Every statement has its truth value, i.e. every statement is either true or false. Truth value of a compound statement can be derived based on truth values of its parts (simple statements)

Truth table - complete list of the possible truth values of a statement:

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Examples: Look at the 3^{rd} column and 3^{rd} row in the table above. Interpretation: if p is true and q is false, than conjunction is false; e.g.: A day has 24 hours and an hour has 70 minutes. Here, p is "A day has 24 hours" and q is "an hour has 70 minutes". p is true, q is false and the compound statement - conjunction - is false, because conjunction requires both simple statements to be true ("p and q" means p is true and at the same time q is true).

Now, let's look at the 5^{th} column and 4^{th} row in the table above. Interpretation: if p is false and q is true, than implication is true; e.g.: If it doesn't rain, I'll go out with you. Here, p is "it doesn't rain" and q is "I'll go out with you". I make a promise to go out only if it does not rain, I don't say a word about what I'll do if it does rain. So $false \Rightarrow true$ is a true statement as well as $false \Rightarrow false$. The only case when implication is $false$ is $true \Rightarrow false$ - it does not rain, but I will not go out. This is the only case when the original statement was a lie.

Negations:

- Conjunction: $p \wedge q \not\Leftrightarrow NOTp \vee NOTq$, where "$\not\Leftrightarrow$" is a symbol for negation
- Disjunction: $p \vee q \not\Leftrightarrow NOTp \wedge NOTq$
- Implication: $p \Rightarrow q \not\Leftrightarrow p \wedge NOTq$
- Equivalence: $p \Leftrightarrow q \not\Leftrightarrow (p \wedge NOTq) \vee (NOTp \wedge q)$

Examples: Statements and their negations:

• John and Susan are sick $\not\Leftrightarrow$ Either John or Susan is not sick

• If a firm has smart CEO then it makes a profit $\not\Leftrightarrow$ Firm has smart CEO and it does not make a profit

• Profit of Microsoft is either \$1000 or \$5000 $\not\Leftrightarrow$ Profit of Microsoft is neither \$1000 nor \$5000

• I had an ice-cream and a cake. $\not\Leftrightarrow$ I did not have an ice-cream or I did not have a cake.

• I went to cinema or for a dinner. $\not\Leftrightarrow$ I did not go to cinema and I did not go for a dinner.

• The flag is on the castle if and only if (iff) the president is inside. $\not\Leftrightarrow$ The flag is on the castle and president is not inside or the flag is not on the castle and president is inside.

Exercise 2: Find negations of the following statements:

• Tom is tall and has dark hair.

• If it rains (then) I will stay at home.

• I had a coffee or tea.

• If you study hard (then) you will get a good grade.

• I will go to cinema if and only if (iff) you go.

2.3 Answers

Exercise 1:

• **At most three** students have blue eyes. True (most of the time). **At least four/more than three** students have blue eyes.

• Every day has 25 hours. False. At least one day does not have 25 hours.

• **All** cars are red. False. **At least one** car is **not** red.

• The semester begins today. False (unless you read this the first day of semester). The semester does not begin today.

Exercise 2:

• Tom is tall and has dark hair. $\not\Leftrightarrow$ Tom is not tall or Tom does not have dark hair.

• If it rains (then) I will stay at home. $\not\Leftrightarrow$ It rains and I will not stay at home.

• I had a coffee or tea. $\not\Leftrightarrow$ I did not have coffee and I did not have tea.

• If you study hard (then) you will get a good grade. $\not\Leftrightarrow$ You study hard and you will not get a good grade.

• I will go to cinema if and only if (iff) you go. $\not\Leftrightarrow$ I will go to cinema and you won't or I will not go to cinema and you will.

3 Algebraic Expressions

Constants and Variables

A constant is something that does not change, over time or otherwise: a fixed value. Mathematical constant, a number that arises naturally in mathematics, such as

$\pi = 3.1415$ (the ratio of any circle's perimeter to its diameter)

and $e = 2.7182$ (Euler's number - the base of the natural logarithm - $\ln x = \log_e x$; with inverse function being exponential function e^x)

These are examples of irrational numbers, i.e. their value cannot be expressed exactly as a fraction m/n, where m and n are integers. Consequently, its decimal representation never ends or repeats.

A variable is a symbol that stands for a value that may vary; the term usually occurs in opposition to constant, which is a symbol for a non-varying value, i.e. completely fixed or fixed in the context of use. For instance, in the formula $x + 1 = 5$, x is a variable which represents an "unknown" number.

3.1 Algebraic Operations

Operations with Negative Numbers

A negative number is a real number that is less than zero, such as -3.

- **Addition and subtraction**: For purposes of addition and subtraction, one can think of negative numbers as debts.

  Adding a negative number is the same as subtracting the corresponding positive number:

  5 + (-3) = 5 - 3 = 2

  Subtracting a negative is equivalent to adding the corresponding positive:

  5 - (-2) = 5 + 2 = 7

- **Multiplication**: The square of a smaller number can be larger than the square of a larger number. For example, how could the square of -3 be larger than the square of -2, since -3 is smaller than -2.

