8 Equations and Inequalities with Absolute value

8.1 Equations with Absolute Value

- if a is some number, then **absolute value** of a, $|a|$, is the distance of a from 0.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples: $|4| = 4$, $|-5| = 5$, $|1-\sqrt{2}| = \sqrt{2}-1, \ldots$

- if a and b are some numbers, then absolute value of $a-b$, $|a-b|$, is the distance of a from b, or the distance between a and b. It holds, that $|a-b| = |b-a|$.

Examples: $|9-4| = |5| = 5$, $|4-9| = |-5| = 5$, $|5| = |5-0| = 5, \ldots$

Problem: Solve $|x-1| = 2$

Solution: We are looking for such number(s) x that the distance of x from 1 is equal to 2. It's clear that there are 2 such numbers: -1 and 3.

Formally: $|x-a| = b \Rightarrow x-a = b$ or $x-a = -b$, i.e. $x-a = \pm b$. Then it follows that $x = a \pm b$.

Problem: Solve $|x+4| = 1$

Solution: Note that $|x+4| = 1$ can be written as $|x-(-4)| = 1$. We are looking for such number(s) x that the distance of x from -4 is equal to 1. It's clear that there are 2 such numbers: -5 and -3.

Note: In the problem $|x+4| = 1$ we are looking for number(s) such that their distance from -4 is equal to 1. Not distance from 4 is equal to 1 !!!

Problem: Solve $|3x-7| = 2$

Solution:

$$\begin{aligned} &3x - 7 = \pm 2 \\ &3x = 7 \pm 2 \\ &x = \frac{7 \pm 2}{3} \\ &x = 3, \frac{5}{3} \end{aligned}$$

Exercise 1: Solve $|2x+5| = 3$

If we have variable x on both sides of the equation, solution is not that easy any more:

Problem: Solve $|x+4| = 3x - 8$

Solution: We distinguish two cases:

- $x + 4 \geq 0 \Leftrightarrow x \geq -4 \ldots x + 4 = 3x - 8 \Rightarrow x = 6$
- $x + 4 \leq 0 \Leftrightarrow x < -4 \ldots -x - 4 = 3x - 8 \Rightarrow x = 1$

In the first case, the initial condition is $x + 4 \geq 0$ and $x = 6$ satisfies this condition. Hence, $x = 6$ is a solution to our equation.

In the second case, the initial condition is $x + 4 \leq 0$. But corresponding solution $x = 1$ does not satisfy this condition. Hence, $x = 1$ is **not** a solution to our equation.

Problem: Solve $|x+4| = |2x - 6|$

Solution: Absolute value keeps the expression the same if it is positive, in changes the sign of the expression if it is negative. Generally, any equation with any number of absolute values can be solved by getting rid of absolute values. To do so, we need to divide the problem into subcases for which we can eliminate the absolute value:

- $x + 4$ is positive for $x \geq -4$ and negative for $x \leq -4$
- $2x - 6$ is positive for $x \geq 3$ and negative for $x \leq 3$

Put the two together:

if $x \in (-\infty, -4]$, both expressions are negative

if $x \in [-4, 3]$, $x + 4$ is positive and $2x - 6$ is negative

if $x \in [3, \infty)$, both expressions are positive

Now we solve following three problems:

$$\begin{array}{lll} x \in (-\infty, -4] & -x - 4 = -2x + 6 & \Rightarrow x = 10 \\ x \in [-4, 3] & x + 4 = -2x + 6 & \Rightarrow x = \dfrac{2}{3} \\ x \in [3, \infty) & x + 4 = 2x - 6 & \Rightarrow x = 10 \end{array}$$

In the firs equality, $x = 10$ does not satisfy the initial condition $x \in (-\infty, -4)$ and therefore this is not a solution. So we have two solutions of our problem $x = 2/3$ and 10.

Exercise 2: Solve $|2x - 2| + |x + 3| = 4$

8.2 Inequalities with Absolute Value

Again, solving inequalities with absolute value is almost the same as solving equations with absolute value. Only multiplying or dividing by a negative number changes the sign of inequality.

Problem: Solve $|x - 5| < 1$

Solution: We are looking for all x such that the difference of x from 5 is less than 1. It's clear that this is true for all $x \in (4, 6)$.

$$|x - 5| < 1$$
$$-1 < x - 5 < 1$$
$$4 < x < 6$$

Indeed, the inequality holds for all $x \in (4, 6)$.

Problem: Solve $|3x - 2| \leq 7$

Solution:

$$|3x - 2| \leq 7$$
$$-7 \leq 3x - 2 \leq 7$$
$$-5 \leq 3x \leq 9$$
$$-5/3 \leq x \leq 3$$

The inequality holds for all $x \in [-5/3, 3]$.

8.3 Answers

Exercise 1: $x = -4, -1$

Exercise 2: $x = 1$

9 Exponential Equations and Logarithmic Equations

9.1 Exponents and Logarithms

Note that there is a difference between x^2 and 2^x. It makes a big difference whether a variable appears as a base with a constant exponent or as an exponent with a constant base.

Exponential function: $f(x) = b^x, b > 0, b \neq 0$

$f(x)$ defines an exponential function for each different constant b, called the base. The independent variable x may assume any real value.

