5 Exponents and logarithms

Note that there is a difference between x^2 and 2^x. It makes a big difference whether a variable appears as a base with a constant exponent or as an exponent with a constant base.

Exponential function: $f(x) = b^x, b > 0, b \neq 0$

$f(x)$ defines an exponential function for each different constant b, called the base. The independent variable x may assume any real value.

We require the base to be positive $(b > 0)$ because if x is for example $1/2$, we have $f(x) = b^x = b^{1/2} = \sqrt{b}$ and we only can have a non negative number under the square root.

Exponential function properties:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$a^x = a^y \text{ if and only if } x = y$$

$$\text{for } x \neq 0, a^x = b^x \text{ if and only if } a = b$$

$$0^x = 0, \quad 1^x = 1, \quad x^0 = 1 \text{ for all } x$$

Example: Simplify:

(a) $\left(\frac{4}{3}\right)^2 \frac{3^3}{4}$

$$\left(\frac{4}{3}\right)^2 \frac{3^3}{4} = \frac{4^2}{3^2}\frac{3^3}{4} = 4^{(2-1)}3^{(3-2)} = 4 \times 3 = 12$$

(b) $\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3}$

$$\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3} = \frac{4a^2}{9b^2}\frac{5}{2^3} = \frac{5}{18}a^2b^{-2}$$

Example: Suppose \$4000 is invested at 10% annual rate compounded annually. How much money will be in the account in 1 year, 2 years, and in 10years?

in one year: $4000 + 0.10 \times 4000 = 4000 \times (1 + 0.1)$

in two years: $4000 \times (1 + 0.1) + 0.1 \times 4000 \times (1 + 0.1) = 4000 \times (1 + 0.1)^2$

in ten years: $4000 \times (1 + 0.1)^{10} \approx 2.6 \times 4000 = 10400$

Generally: If P is the amount of money invested (principal) at an annual rate r (expressed in decimal form), then the amount A in the account at the end of t years is given by:

$$A = P\,(1+r)^t$$

Example: How much do you have to invest if you want to have \$ 5000 in 3 years at 5 % compounded annually?

$$\begin{aligned} A &= P\,(1+r)^t \\ 5000 &= P\,(1+0.05)^3 \\ P &= \frac{5000}{1.05^3} = 4320 \end{aligned}$$

Example: Suppose you deposit \$1000 in a savings and loan that pays 8% compounded semiannually. How much will the savings and loan owe you at the end of 2 years? 10 years?

$$\begin{aligned} &\text{in half a year: } 1000 + \frac{0.08}{2} \times 1000 = 1000 \times (1+0.04) \\ &\text{in a year: } 1000 \times (1+0.04) + \frac{0.08}{2} \times 1000(1+0.04) = 1000 \times (1+0.04)^2 \\ &\text{in two years: } 1000 \times (1+0.04)^4 \\ &\text{in ten years: } 1000 \times (1+0.04)^{20} \end{aligned}$$

Generally: If a principal P is invested at an annual rate r (expressed in decimal form) compounded n times a year, then the amount A in the account at the end of t years is given by:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Exponential function with base e: $f(x) = e^x$, where $e = 2.7182$

Now, let's get back to the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$. What happens if n increases to infinity? In other words, what it an annual rate r is compounded continuously?

$$A = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{1}{n/r}\right)^{(n/r)rt} = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} = Pe^{rt}$$

For a very large values of m, $m \to \infty$, $(1+1/m)^m \approx e$

Continuous compound interest formula: If a principal P is invested at an annual rate r (expressed as a decimal) compounded continuously, then the amount A in the account at the end of t years is given by:

$$A = Pe^{rt}$$

This formula is widely used in business, banking and economics.

Definition of logarithmic function: For $b > 0$ and $b \neq 1$,

logarithmic form		exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

For example,

$$y = \log_{10} x \quad \text{is equivalent to} \quad x = 10^y$$
$$y = \log_e x \quad \text{is equivalent to} \quad x = e^y$$

Example: Change each logarithmic form to an equivalent exponential form:

$$\begin{aligned}
\log_2 8 = 3 &\quad \text{is equivalent to} \quad 8 = 2^3 \\
\log_{25} 5 = 1/2 &\quad \text{is equivalent to} \quad 5 = 25^{1/2} \\
\log_2 1/4 = -2 &\quad \text{is equivalent to} \quad 1/4 = 2^{-2} \\
\log_3 27 = 3 &\quad \text{is equivalent to} \quad 27 = 3^3 \\
\log_{36} 6 = 1/2 &\quad \text{is equivalent to} \quad 6 = 36^{1/2} \\
\log_3 1/9 = -2 &\quad \text{is equivalent to} \quad 1/9 = 3^{-2}
\end{aligned}$$

Example: Change each exponential form to an equivalent logarithmic form:

$$\begin{aligned}
49 = 7^2 &\quad \text{is equivalent to} \quad \log_7 49 = 2 \\
3 = \sqrt{9} &\quad \text{is equivalent to} \quad \log_9 3 = 1/2 \\
1/5 = 5^{-1} &\quad \text{is equivalent to} \quad \log_5 1/5 = -1 \\
16 = 4^2 &\quad \text{is equivalent to} \quad \log_4 16 = 2 \\
3 = 27^{1/3} &\quad \text{is equivalent to} \quad \log_{27} 3 = 1/3 \\
4 = 16^{1/2} &\quad \text{is equivalent to} \quad \log_{16} 4 = 1/2
\end{aligned}$$

Properties of logarithmic functions: If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then:

$$\begin{aligned}
\log_b 1 = 0 &\qquad \log_b MN = \log_b M + \log_b N \\
\log_b b = 1 &\qquad \log_b \frac{M}{N} = \log_b M - \log_b N \\
\log_b b^x = x &\qquad \log_b M^p = p \log_b M \\
b^{\log_b x} = x, x > 0 &\qquad \log_b M = \log_b N \ \text{ iff } \ M = N
\end{aligned}$$