  Multiplication of a negative number by a positive number yields a negative result. Multiplication of two negative numbers yields a positive result: (-4) × (-3) = 12.

- **Division**: Division is similar to multiplication. A positive number divided by a negative number is negative. If dividend and divisor have the same sign, the result is positive, even if both are negative.

Operations with Fractions

A fraction is a number that can represent part of a whole and is usually written as a pair of numbers, the top number called the **numerator** and the bottom number called the **denominator**.

- **Addition**: The first rule of addition is that only like quantities can be added; for example, various quantities of quarters. Unlike quantities, such as adding thirds to quarters, must first be converted to like quantities as described below: Imagine a pocket containing two quarters, and another pocket containing three quarters; in total, there are five quarters. Since four quarters is equivalent to one (dollar), this can be represented as follows:

  $$\frac{2}{4} + \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}$$

  If $\frac{1}{2}$ of a cake is to be added to $\frac{1}{4}$ of a cake, the pieces need to be converted into comparable quantities, such as cake-eighths or cake-quarters.

- **Adding unlike quantities**: To add fractions containing unlike quantities (e.g. quarters and thirds), it is necessary to convert all amounts to like quantities. It is easy to work out the type of fraction to convert to; simply multiply together the two denominators (bottom number) of each fraction.

  For adding quarters to thirds, both types of fraction are converted to $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$ (twelfths).

  Consider adding the following two quantities:

  $$\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12} = 1\frac{5}{12}$$

  This method always works, but sometimes there is a smaller denominator that can be used (a least common denominator). For example, to add $\frac{3}{4}$ and $\frac{5}{12}$ the denominator 48 can be used (the product of 4 and 12), but the smaller denominator 12 may also be used, being the least common multiple of 4 and 12.

- **Subtraction**: The process for subtracting fractions is, in essence, the same as that of adding them: find a common denominator, and change each fraction to an equivalent fraction with the chosen common denominator. The resulting fraction will have that denominator, and its numerator will be the result of subtracting the numerators of the original fractions.

- **Multiplication**: When multiplying or dividing, it may be possible to choose to cancel down crosswise multiples (often simply called, 'canceling tops and bottom lines') that share a common factor. For example:

  $$2/7 \times 7/8 = 1/1 \times 1/4 = 1/4$$

  A two is a common factor in both the numerator of the left fraction and the denominator of the right so is divided out of both. A seven is a common factor of the left denominator and right numerator.

3.2 Algebraic Expressions

Algebraic expressions are formed using constants, variables and operators
e.g. $\sqrt{x^3+5}$, $x+y-7$, $(2x-y)^2$, ...

Polynomials are special algebraic expressions which include only addition, subtraction, multiplication and raising to a natural number powers
e.g. $4x^3-2x+7$ (polynomial of 3^{rd} degree), $x^3-3x^2y+xy^2+2y^7$ (7^{th} degree), $2x^3y^2-5x-2y^2$ (5^{th} degree), ...

Basic operations on polynomials

Addition: $(3x^3+2x+1)+(7x^2-x+3) = 3x^3+7x^2-x+4$

Subtraction: $(3x^3 + 2x + 1) - (7x^2 - x + 3) = 3x^3 + 7x^2 + 3x - 2$

Multiplication: $(2x-3)(3x^2-2x+3) = 2x(3x^2-2x+3) - 3(3x^2-2x+3) = 6x^3 - 4x^2 + 6x - 9x^2 + 6x - 9 = 6x^3 - 13x^2 + 12x - 9$

Special products: $(a+b)^2 = a^2 + 2ab + b^2$ NOT $a^2 + b^2$!!!

$(a-b)^2 = a^2 - 2ab + b^2$ NOT $a^2 - b^2$!!!

$a^2 - b^2 = (a+b)(a-b)$

Factoring: Factor of an algebraic expression is one of two or more algebraic expressions whose product is the given algebraic expression; e.g. $x^2 - 4 = (x+2)(x-2)$. $(x+2)$ and $(x-2)$ are factors.

3.3 Rational Expressions

Rational expressions are fractional expressions whose numerator and denominator are polynomials.

Problem: Simplify $\frac{x^2-6x+9}{x^2-9}$.

Solution:

$$\frac{x^2-6x+9}{x^2-9} = \frac{(x-3)^2}{(x+3)(x-3)} = \frac{x-3}{x+3} \quad \text{for all} \quad x \neq \pm 3$$

Note: We use condition $x \neq \pm 3$ because for ± 3 there would be zero in denominator and the expression would not be well defined. Details will be explained in the section 3.6 below.

Problem: Reduce $\frac{6x^4(x^2+1)^2 - 3x^2(x^2+1)^3}{x^6}$ to the lowest terms.

