14 Review Lecture

1. Solve the following equations and inequalities for x:

$$
\begin{aligned}
(i)\quad & 5-3(x-6)=2(x-6)\\
& 5-3x+18=2x-12\\
& 5x=35\\
& x=7\\
(ii)\quad & \frac{3}{x}-\frac{1}{4}=\frac{1}{3}\\
& \frac{3}{x}=\frac{1}{4}+\frac{1}{3}=\frac{3}{12}+\frac{4}{12}=\frac{7}{12}\\
& 3\cdot 12=7x\\
& x=\frac{36}{7}\\
(iii)\quad & 3x^2-5x-2=0\\
& D=b^2-4ac=(-5)^2-4\cdot 3\cdot(-2)=49\\
& x=\frac{-b\pm\sqrt{D}}{2a}=\frac{5\pm 7}{6}=2,-\frac{1}{3}\\
(iv)\quad & \log_x 32=5\\
& x^5=32\\
& x=\sqrt[5]{32}=2\\
(v)\quad & 3x-2<-x+22\\
& 4x<24\\
& x<6\\
(vi)\quad & |x+2|<7\\
& -7<x+2<7\\
& -9<x<5\\
(vii)\quad & \log_3(x-1)=1\\
& \log_3(x-1)=\log_3 3\\
& x-1=3\\
& x=4
\end{aligned}
$$

2. Solve the following systems of equations for both x and y:

$$
(a)\quad \begin{aligned} 2x+3y&=2\\ x-y&=\frac{1}{6}\end{aligned} \qquad (b)\quad \begin{aligned} x-y&=3\\ x+y&=7\end{aligned}
$$

Solution:

$$
\begin{aligned}
(a)\quad & 2x+3y=2\\
& x-y=\frac{1}{6} \quad\longrightarrow\quad x=y+\frac{1}{6}\\
& 2x+3y=2 \quad\longrightarrow\quad 2\left(y+\frac{1}{6}\right)+3y=2\\
& 2y+\frac{1}{3}+3y=2\\
& 5y=\frac{5}{3}\\
& y=\frac{1}{3} \quad\Rightarrow\quad x=y+\frac{1}{6}=\frac{1}{3}+\frac{1}{6}=\frac{1}{2}\\
(b)\quad & x-y=3\\
& x+y=7\\
\hline
& 2x=10\\
& x=5,\quad y=2
\end{aligned}
$$

3. Find inverse matrix to the following matrices and check that your answer is correct:

(a) $\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

Solution:

(a)

$$
\left(\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \div 4 \sim \left(\begin{array}{cc|cc} 1 & -1/4 & 1/4 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \searrow\swarrow \sim \left(\begin{array}{cc|cc} 1 & -1/4 & 1/4 & 0 \\ 0 & 7/4 & 1/4 & 1 \end{array}\right) \times 4/7 \;\nwarrow \sim
$$

$$
\left(\begin{array}{cc|cc} 1 & -1/4 & 1/4 & 0 \\ 0 & 1 & 1/7 & 4/7 \end{array}\right) \times 1/4 \sim \left(\begin{array}{cc|cc} 1 & 0 & 2/7 & 1/7 \\ 0 & 1 & 1/7 & 4/7 \end{array}\right)
$$

Check:

$$
\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2/7 & 1/7 \\ 1/7 & 4/7 \end{pmatrix} = \begin{pmatrix} (4,-1)\begin{pmatrix} 2/7 \\ 1/7 \end{pmatrix} & (4,-1)\begin{pmatrix} 1/7 \\ 4/7 \end{pmatrix} \\ (-1,2)\begin{pmatrix} 2/7 \\ 1/7 \end{pmatrix} & (-1,2)\begin{pmatrix} 1/7 \\ 4/7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

(b)

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{cc} /\times(-1) & /\times(-2) \\ \swarrow & \downarrow \\ & \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -6 & 1 & -2 & 0 & 1 \end{array}\right) \times(-1) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & -6 & 1 & -2 & 0 & 1 \end{array}\right) \begin{array}{c} \\ \times 6 \\ \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -5 & 4 & -6 & 1 \end{array}\right) \begin{array}{c} \\ \\ \div(-5) \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4/5 & 6/5 & -1/5 \end{array}\right) \begin{array}{c} \nwarrow \\ \nearrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/5 & 1/5 & -1/5 \\ 0 & 0 & 1 & -4/5 & 6/5 & -1/5 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-3) \\ \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & -3/5 & 3/5 \\ 0 & 1 & 0 & 1/5 & 1/5 & -1/5 \\ 0 & 0 & 1 & -4/5 & 6/5 & -1/5 \end{array}\right)$$

Check:

$$\begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2/5 & -3/5 & 3/5 \\ 1/5 & 1/5 & -1/5 \\ -4/5 & 6/5 & -1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Find the following determinants:

(a) $$\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix}$$

(b) $$\begin{vmatrix} 1 & 3 & 3 \\ -2 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix}$$

Solution:

