AAC - Business Mathematics I
Lecture #3, October 20, 2007
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3 Equations: linear, quadratic, rational

Equation: mathematical statement that relates two algebraic expressions involving at least one variable.

- $5x + 3 = 2 - x$
- $x^3 + 3x^2 - 1 = 7 + x - x^2$
- $\frac{3}{x^2-x+1} = x + 2$

Domain: the set of numbers that are permitted to replace the variable (no "0" in the denominator, no negative number under the square root).

- $\frac{3}{x-1} = \frac{x+2}{x}$

Here, the domain is the set of all real numbers except 0 and 1 for which we would have 0 in the denominator.

Properties of equality:

1. if $a = b$ then $a + c = b + c$ — addition
2. if $a = b$ then $a - c = b - c$ — subtraction
3. if $a = b$ then $ca = cb$, $c \neq 0$ — multiplication
4. if $a = b$ then $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$ — division
5. if $a = b$ then they can be used interchangeably — substitution

LINEAR EQUATIONS - $ax + b = 0$

To solve linear equations in one variable we use the properties of equality. Remember, that whatever you do with one side of the equation has to be done with the other side as well.

$7x - 4 = 3$ add 4 to both sides of equation
$7x - 4 + 4 = 3 + 4$
$7x = 7$ dived both sides of equation by 7
$$\frac{7x}{7} = \frac{7}{7}$$
$x = 1$

Example: Solve the following equation and check.

$$6x + 2 = 2x + 14$$
$$6x - 2x + 2 = 14$$
$$4x + 2 = 14$$
$$4x = 14 - 2$$
$$4x = 12$$
$$x = 3$$

Check: we substitute 3 for x in the original equation in order to check that our solution is correct:

$$6x + 2 =^{?} 2x + 14$$
$$6 \times 3 + 2 =^{?} 2 \times 3 + 14$$
$$20 =^{\checkmark} 20$$

So indeed, $x = 3$ is a solution to our equation.

Problem: Find 5 consecutive natural numbers such that their sum is 50.

Solution: Let's denote the first number x. Then the four remaining numbers are $x+1, x+2, x+3$ and $x + 4$. Their sum is supposed to be equal to 50. So we have the following equation:

$$x + (x + 1) + (x + 2) + (x + 3) + (x + 4) = 50$$
$$5x + 10 = 50$$
$$5x = 40$$
$$x = 8$$

Hence the numbers are 8, 9, 10, 11 and 12.

Problem: Find 4 consecutive odd integers such that the sum of the last two is equal to 2 times the sum of the first two numbers.

Solution: Let's denote the first number x. Then the three remaining numbers are $x+2, x+4$ and $x+6$. Sum of the first two numbers is $x+(x+2)$ and the sum of two last numbers is $(x+4)+(x+6)$. Sum of the last two is 2 times the sum of the first two numbers. Therefore, to have an equality we have to multiply the sum of the first two numbers by 2:

$$2[x + (x + 2)] = (x + 4) + (x + 6)$$
$$4x + 4 = 2x + 10$$
$$2x = 6$$
$$x = 3$$

Hence the numbers are 3, 5, 7 and 9.

SYSTEM OF TWO EQUATIONS IN TWO VARIABLES

$$3x + 2y = 12$$
$$4x - y = 5$$

SOLVING BY SUBSTITUTION:

Eliminate one of the variables by replacement when solving a system of equations. Think of it as "grabbing" what one variable equals from one equation and "plugging" it into the other equation.

Solution:

$$3x + 2y = 12$$
$$4x - y = 5 \qquad \Rightarrow y = 4x - 5$$

Now, plug $4x - 5$ for y in the first equation:

$$3x + 2(4x - 5) = 12$$
$$3x + 8x - 10 = 12$$
$$11x = 22$$
$$x = 2$$

Now we get back to $y = 4x - 5$ and therefore $y = 4 \times 2 - 5 = 3$.

Problem: Solve the system of linear equations given below using substitution.

