix are always given with the number of rows first, then the number of columns.

Almost always capital letters denote matrices with the corresponding lower-case letters with two indices representing the entries. For example, the entry of a matrix $A$ that lies in the $i$-th row and the $j$-th column is written as $a_{i,j}$ and called the $i,j$ entry or $(i,j)$-th entry of $A$.

*Example:*

$$A=\begin{pmatrix} 8 & 9 & 6 \\ 1 & 2 & 7 \\ 9 & 2 & 4 \\ 6 & 0 & 5 \end{pmatrix}$$

is a $4\times 3$ matrix. The element $a_{2,3}$ is 7.

$a_{1,1}=8, a_{1,2}=9, a_{1,3}=6$
$a_{2,1}=1, a_{2,2}=2, a_{2,3}=7$
$a_{3,1}=9, a_{3,2}=2, a_{3,3}=4$
$a_{4,1}=6, a_{4,2}=0, a_{4,3}=5$

*More examples:*

| | | | | |
|---|---|---|---|---|
| $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ | $\begin{pmatrix} 1 & 3 & 5 & 7 \end{pmatrix}$ | $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| Square matrix $3\times 3$ | Column matrix $4\times 1$ | Row matrix $1\times 4$ | Zero matrix $2\times 3$ | Identity matrix $3\times 3$ |

*Relationship between system of equations and matrix:*

$$2x - 3y = 5$$
$$x + 2y = -3$$

In matrix notation:

$$\left(\begin{array}{cc|c} 2 & -3 & 5 \\ 1 & 2 & -3 \end{array}\right)$$

## 11.2 Operations on/with Matrices

**Elementary Row Operations producing Row-Equivalent Matrices:**

1. Two rows are interchanged
2. A row is multiplied by a non-zero constant
3. A constant multiple of one row is added to another row.

*Example:* Solve the following system by using matrix:

$$3x + 4y = 1$$
$$x - 2y = 7$$

*Solution:* We start by writing corresponding matrix form:

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right)$$

Our objective is to use row operations as described above to transform matrix into the following form (which is called *reduced form*):

$$\left(\begin{array}{cc|c} 1 & 0 & \text{m} \\ 0 & 1 & \text{n} \end{array}\right)$$

where $m$ and $n$ are some real numbers. The solution to our system is then obvious because if we rewrite the matrix form into the system form we get:

$$1x + 0y = m$$
$$0x + 1y = n$$

or equivalently

$$x = m$$
$$y = n$$

which is the solution that we were looking for. So the only problem to be solved is to transform

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & \text{-}2 & 7 \end{array}\right) \quad \text{into} \quad \left(\begin{array}{cc|c} 1 & 0 & \text{m} \\ 0 & 1 & \text{n} \end{array}\right)$$

*Step 1:* To get 1 in the upper left corner, we interchange rows 1 and 2:

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & \text{-}2 & 7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & \text{-}2 & 7 \\ 3 & 4 & 1 \end{array}\right)$$

*Step 2:* To get 0 in the lower left corner, we multiply row 1 by (-3) and add to row 2:

$$\left(\begin{array}{cc|c} 1 & \text{-}2 & 7 \\ 3 & 4 & 1 \end{array}\right) \begin{array}{c} (-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & \text{-}2 & 7 \\ 0 & 10 & \text{-}20 \end{array}\right)$$

*Step 3:* To get 1 in the second row, second column, we multiply row 2 by 1/10:

$$\left(\begin{array}{cc|c} 1 & \text{-}2 & 7 \\ 0 & 10 & \text{-}20 \end{array}\right) \begin{array}{c} \\ 1/10 \end{array} \sim \left(\begin{array}{cc|c} 1 & \text{-}2 & 7 \\ 0 & 1 & \text{-}2 \end{array}\right)$$

*Step 4:* To get 0 in the first row, second column, we multiply row 2 by 2 and add the result to row 1:

$$\left(\begin{array}{cc|c} 1 & \text{-}2 & 7 \\ 0 & 1 & \text{-}2 \end{array}\right) \begin{array}{c} \nwarrow \\ 2 \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & \text{-}2 \end{array}\right)$$

The last matrix is the matrix for:

$x = 3$

$y = -2$

**Exercise 1:** Solve the following system by using matrix method:

$$\begin{aligned} 2x + 3y &= 11 \\ x - 2y &= 2 \end{aligned}$$

*Example:* Solve the following system by using matrix method:

$$\begin{aligned} x + y + z &= 6 \\ 2x + y - z &= 1 \\ 3x + y + z &= 8 \end{aligned}$$