We require the base to be positive ($b > 0$) because if x is for example $1/2$, we have $f(x) = b^x = b^{1/2} = \sqrt{b}$ and we only can have a non negative number under the square root.

Exponential function properties:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$a^x = a^y \text{ if and only if } x = y$$

$$\text{for } x \neq 0, a^x = b^x \text{ if and only if } a = b$$

$$0^x = 0, \quad 1^x = 1, \quad x^0 = 1 \text{ for all } x$$

Example: Simplify:

(a) $\left(\frac{4}{3}\right)^2 \frac{3^3}{4}$

$$\left(\frac{4}{3}\right)^2 \frac{3^3}{4} = \frac{4^2}{3^2}\frac{3^3}{4} = 4^{(2-1)}3^{(3-2)} = 4 \times 3 = 12$$

(b) $\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3}$

$$\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3} = \frac{4a^2}{9b^2}\frac{5}{2^3} = \frac{5}{18}a^2b^{-2}$$

Example: Suppose \$4000 is invested at 10% annual rate compounded annually. How much money will be in the account in 1 year, 2 years, and in 10years?

in one year: $4000 + 0.10 \times 4000 = 4000 \times (1 + 0.1)$

in two years: $4000 \times (1 + 0.1) + 0.1 \times 4000 \times (1 + 0.1) = 4000 \times (1 + 0.1)^2$

in ten years: $4000 \times (1 + 0.1)^{10} \approx 2.6 \times 4000 = 10400$

Generally: If P is the amount of money invested (principal) at an annual rate r (expressed in decimal form), then the amount A in the account at the end of t years is given by:

$$A = P\,(1 + r)^t$$

Example: How much do you have to invest if you want to have \$ 5000 in 3 years at 5 % compounded annually?

$$A = P\,(1 + r)^t$$

$$5000 = P\,(1 + 0.05)^3$$

$$P = \frac{5000}{1.05^3} = 4320$$

Example: Suppose you deposit \$1000 in a savings and loan that pays 8% compounded semiannually. How much will the savings and loan owe you at the end of 2 years? 10 years?

$$\text{in half a year: } 1000 + \frac{0.08}{2} \times 1000 = 1000 \times (1 + 0.04)$$
$$\text{in a year: } 1000 \times (1 + 0.04) + \frac{0.08}{2} \times 1000(1 + 0.04) = 1000 \times (1 + 0.04)^2$$
$$\text{in two years: } 1000 \times (1 + 0.04)^4$$
$$\text{in ten years: } 1000 \times (1 + 0.04)^{20}$$

Generally: If a principal P is invested at an annual rate r (expressed in decimal form) compounded n times a year, then the amount A in the account at the end of t years is given by:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Exponential function with base e: $f(x) = e^x$, where $e = 2.7182$

Now, let's get back to the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. What happens if n increases to infinity? In other words, what it an annual rate r is compounded continuously?

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{n/r}\right)^{(n/r)rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} = Pe^{rt}$$

For a very large values of m, $m \to \infty$, $(1 + 1/m)^m \approx e$

Continuous compound interest formula: If a principal P is invested at an annual rate r (expressed as a decimal) compounded continuously, then the amount A in the account at the end of t years is given by:

$$A = Pe^{rt}$$

This formula is widely used in business, banking and economics.

Definition of logarithmic function: For $b > 0$ and $b \neq 1$,

logarithmic form		exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

For example,

$y = \log_{10} x$	is equivalent to	$x = 10^y$
$y = \log_e x$	is equivalent to	$x = e^y$

Example: Change each logarithmic form to an equivalent exponential form:

$$\begin{aligned} \log_2 8 = 3 &\quad \text{is equivalent to} \quad 8 = 2^3 \\ \log_{25} 5 = 1/2 &\quad \text{is equivalent to} \quad 5 = 25^{1/2} \\ \log_2 1/4 = -2 &\quad \text{is equivalent to} \quad 1/4 = 2^{-2} \end{aligned}$$

Exercise 1: Change each logarithmic form to an equivalent exponential form:

$$\begin{aligned} \log_3 27 &= 3 \\ \log_{36} 6 &= 1/2 \\ \log_3 1/9 &= -2 \end{aligned}$$

Example: Change each exponential form to an equivalent logarithmic form:

$$\begin{aligned} 49 = 7^2 &\quad \text{is equivalent to} \quad \log_7 49 = 2 \\ 3 = \sqrt{9} &\quad \text{is equivalent to} \quad \log_9 3 = 1/2 \\ 1/5 = 5^{-1} &\quad \text{is equivalent to} \quad \log_5 1/5 = -1 \end{aligned}$$

Exercise 2: Change each exponential form to an equivalent logarithmic form:

$$\begin{aligned} 16 &= 4^2 \\ 3 &= 27^{1/3} \\ 4 &= 16^{1/2} \end{aligned}$$

Properties of logarithmic functions: If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then:

$$\begin{aligned} \log_b 1 = 0 &\qquad \log_b MN = \log_b M + \log_b N \\ \log_b b = 1 &\qquad \log_b \frac{M}{N} = \log_b M - \log_b N \\ \log_b b^x = x &\qquad \log_b M^p = p \log_b M \\ b^{\log_b x} = x, x > 0 &\qquad \log_b M = \log_b N \ \text{ iff } \ M = N \end{aligned}$$