Solution:

$$\frac{6x^4(x^2+1)^2 - 3x^2(x^2+1)^3}{x^6} = \frac{(x^2+1)^2[6x^4 - 3x^2(x^2+1)]}{x^6} =$$

$$= \frac{(x^2+1)^2 3x^2[2x^2 - x^2 - 1]}{x^6} = \frac{3(x^2+1)^2(x^2-1)}{x^4} \quad \text{for all} \quad x \neq 0$$

3.4 Polynomial Functions and their Factorization

Polynomial function has the following general form:

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

in which each term contains a coefficient as well as a nonnegative-integer power of the variable x. (We can write $x^1 = x$ and $x^0 = 1$, thus the first two terms may be taken to be $a_0 x^0$ and $a_1 x^1$ respectively.)

Depending on the value of the integer n (which specifies the highest power of x), we have several subclasses of polynomial function:

Case of $n = 0$:	$y = a_0$	*constant* function
Case of $n = 1$:	$y = a_0 + a_1 x$	*linear* function
Case of $n = 2$:	$y = a_0 + a_1 x + a_2 x^2$	*quadratic* function
Case of $n = 3$:	$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$	*cubic* function

Factoring: Factor of an algebraic expression is one of two or more algebraic expressions whose product is the given algebraic expression; e.g. $x^3 - x^2 - 4x + 4 = (x-1)(x+2)(x-2)$. $(x-1)$, $(x+2)$ and $(x-2)$ are factors.

In general, quadratic function $y = a_0 + a_1 x + a_2 x^2$ can be written as $(x - r_1)(x - r_2)$; cubic function $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ can be written as $(x - r_1)(x - r_2)(x - r_3)$ and so on. Numbers r_1, r_2, r_3 are called *roots*. Note that if all roots are integers then all roots must be divisors of a_0 (Why?).

Example: Find all integer roots of the (polynomial) cubic function $x^3 - 2x^2 - 9x + 18$. All the roots have to be solutions of equation $x^3 - 2x^2 - 9x + 18 = 0$.

First of all, note that there will be three roots in this example. All possible roots have to be divisors of 18, so it can be: ± 1, ± 2, ± 3, ± 6, ± 9. We have to check all possible number to check if they are solution of above mentioned equation.

-1: $(-1)^3 - 2(-1)^2 - 9(-1) + 18 = 22 \neq 0$

1: $1^3 - 2 * 1^2 - 91 + 18 = 10 \neq 0$

-2: $(-2)^3 - 2(-2)^2 - 9(-2) + 18 = 20 \neq 0$

2: $2^3 - 2 * 2^2 - 92 + 18 = 0$

-3: $(-3)^3 - 2(-3)^2 - 9(-3) + 18 = 0$

3: $3^3 - 2 * 3^2 - 93 + 18 = 0$

We already found three roots, so all the integer roots are: $2, -3$, and 3.

Factoring quadratic polynomials:

Quadratic polynomial can sometimes be factored into two binomials with simple integer coefficients without the need to use the quadratic formula. In a quadratic equation, this will expose its two roots. The formula

$$ax^2 + bx + c$$

would be factored into:

$$(mx + p)(nx + q)$$

where

$$mn = a$$
$$pq = c$$
$$pn + mq = b$$

and m, n, p, and q are integers.

Example: Consider $x^2 - 5x + 6$. In this case: $a = 1$, $b = -5$ and $c = 6$.

This polynomial would be factored into:

$$(mx + p)(nx + q)$$

where

$$\begin{aligned} &mn = 1 \quad \Rightarrow \quad m = n = 1 \\ &pq = 6 \\ &pn + mq = -5 \quad \Rightarrow \quad p + q = -5 \end{aligned}$$

This only holds for p and q being equal to -2 and -3. Hence, polynomial $x^2 - 5x + 6$ can be written as $(x - 2)(x - 3)$ and therefore has two roots: 2, and 3; and two factors $(x - 2)$ and $(x - 3)$.

Note: The first condition, $mn = 1$, has one more solution: $m = n = -1$. This, however, leads to the same result, only in a slightly different way: $x^2 - 5x + 6 = (-x + 2)(-x + 3)$ which is the same as $(x - 2)(x - 3)$.

Factoring by grouping

Here, the general idea, as the name of the method suggests, is to group in some sense similar terms together. The method is illustrated on the following examples:

Example: $x^3 + x^2 + x + 1 = x^2(x + 1) + (x + 1) = (x + 1)(x^2 + 1)$

Example: $x^3 - x^2 - 4x + 4 = x^2(x - 1) - 4(x - 1) = (x - 1)(x^2 - 4) = (x - 1)(x - 2)(x + 2)$

3.5 Least Common Denominator

Least common denominator: is found as follows: Factor each denominator completely; identify each different prime factor from all the denominators; form a product using each different factor to the highest power that occurs in any one denominator. This product is the LCD.

Example:
$$\begin{aligned} &\frac{x^2}{x^2 + 2x + 1} + \frac{x - 1}{3x + 3} - \frac{1}{6} = \frac{x^2}{(x + 1)^2} + \frac{x - 1}{3(x + 1)} - \frac{1}{6} = \\ &= \frac{6x^2 + 2(x + 1)(x - 1) - (x + 1)^2}{6(x + 1)^2} = \frac{7x^2 - 2x - 3}{6(x + 1)^2} \quad \text{for all} \quad x \neq -1 \end{aligned}$$