11.6 Cramer's Rule

Given the system:

$$\begin{array}{l} a_{11}x + a_{12}y = k_1 \\ a_{21}x + a_{22}y = k_2 \end{array} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

Problem: Solve the following system using: 1. matrix method, 2. inverse matrix, 3. Cramer's rule.

$-2x + y = 6$
$x - y = -5$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} -2 & 1 & 6 \\ 1 & -1 & -5 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ -2 & 1 & 6 \end{array}\right) \begin{array}{c}/2\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & -1 & -4 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & 1 & 4 \end{array}\right) \begin{array}{c}\nwarrow\\ \nearrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right) \Rightarrow \begin{array}{c} x = -1 \\ y = 4 \end{array}$$

2. using inverse matrix: First we find the inverse matrix:

$$\left(\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array}\right) \begin{array}{c}/2\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right) \begin{array}{c}\nwarrow\\ \nearrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

Now, using the inverse matrix we get x and y:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \Rightarrow \begin{array}{c} x = -1 \\ y = 4 \end{array}$$

3. Cramer's rule:

$$x = \frac{\begin{vmatrix} 6 & 1 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-1}{1} = -1$$

$$y = \frac{\begin{vmatrix} -2 & 6 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{4}{1} = 4$$

Problem: Solve the following system using: 1. matrix method, 2. inverse matrix, 3. Cramer's rule.

$3x - 2y = 0$
$x + 2y = 8$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} 3 & -2 & 0 \\ 1 & 2 & 8 \end{array}\right) \begin{array}{c} \nwarrow \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -2 & 0 \end{array}\right) \begin{array}{c} /(-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -8 & -24 \end{array}\right) \; / \div 3 \sim$$

$$\left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right) \begin{array}{c} \nwarrow \\ /(-2) \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right) \quad \Rightarrow \quad \begin{array}{c} x = 2 \\ y = 3 \end{array}$$

2. using inverse matrix: First we find the inverse matrix:

$$\left(\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \begin{array}{c} \nwarrow \\ \swarrow \end{array} \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & -2 & 1 & 0 \end{array}\right) \begin{array}{c} /(-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -8 & 1 & -3 \end{array}\right) \; / \div (-8) \sim$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/8 & 3/8 \end{array}\right) \begin{array}{c} \nwarrow \\ /(-2) \end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/4 & 1/4 \\ 0 & 1 & -1/8 & 3/8 \end{array}\right)$$

Now, using the inverse matrix we get x and y:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} x = 2 \\ y = 3 \end{array}$$

3. Cramer's rule:

$$x = \frac{\begin{vmatrix} 0 & -2 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{16}{8} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 0 \\ 1 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{24}{8} = 3$$

The rules for 3×3 matrices are similar. Given:

$$\begin{aligned} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l \end{aligned}$$

x, y and z can be found as follows:

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} \quad \text{and} \quad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

Problem: Solve the following system using Cramer's rule.
$2x - 2y - z = 1$
$4x + 2y + 3z = 6$
$2x - y - z = 0$

Solution: x=1/2 y = -1 z=2

$$x = \frac{\begin{vmatrix} 1 & -2 & -1 \\ 6 & 2 & 3 \\ 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -2 & -1 \\ 4 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix}} = \frac{-5}{-10} = \frac{1}{2}$$

$$y = \frac{\begin{vmatrix} 2 & 1 & -1 \\ 4 & 6 & 3 \\ 2 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & -2 & -1 \\ 4 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix}} = \frac{10}{-10} = -1$$

$$z = \frac{\begin{vmatrix} 2 & -2 & 1 \\ 4 & 2 & 6 \\ 2 & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -2 & -1 \\ 4 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix}} = \frac{-20}{-10} = 2$$

Alternative way to solve the system is Gauss elimination method (did not work out during the lecture because of some numerical mistake):

$$\left(\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 4 & 2 & 3 & 6 \\ 2 & -1 & -1 & 0 \end{array}\right) \begin{array}{cc} (-2) \searrow & (-1) \searrow \\ \swarrow & \\ & \swarrow \end{array} \sim \left(\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & 6 & 5 & 4 \\ 0 & 1 & 0 & -1 \end{array}\right) \begin{array}{c} \\ (-1/6) \searrow \\ \swarrow \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 2 & -2 & -1 & 1 \\ 0 & 6 & 5 & 4 \\ 0 & 0 & -5/6 & -10/6 \end{array}\right)$$

Now we can find the solution to this system by going from the bottom to the top row of the matrix. The bottom row represents the following equation: $-5/6z = -10/6 \Rightarrow z = 2$.

The second row: $6y + 5z = 4 \Rightarrow 6y + 10 = 4 \Rightarrow y = -1$.

The top row: $2x - 2y - z = 1 \Rightarrow 2x + 2 - 2 = 1 \Rightarrow x = 1/2$.

11.7 Inverse Matrices and Determinants

In the previous section we learned how to find an inverse matrix (if it exists) to any square matrix M. Here we will provide an alternative way to find an inverse matrix by using determinants and cofactors.

Writing the transpose of the matrix of cofactors can also be an efficient way to calculate the inverse of small matrices. To determine the inverse, we calculate a matrix of cofactors:

$$M^{-1} = \frac{1}{|M|} C = \frac{1}{|M|} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

where $|M|$ is the determinant of M and C is the matrix of cofactors. Note, however, that matrix of cofactors is so called transposed what means that the rows and columns are switched. Normally, the first number in the index refers to row and second to column but here it is the other way around. So, for example, in the **second** row and **first** column of the C matrix we have a cofactor to the element in the **first** row and the **second** column of the original matrix M.