(a) $$\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 3 \times 1 - 0 \times (-2) = 3$$

(b) $$\begin{vmatrix} 1 & 3 & 3 \\ -2 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix} =$$

$$= -2 + 18 - 12 = 4$$

5. Use matrix method **or** inverse matrix **or** Cramer's rule to solve the following systems:

$$(a)\quad \begin{aligned} 2x+3y&=2\\ x-y&=1/6 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} x-y&=3\\ x+y&=7 \end{aligned}$$

Solution:

(a)

Matrix method:

$$\left(\begin{array}{cc|c} 2 & 3 & 2\\ 1 & -1 & 1/6 \end{array}\right) \begin{matrix}\nwarrow\\ \swarrow\end{matrix} \sim \left(\begin{array}{cc|c} 1 & -1 & 1/6\\ 2 & 3 & 2 \end{array}\right) \begin{matrix}\times(-2)\\ \swarrow\end{matrix} \sim \left(\begin{array}{cc|c} 1 & -1 & 1/6\\ 0 & 5 & 10/6 \end{array}\right) \div 5 \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & 1/6\\ 0 & 1 & 1/3 \end{array}\right) \begin{matrix}\nwarrow\\ \nearrow\end{matrix} \sim \left(\begin{array}{cc|c} 1 & 0 & 1/2\\ 0 & 1 & 1/3 \end{array}\right) \Rightarrow \begin{array}{c} x=1/2\\ y=1/3 \end{array}$$

Inverse matrix:

$$A=\begin{pmatrix} 2 & 3\\ 1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0\\ 1 & -1 & 0 & 1 \end{array}\right) \begin{matrix}\nwarrow\\ \swarrow\end{matrix} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1\\ 2 & 3 & 1 & 0 \end{array}\right) \begin{matrix}\times(-2)\\ \swarrow\end{matrix} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1\\ 0 & 5 & 1 & -2 \end{array}\right) \div 5 \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1\\ 0 & 1 & 1/5 & -2/5 \end{array}\right) \begin{matrix}\nwarrow\\ \nearrow\end{matrix} \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & 3/5\\ 0 & 1 & 1/5 & -2/5 \end{array}\right)$$

$$A^{-1}=\begin{pmatrix} 1/5 & 3/5\\ 1/5 & -2/5 \end{pmatrix}$$

Check:

$$AA^{-1}=\begin{pmatrix} 2 & 3\\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1/5 & 3/5\\ 1/5 & -2/5 \end{pmatrix}=$$

$$=\begin{pmatrix} 2\times 1/5+3\times 1/5 & 2\times 3/5+3\times(-2/5)\\ 1\times 1/5+(-1)\times 1/5 & 1\times 3/5+(-1)\times(-2/5) \end{pmatrix}=\begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$

$$AX=B\longrightarrow \begin{pmatrix} 2 & 3\\ 1 & -1 \end{pmatrix}\begin{pmatrix} x\\ y \end{pmatrix}=\begin{pmatrix} 2\\ 1/6 \end{pmatrix}$$

$$X=A^{-1}B\longrightarrow \begin{pmatrix} x\\ y \end{pmatrix}=\begin{pmatrix} \frac{1}{5} & \frac{3}{5}\\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} 2\\ 1/6 \end{pmatrix}=\begin{pmatrix} \frac{1}{5}\times 2+\frac{3}{5}\times\frac{1}{6}\\ \frac{1}{5}\times 2-\frac{2}{5}\times\frac{1}{6} \end{pmatrix}=\begin{pmatrix} 1/2\\ 1/3 \end{pmatrix}\Rightarrow \begin{array}{c} x=1/2\\ y=1/3 \end{array}$$

Cramer's rule:

$$x = \frac{\begin{vmatrix} 2 & 3 \\ 1/6 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}} = \frac{-5/2}{-5} = 1/2$$

$$y = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 1/6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}} = \frac{-5/3}{-5} = 1/3$$

(b)

Matrix method:

$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 1 & 1 & 7 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-1) \end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & -2 & -4 \end{array}\right) \div(-2) \sim \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 2 \end{array}\right) \begin{array}{c} \nwarrow \\ \nearrow \end{array} \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right) \Rightarrow \begin{array}{c} x = 5 \\ y = 2 \end{array}$$

Inverse matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} \times(-1) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right) \div 2 \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right) \begin{array}{c} \nwarrow \\ \nearrow \end{array} \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AX = B \longrightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$X = A^{-1}B \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times 3 + \frac{1}{2} \times 7 \\ -\frac{1}{2} \times 3 + \frac{1}{2} \times 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \Rightarrow \begin{array}{c} x = 5 \\ y = 2 \end{array}$$

Cramer's rule:

$$x = \frac{\begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{10}{2} = 5$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

6. Find AB, where

$$A = \begin{pmatrix} 2 & 0 & -1 \\ -4 & 3 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 3 & -2 \\ 2 & 3 \end{pmatrix}$$

Solution:

$$AB = \begin{pmatrix} 1 & 4 & -2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} -15 & 7 \\ -3 & 6 \end{pmatrix}$$

7. A farmer consumes both apples and honey. Suppose the price of apples in the market is $p_A = 10\$$ per kilogram and price of one liter of honey is $p_H = 25\$$ and farmer's income is 300\$.