*Solution:*

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 3 & 1 & 1 & 8 \end{array}\right) \begin{array}{cc} (-2) & (-3) \\ \swarrow & \\ & \swarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & -2 & -2 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 2 & 2 & 10 \end{array}\right) \begin{array}{c} \\ (-2) \\ \swarrow \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -4 & -12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{c} \\ \nwarrow \\ (-3) \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{c} \nwarrow \\ (-1) \\ (-1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \Longrightarrow \begin{array}{c} x = 1 \\ y = 2 \\ z = 3 \end{array}$$

**MATRICES: BASIC OPERATIONS**

**1. Addition and Subtraction**

The sum of two matrices of the same size is a matrix, with elements that are the sums of the corresponding elements of the two given matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} (a \pm w) & (b \pm x) \\ (c \pm y) & (d \pm z) \end{pmatrix}$$

*Example:*

$$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 8 & 7 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} (2+8) & (1+7) \\ (3+0) & (5+4) \end{pmatrix} = \begin{pmatrix} 10 & 8 \\ 3 & 9 \end{pmatrix}$$

**2. Multiplication of a Matrix by a Number**

The product of a number $k$ and a matrix $M$, denoted by $kM$, is a matrix formed by multiplying each element of $M$ by $k$.

$$k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

*Example:*

$$2 \begin{pmatrix} 1 & 3 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 12 & 14 \end{pmatrix}$$

## 3. Matrix Product

*Product of a row and a column matrix:* is given by:

$$(a_1\ a_2 \dots a_n)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (a_1b_1 + a_2b_2 + \cdots + a_nb_n)$$

*Example:*

$$(2 - 3\ 0)\begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix} = (2(-5) + (-3)2 + 0(-2)) = -10 - 6 = -16$$

*Example:* A factory produces a slalom water ski that requires 4 labor-hours in the fabricating department and 1 labor-hour in the finishing department. Fabricating personnel receive \$10 per hour, and finishing personnel receive \$8 per hour. Total labor cost per ski is given by the product:

$$(4, 1)\begin{pmatrix} 10 \\ 8 \end{pmatrix} = (4 \times 10 + 1 \times 8) = 40 + 8 = \$48 \text{ per ski.}$$

Now the factory also produces a trick water ski that requires 6 labor-hours in the fabricating department and 1.5 labor-hours in the finishing department. Compute the cost for the trick water ski and the total cost.

$$(6, 1.5)\begin{pmatrix} 10 \\ 8 \end{pmatrix} = (6 \times 10 + 1.5 \times 8) = 60 + 12 = \$72 \text{ per trick ski.}$$

The total cost is 48+72 = \$120.

*Matrix product:* If $A$ is an $m \times p$ matrix and $B$ is a $p \times n$ matrix, then the matric product of $A$ and $B$, denoted $AB$, is an $m \times n$ matrix whose element in the $i$th row and $j$th column is the real number obtained from the product of the $i$th row of $A$ and the $j$th column of $B$. If the number of columns in $A$ does not equal the number of rows in $B$, then the matrix product $AB$ is not defined.

*Example:*

$$\begin{pmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} (\ 2\ \ 3\ \ -1\ )\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & (\ 2\ \ 3\ \ -1\ )\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \\ (\ -2\ \ 1\ \ 2\ )\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & (\ -2\ \ 1\ \ 2\ )\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 9 & 4 \\ -2 & -2 \end{pmatrix}$$

## 11.3 Properties of Matrices

***Addition properties:***

- Associative: $(A + B) + C = A + (B + C)$
- Commutative: $A + B = B + A$
- Additive identity: $A + 0 = 0 + A = A$
- Additive inverse: $A + (-A) = 0$

***Multiplication properties:***

- Associative: $(AB)C = A(BC)$
- Multiplicative identity: $AI = IA = A$
- Multiplicative inverse: If $A$ is a square matrix and $A^{-1}$ exists, then $AA^{-1} = A^{-1}A = I$
- *Note*: $AB \neq BA$ (see the following example)

*Example:*

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 11 & -10 \\ 24 & -10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 13 \\ -10 & 0 \end{pmatrix}$$

***Equality:***

- Addition: If $A = B$, then $A + C = B + C$
- Left Multiplication: If $A = B$, then $CA = CB$
- Right multiplication: If $A = B$, then $AC = BC$