Inversion of 2 × 2 matrices

The cofactor equation listed above yields the following result for 2 × 2 matrices. Inversion of these matrices can be done easily as follows:

$$M^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example: Using determinants and cofactors find the inverse matrix to matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Example: Using determinants and cofactors find the inverse matrix to matrix

$$B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \frac{1}{5} \begin{bmatrix} 2 & -6 & 3 \\ 1 & 2 & -1 \\ -1 & 3 & 1 \end{bmatrix}$$

11.8 Answers

Exercise 1: $x = 4$, $y = 1$

12 Arithmetic and Geometric Sequence

Finite sequence: 1, 5, 9, 13, 17

Finite series: 1 + 5 + 9 + 13 +17

Infinite sequence: 1, 2, 4, 8, 16, . . .

Infinite series: 1 + 2 + 4 + 8 + 16 + . . .

When Gauss was a boy, the teacher ran out of stuff to teach and asked them, in the remaining time, to compute the sum of all the numbers from 1 to 40.

Gauss thought that 1+40 is 41. And 2+39 is also 41. And this is true for all the similar pairs, of which there are 20. So... the answer is 820.

One can wonder what would have happened had the teacher asked for the sum of the numbers from 1 to 39. Perhaps Gauss would have noted that 1+39 is 20, as is 2+38. This is true for all the pairs, of which there are 19, and the number 20 is left on its own. Nineteen 40's is 760 and the remaining 20 makes 780.

Example: Let's consider the series 3+5+7+9+11+13+15+17. If we add the first term to the last we get 20. If we add the second term to the second-to-last we get 20 again. Now we see that the series adds up to four 20s, or 80.

Now the question is - will this trick work for all series? If so, why? If not, which series will it work for? Answer: It will work for all **arithmetic series**. The reason that the second pair added up the same as the first pair was that we went up by two on the left, and down by two on the right. As long as you go up by the same as you go down, the sum will stay the same and this is just what happens for arithmetic series.

Arithmetic sequence: is a sequence $a_1, a_2, \dots a_n$ such that $a_n - a_{n-1} = d$ for all n. So the distance between the two following elements of the sequence is constant.

For example: 1,2,3, ... ($d = 1$); 2,4,6, ... 16 ($d = 2$); 0,3,6, ... 18 ($d = 3$)

Arithmetic series: is a sum of elements of arithmetic sequence. The sum is given by:

$$\Sigma_{i=1}^{n} a_i = S_n = \frac{(a_1 + a_n) \cdot n}{2}$$

Example: Find the sum of the following arithmetic series: 1 + 5 + 9 + 13 +17.

a_1=1, a_5=17, d=4, n=5.

$$\Sigma_{i=1}^{n} a_i = \Sigma_{i=1}^{5} a_i = \frac{(a_1 + a_n) \cdot n}{2} = \frac{(1 + 17) \cdot 5}{2} = 45$$

Example: Now consider the following sum: 2+6+18+54+162+486+1458. Clearly the "arithmetic series trick" will not work here: 2+1458 is not 6+486. We need a whole new trick. Here it comes.

$S = 2 + 6 + 18 + 54 + 162 + 486 + 1458$

where S is the sum we are looking for. Now, we multiply the whole equation by 3:

$3S = 6 + 18 + 54 + 162 + 486 + 1458 + 4374$

Now let's subtract the first equation from the second one:

$2S = 4374 - 2$ which means that $S = 2186$.

This trick will work for all **geometric series**.

Geometric sequence: is a sequence $a_1, a_2, \dots a_n$ such that $\frac{a_n}{a_{n-1}} = r$ for all n. So the ratio between the two following elements is constant.

For example: 2,4,8, ... ($r = 2$); 1,3,9,27,51 ($r = 3$)

Geometric series: is a sum of elements of geometric sequence. The sum is given by:

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

Example: Find the following sum: 1 + 2 + 4 + 8 + 16

In this example, $a_1 = 1$, $a_5 = 16$, $n = 5$, $r = 2$.

$$S_n = a_1 \frac{1 - r^n}{1 - r} = 1 \frac{1 - 2^5}{1 - 2} = 31$$

Problem: Find the sum of the numbers: 3, 7, 11, 15, ..., 99.

Solution: In this example, $a_1 = 3$, $a_n = 99$, and $d = 4$. To find n we solve the following equation:

$$3 + (n-1)4 = 99$$

to get $n = 25$. Then the sum of the numbers is:

$$\Sigma_{i=1}^{n} = \frac{(a_1 + a_n) \cdot n}{2} = \frac{(3+99) \cdot 25}{2} = 1275$$

Example: Infinite geometric series. Find the sum of the numbers: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
In this case $r = \frac{1}{2} < 1$. If $r < 1$ then the sum of infinite geometric series exists and it can be found as:

$$\Sigma_{i=1}^{\infty} = \frac{a_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

(If $r > 1$ then the sum is equal to infinity.)

Example: Find a decimal form of the number $0.77\overline{7}$.
Notice that:

$$0.77\overline{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$$

So r in this case is $\frac{1}{10}$ and hence:

$$\Sigma_{i=1}^{\infty} = \frac{a_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$$

This means that $\frac{7}{9} = 0.77\overline{7}$.
Similarly, $0.090909\overline{09} = \frac{09}{99} = \frac{1}{11}$; $0.1438\overline{1438} = \frac{1438}{9999}$

Problem: Find the sum of the first 30 terms of 5 + 9 + 13 + 17 + ...

Solution: We know that $n = 30$ and $a_1 = 5$, and we need the 30th term. Use the definition of an arithmetic sequence.
$a_{30} = 5 + 29 \times 4 = 121$. Therefore, $S_{30} = 30(5 + 121)/2 = 1890$.