4 Inequalities: linear, quadratic, rational

$3(x-5) \geq 5(x+7)$, $-4 \leq 3-2x < 7$, ...

Properties of inequality:

1. if $a < b$ then $a+c < b+c$ — addition
2. if $a < b$ then $a-c < b-c$ — subtraction
3. if $a < b$ then $ca < cb$ for $c > 0$
   $ca > cb$ for $c < 0$ — multiplication
4. if $a < b$ then $a/c < b/c$ for $c > 0$
   $a/c > b/c$ for $c < 0$ — division
5. if $a < b$ and $b < c$ then $a < c$ — transitivity

Problem: Solve $2(2x+3)-10 < 6(x-2)$

Solution: Solving linear inequalities is almost the same as solving linear equalities. Only if you multiply or divide by a negative number, change the sign!!!

$$\begin{aligned} 2(2x+3)-10 &< 6(x-2) \\ 4x+6-10 &< 6x-12 \\ -2x &< -8 \qquad /(-2) \qquad \text{Change the sign of the inequality!} \\ x &> 4 \end{aligned}$$

The inequality holds for all $x \in (4, \infty)$.

Problem: Solve $-6 < 2x+3 \leq 5x-3$

Solution: We divide this problem into two parts and solve simultaneously these two inequalities: $-6 < 2x+3$ and $2x+3 \leq 5x-3$

$$\begin{array}{rl} -6 < 2x+3 & 2x+3 \leq 5x-3 \\ -9 < 2x & -3x \leq -6 \\ -9/2 < x & x \geq 2 \end{array}$$

The inequality holds for all $x \in (-9/2, \infty)$ and at the same time $x \in [2, \infty)$. So the solution is $x \in [2, \infty)$.

Problem: Apple Inc. produces 100% apple juice. Its production function is $J \leq 12A - 4$, where J is quantity of juice in liters and A is quantity of apples in kilograms. For what quantity does Apple Inc. produce at most 20 liters of juice?

Solution:

$$\begin{aligned} J &\leq 20 \\ 12A - 4 &\leq 20 \\ 12A &\leq 24 \\ A &\leq 2 \end{aligned}$$

In order to produce at most 20 liters of juice, Apple Inc. can use at most 2 kilograms of apples.

INEQUALITIES WITH ABSOLUTE VALUE:

Again, solving inequalities with absolute value is almost the same as solving equations with absolute value. Only multiplying or dividing by a negative number changes the sign of inequality.

Problem: Solve $|x - 5| < 1$

Solution: We are looking for all x such that the difference of x from 5 is less than 1. It's clear that this is true for all $x \in (4, 6)$.

$$|x - 5| < 1$$
$$-1 < x - 5 < 1$$
$$4 < x < 6$$

Indeed, the inequality holds for all $x \in (4, 6)$.

Problem: Solve $|3x - 2| < 7$

Solution:

$$|3x - 2| < 7$$
$$-7 < 3x - 2 < 7$$
$$-5 < 3x < 9$$
$$-5/3 < x < 3$$

The inequality holds for all $x \in (-5/3, 3)$.

QUADRATIC INEQUALITIES - $ax^2 + bx + c > 0$

We always start solving quadratic inequality with solving a corresponding quadratic equality. This allows for sketching a graph of out quadratic function (parabola). From the graph we can determine the solution easily.

Problem: Solve $x^2 + 5x + 6 > 0$

Solution:

$$
\begin{aligned}
&x^2 + 5x + 6 = 0 \\
&(x+2)(x+3) = 0 \\
&x = -2, -3
\end{aligned}
$$

Therefore, $x^2 + 5x + 6 > 0$ holds for all $x \in (-\infty, -2) \cup (-3, \infty)$.

Problem: Solve $x^2 - 5x + 4 > 0$

Solution:

$$
\begin{aligned}
&x^2 - 5x + 4 = 0 \\
&(x-1)(x-4) = 0 \\
&x = 1, 4
\end{aligned}
$$

Therefore, $x^2 - 5x + 4 > 0$ holds for all $x \in (1, 4)$.

RATIONAL INEQUALITIES

$$\frac{x+1}{x-3} > 1, \quad \frac{x+1}{x^2-3x+5} < 0, \quad \frac{x^2-x-1}{2x^2+4x-3} > 5, \ \ldots$$

Problem: Solve $\frac{2x}{x+2} > 1$

Solution: Note that we can **not** just multiply the inequality by $(x+2)$ and solve the resulting linear inequality $2x > x + 2$, because we do not know whether $x + 2$ is positive or negative and hence we do not know whether we should change the sign of inequality or not. So we distinguish two cases:

- $x + 2 > 0 \quad \Rightarrow \quad x > -2 \quad \ldots \quad 2x > x + 2 \quad \Rightarrow \quad x > 2$
- $x + 2 < 0 \quad \Rightarrow \quad x < -2 \quad \ldots \quad 2x < x + 2 \quad \Rightarrow \quad x < 2$

Alternative solution:

$$
\begin{aligned}
&\frac{2x}{x+2} > 1 \\
&\frac{2x}{x+2} - 1 > 0 \\
&\frac{x-2}{x+2} > 0
\end{aligned}
$$

Here, we need the ratio of two numbers be positive. This is the case if both numerator and denominator are positive or if both numerator and denominator are negative:

- $x - 2 > 0$ and $x + 2 > 0 \quad \Leftrightarrow \quad x > 2$ and $x > -2 \quad \Rightarrow x > 2$

OR

- $x - 2 < 0$ and $x + 2 < 0 \quad \Leftrightarrow \quad x < 2$ and $x < -2 \quad \Rightarrow x < -2$

Hence the solution to the problem is $x \in (-\infty, -2) \cup (2, \infty)$.

Problem: Solve and $\frac{x^2-3x-10}{1-x} \geq 2$

Solution:

$$\frac{x^2 - 3x - 10}{1 - x} \geq 2$$
$$\frac{x^2 - 3x - 10 - 2 + 2x}{1 - x} \geq 0$$
$$\frac{x^2 - x - 12}{1 - x} \geq 0$$

This fraction is greater or equal to 0 if both numerator and denominator are positive or if both are negative. In this problem, numerator can be equal to 0 as well. Denominator can never be equal to 0!

$$\frac{(x + 3)(x - 4)}{1 - x} \geq 0$$

- $(x + 3)(x - 4) \geq 0$ and $1 - x > 0 \quad \Leftrightarrow \quad x \in (-\infty, -3] \cup [4, \infty)$ and $x < 1 \quad \Rightarrow x \in (-\infty, -3]$

OR

- $(x + 3)(x - 4) \leq 0$ and $1 - x < 0 \quad \Leftrightarrow \quad x \in [-3, 4]$ and $x > 1 \quad \Rightarrow x \in (1, 4]$

Therefore, the solution to this problem is $x \in (-\infty, -3] \cup (1, 4]$.