Suppose there is a piggybank that contains 57 coins, which are only quarters and dimes. The total number of coins in the bank is 57, and the total value of these coins is \$9.45. This information can be represented by the following system of equations:

$$D + Q = 57$$
$$00.10D + 0.25Q = 9.45$$

Determine how many of the coins are quarters and how many are dimes.

Solution:

$$D + Q = 57 \qquad \Rightarrow D = 57 - Q$$
$$00.10D + 0.25Q = 9.45$$

Plug $57 - Q$ for D in the second equation

$$\begin{aligned} 00.10(57 - Q) + 0.25Q &= 9.45 \\ 5.7 - 0.1Q + 0.25Q &= 9.45 \\ 0.15Q &= 3.75 \\ Q &= 25 \qquad D = 57 - Q = 57 - 25 = 32 \end{aligned}$$

SOLVING BY ADDITION (ELIMINATION) METHOD:

The addition method says we can just add everything on the left hand side and add everything on the right hand side and keep the equal sign in between.

$$3x + y = 14$$
$$4x - y = 14$$

Solution: Add the two equations; i.e sum left hand sides, sum right hand sides and keep equal sign in between. This way, we eliminate variable y and get only one equation in one variable x:

$$3x + 4x + y - y = 14 + 14$$
$$7x = 28$$
$$x = 4$$

Now we plug 4 for x and use any of two equations to determine y:

$$3x + y = 14$$
$$3 \times 4 + y = 14$$
$$y = 2$$

Check:

$$3x + y = 14 \ldots 3 \times 4 + 2 =^? 14 \ldots 14 =^{\checkmark} 14$$
$$4x - y = 14 \ldots 4 \times 4 - 2 =^? 14 \ldots 14 =^{\checkmark} 14$$

Problem:

$$2x + 2y = 12$$
$$3x - y = 14$$

Solution: First multiply the second equation by 2 so that we can use the addition method.

$$2x + 2y = 12$$
$$6x - 2y = 28$$

Adding the two equations we get:

$$8x = 40$$
$$x = 5$$

Now we plug 5 for x and use any of two equations to determine y:

$$\begin{aligned}
&2x + 2y = 12\\
&2 \times 5 + 2y = 12\\
&2y = 2\\
&y = 1
\end{aligned}$$

Check:

$$\begin{aligned}
&2x + 2y = 12 \ldots 2 \times 5 + 2 \times 1 \stackrel{?}{=} 12 \ldots 12 \stackrel{\checkmark}{=} 12\\
&3x - y = 14 \ldots 3 \times 5 - 1 \stackrel{?}{=} 14 \ldots 14 \stackrel{\checkmark}{=} 14
\end{aligned}$$

Problem: Find the equilibrium price of apple and equilibrium quantity consumed if demand and supply equations are as follows:

$$\begin{aligned}
&p = -q + 20 \quad \text{Demand equation (consumer)}\\
&p = 4q - 55 \quad \text{Supply equation (supplier)}
\end{aligned}$$

Solution:

$$\begin{aligned}
&p = -q + 20\\
&p = 4q - 55 \quad \Rightarrow \quad -q + 20 = 4q - 55 \quad \Rightarrow \quad 5q = 75 \quad \Rightarrow \quad q = 15\\
&p = -q + 20 = -15 + 20 = 5
\end{aligned}$$

QUADRATIC EQUATIONS - $ax^2 + bx + c = 0$

Equations with the second power of a variable; e.g.

$x^2 - 6x + 9 = 0$

$y^2 + 3y - 1 = 2y^2 - 4y - 3$

SOLVING BY SQUARE ROOT:

$$\begin{aligned}
&3x^2 - 27 = 0\\
&3x^2 = 27\\
&x^2 = 9\\
&x = \pm\sqrt{9}\\
&x_{1,2} = \pm 3
\end{aligned}$$

Note: if $a^2 = b$, then $a \neq \sqrt{b}$!!!, but $a = \pm\sqrt{b}$

SOLVING BY FACTORING:

$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$
$$x_1 = -2 \quad x_2 = 3$$

SOLVING BY QUADRATIC FORMULA:

$$ax^2 + bx + c = 0$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \qquad \text{where} \quad D = b^2 - 4ac$$

$$x^2 - x - 6 = 0$$
$$D = b^2 - 4ac = 1 - 4 \times 1 \times (-6) = 25$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 5}{2} = -2, 3$$

EQUATIONS WITH ABSOLUTE VALUE:

- if a is some number, than absolute value of a, $|a|$, is the distance of a from 0.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples: $|4| = 4$, $|-5| = 5$, $|1 - \sqrt{2}| = \sqrt{2} - 1$, ...