(i) Draw farmer's budget set.

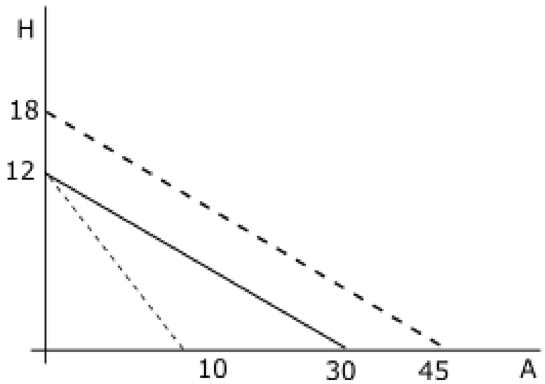
(ii) Suppose the price of apples increases to $p_A = 30\$$. Draw the budget set.

(iii) Now imagine that the price of apples is back to $p_A = 10\$$ per kilogram and farmer's wife gives him additional 150\$. Draw farmer's budget set.

If you spend all the money on apples you can buy $300/10 = 30$ units. Similarly, you can buy at most $300/25 = 12$ units of honey.

When the price of apples goes up to 30 you can afford to buy at most 10 units.

Finally, with original prices and the budget increase to 450 the whole budget line shifts outwards and you can buy at most $450/10 = 45$ units of apples and $450/25 = 18$ units of honey.



8. Solve the following system numerically and graphically:

$x - 2y = 1$

$x + y = 10$

Solution: Numerical solution:

$$x - 2y = 1 \quad \rightarrow \quad x = 1 + 2y$$
$$x + y = 10$$

$$(1 + 2y) + y = 10 \quad \Rightarrow \quad 1 + 3y = 10$$
$$3y = 9 \quad \Rightarrow \quad y = 3$$
$$x = 1 + 2y = 1 + 6 = 7$$

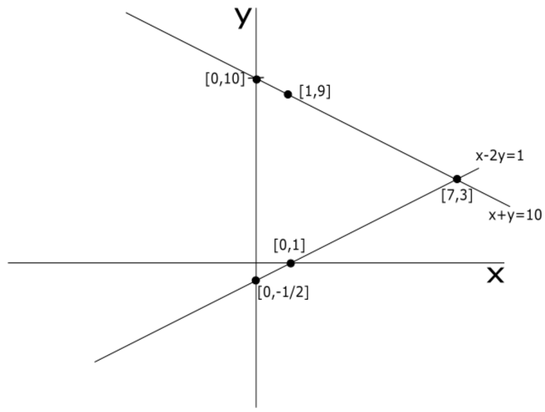
Graphical solution:

$x - 2y = 1 \Rightarrow y = \frac{1}{2}(x - 1)$

x	0	1
y	$-1/2$	0

$x + y = 10 \Rightarrow y = 10 - x$

x	0	1
y	10	9



9. Suppose that the interest rate is 5%. How much money do you have to deposit today if you want to get 75000CZK back in 5 years?

Solution:

$$A = P(1+i)^n$$
$$P = \frac{A}{(1+i)^n}$$
$$P = \frac{75000}{(1+0.05)^5} = 58764$$

10. Consider a 20 year mortgage where the principal amount P is 50,000 $ and the annual interest rate is 6%. What is the monthly payment to be paid?

Solution: The number of monthly payments is

$$n = 20 \text{ years} \times 12 \text{ months} = 240 \text{ months}$$

The monthly interest rate is

$$i = \frac{6\% \text{ per year}}{12 \text{ monhs per year}} = 0.5\% \text{ per month}$$

$$PV(A) = A\frac{1-\frac{1}{(1+i)^n}}{i} \quad \Rightarrow A = PV(A)\frac{i}{1-\frac{1}{(1+i)^n}} = PV(A)\frac{i(1+i)^n}{(1+i)^n-1}$$

$$A = 50000\frac{0.005(1+0.005)^{240}}{(1+0.005)^{240}-1} = \$358.22 \text{ per month.}$$

11. What is the price of the following bond?

Face Value: \$1,000; Maturity: 50 years; Coupon Rate: 10%; Discount Rate: 12%

Solution:

$$\frac{100}{0.12}\left[1-\frac{1}{1.12^{50}}\right]+\frac{1000}{1.12^{50}}=\$833.91$$

12. What is the value of the following semi-annual bond?

Face Value: \$1,000; Maturity: 20 years; Coupon Rate: 9%; Discount Rate: 10%

Solution:

$$\frac{45}{0.05}\left[1-\frac{1}{1.05^{20\cdot 2}}\right]+\frac{1000}{1.05^{20\cdot 2}}=\$914.20$$