## 11.4 Inverse Matrix

**Inverse of a square matrix:** If $M$ is a square matrix of order $n$ and if there exists a matrix $M^{-1}$ such that

$$MM^{-1} = M^{-1}M = I$$

then $M^{-1}$ is called the inverse of $M$.

**How to find an inverse matrix:**

$$M = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

We are looking for

$$M^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

such that $MM^{-1} = M^{-1}M = I$

So we have:

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a+3b & 2c+3d \\ a+2b & c+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These matrices represent the following system:

$$\begin{aligned} 2a+3b &= 1 \qquad & 2c+3d &= 0 \\ a+2b &= 0 \qquad & c+2d &= 1 \end{aligned}$$

Solving these two systems we find that $a = 2, b = -1, c = -3, d = 2$ and therefore

$$M^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

We can check if $M^{-1}$ is really inverse matrix to $M$:

$$MM^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 2+3\times(-1) & 2\times(-3)+3\times 2 \\ 1\times 2+2\times(-1) & 1\times(-3)+2\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This method to find an inverse matrix gets more complicated for larger matrices, but we can use an alternative method:

*Example:* Find the inverse of the following matrix:

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$$

We start as before:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This holds if

$$\begin{aligned} a - b + c = 1 \quad & d - e + f = 0 \quad & g - h + i = 0 \\ 2b - c = 0 \quad & 2e - f = 1 \quad & 2h - i = 0 \\ 2a + 3b = 0 \quad & 2d + 3e = 0 \quad & 2g + 3h = 1 \end{aligned}$$

with corresponding matrices:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 3 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

Since the left side of all matrices is the same we would use the same operations to transform them into identity matrices in order to get the solution. This process can be facilitated by combining all three matrices into the single one:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now, we transform matrix to the left from the vertical line into the identity matrix and the new matrix to the right from the vertical line is the inverse matrix that we are looking for.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} (-2) \\ | \\ \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right) \; /2 \; \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{c} \\ (-5) \\ \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & -2 & -5/2 & 1 \end{array}\right) \begin{array}{cc} \nwarrow & \\ \nearrow & \nwarrow \\ & \nearrow \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1/2 & -2 & -5/2 & 1 \end{array}\right) \begin{array}{cc} & \nwarrow \\ & \\ (-1) \nearrow & \times 2 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right)$$

Hence,

$$M^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

Again, you can check that $MM^{-1} = I$.

**Solving a Matrix Equation**

**Problem:** Given an $n \times n$ matrix $A$ and $n \times 1$ column matrices $B$ and $X$, solve $AX = B$ for $X$.

**Solution:**

$$\begin{aligned} &AX = B \\ &A^{-1}(AX) = A^{-1}B \\ &(A^{-1})AX = A^{-1}B \\ &IX = A^{-1}B \\ &X = A^{-1}B \end{aligned}$$

**Using inverses to solve systems of equations:**

Solve the following system of linear equations:

$$\begin{aligned} x - y + z &= 1 \\ 2y - z &= 1 \\ 2x + 3y &= 1 \end{aligned}$$

*Solution:* We start by rewriting this system into the matrix form:

$$\underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{B}$$

From the previous example we know, that the inverse matrix to matrix $A$ is:

$$A^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

Thus we have:

$$\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -7 \end{pmatrix}$$

Therefore the solution to our system is $x = 5, y = -3$ and $z = -7$.

## 11.5 Determinant

**Determinants:** Determinant is a real number associated with each square matrix. If $A$ is a square matrix, then the determinant of $A$ is denoted by **det A** of by writing the array of elements in $A$ using vertical lines in place of square brackets. For example, if

$$A = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

then the determinant is denoted

$$det \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} = \begin{vmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{vmatrix}$$

**Value of a second-order determinant:**

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

*Examples:*

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 2 = -2$$

$$\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = (-1) \times (-4) - (-3) \times 2 = 10$$

**Value of a third-order determinant:**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

*Note:* you do not need to remember this formula, there are two options how to calculate a third-order determinant:

*Option 1:* Copy the first two lines of the matrix below it:

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

Now the determinant is just a sum of products of elements on main diagonals with positive sign and elements on secondary diagonals with negative signs:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

*Option 2:* Using minors and cofactors: The ***minor*** of an element in a third-order determinant is a second-order determinant obtained by deleting the row and column that contains that element. E.g.:

$$\text{Minor of } \; a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{31} \end{vmatrix}, \; \text{Minor of } \; a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

The **cofactor** of an element $a_{ij}$ is a product of the minor of $a_{ij}$ and $(-1)^{i+j}$. E.g.:

$$\text{Cofactor of } \; a_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{31} \end{vmatrix}, \; \text{Cofactor of } \; a_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Value of a third-order determinant is the sum of three products obtained by multiplying each element of any one row (or any one column) by its cofactor.

*Example:* Find determinant by using cofactors:

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 2(-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} + (-2)(-1)^{1+2} \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} + 0(-1)^{1+2} \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} =$$

$$= 2 \times 5 + 2 \times 1 + 0 = 12$$

**Properties of determinants:**

• **Multiplying a row by a constant:** If each element of any row (or column) of a determinant is multiplied by a constant $k$, the new determinant is $k$ times the original.

*Example:*

$$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 1 = 4$$

$$\begin{vmatrix} 2 \times 2 & 2 \times 1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 4 \times 3 - 2 \times 2 = 8 = 2 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

• **Row of zeros:** If every element in a row (or column) is 0, then the value of the determinant is 0.

*Example:*

$$\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 2 \times 0 - 0 \times 1 = 0$$

• **Interchanging rows:** If two rows (or columns) are interchanged, the new determinant is the negative of the original.

*Example:*

$$\begin{vmatrix} 1 & 0 & 9 \\ -2 & 1 & 5 \\ 3 & 0 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 9 & 0 \\ -2 & 5 & 1 \\ 3 & 7 & 0 \end{vmatrix}$$

• **Equal rows:** If the corresponding elements are equal in two rows (or columns), the value of the determinant is 0.

• **Addition of rows:** If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.