- if a and b are some numbers, than absolute value of $a - b$, $|a - b|$, is the distance of a from b, or the distance between a and b. It holds, that $|a - b| = |b - a|$.

Examples: $|9 - 4| = |5| = 5$, $|4 - 9| = |-5| = 5$, $|5| = |5 - 0| = 5$, ...

Problem: Solve $|x - 1| = 2$

Solution: We are looking for such number(s) x that the distance of x from 1 is equal to 2. It's clear that there are 2 such numbers: -1 and 3.

Formally: $|x - a| = b \Rightarrow x - a = b$ or $x - a = -b$, i.e. $x - a = \pm b$. Then it follows that $x = a \pm b$.

Problem: Solve $|x + 4| = 1$

Solution: Note that $|x + 4| = 1$ can be written as $|x - (-4)| = 1$. We are looking for such number(s) x that the distance of x from -4 is equal to 1. It's clear that there are 2 such numbers: -5 and -3.

Note: In the problem $|x+4| = 1$ we are looking for number(s) such that their distance from -4 is equal to 1. Not distance from 4 is equal to 1 !!!

Problem: Solve $|3x-7| = 2$

Solution:

$$3x - 7 = \pm 2$$
$$3x = 7 \pm 2$$
$$x = \frac{7 \pm 2}{3}$$
$$x = 3, \frac{5}{3}$$

Problem: Solve $|2x+5| = 3$

Solution:

$$|2x+5| = 3$$
$$2x + 5 = \pm 3$$
$$2x = -5 \pm 3$$
$$x = \frac{-5 \pm 3}{2}$$
$$x = -4, -1$$

If we have variable x on both sides of the equation, solution is not that easy any more:

Problem: Solve $|x+4| = 3x - 8$

Solution: We distinguish two cases:

- $x + 4 \geq 0 \Leftrightarrow x \geq -4 \ldots x + 4 = 3x - 8 \Rightarrow x = 6$
- $x + 4 < 0 \Leftrightarrow x < -4 \ldots -x - 4 = 3x - 8 \Rightarrow x = 1$

In the first case, the initial condition is $x + 4 \geq 0$ and $x = 6$ satisfies this condition. Hence, $x = 6$ is a solution to our equation.

In the second case, the initial condition is $x + 4 < 0$. But corresponding solution $x = 1$ does not satisfy this condition. Hence, $x = 1$ is **not** a solution to our equation.

Problem: Solve $|x+4| = |2x-6|$

Solution: Absolute value keeps the expression the same if it is positive, in changes the sign of the expression if it is negative. Generally, any equation with any number of absolute values can be solved by getting rid of absolute values. To do so, we need to divide the problem into subcases for which we can eliminate the absolute value:

- $x + 4$ is positive for $x \geq -4$ and negative for $x < -4$
- $2x - 6$ is positive for $x \geq 3$ and negative for $x < 3$

Put the two together:

if $x \in (-\infty, -4)$, both expressions are negative

if $x \in [-4, 3)$, $x + 4$ is positive and $2x - 6$ is negative

if $x \in [3, \infty)$, both expressions are positive

Now we solve following three problems:

$$
\begin{array}{lll}
x \in (-\infty, -4) & -x - 4 = -2x + 6 & \Rightarrow x = 10 \\
x \in [-4, 3) & x + 4 = -2x + 6 & \Rightarrow x = \dfrac{2}{3} \\
x \in [3, \infty) & x + 4 = 2x - 6 & \Rightarrow x = 10
\end{array}
$$

In the firs equality, $x = 10$ does not satisfy the initial condition $x \in (-\infty, -4)$ and therefore this is not a solution. So we have two solutions of our problem $x = 2/3$ and 10.