11 Combinatorial mathematics

Loosely speaking, combinatorics is a branch of mathematics dealing with counting objects satisfying certain criteria. Since this theory relies on factorials and combinatorial symbols we start with their definitions.

Factorial: For n a natural number, n factorial denoted by $n!$ is the product of the first n natural numbers. Zero factorial is defined to be 1.

For n a natural number:

$$n! = n \times (n-1) \times (n-2) \dots 2 \times 1$$

Note that:

$$n! = n \times (n-1)!$$

Example:

$$4! = 4 \times 3! = 4 \times 3 \times 2! = 4 \times 3 \times 2 \times 1! = 4 \times 3 \times 2 \times 1 = 24$$

$$\frac{7!}{6!} = \frac{7 \times 6!}{6!} = 7$$

$$\frac{6!}{3!} \neq 2! \qquad \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 120$$

Note:

$$\binom{n}{0} = \binom{n}{n} = 1 \quad \text{for all } n$$

$$\frac{m!}{n!} \neq \left(\frac{m}{n}\right)!$$

$$m!n! \neq (m \times n)!$$

Binomial formula:

$(a+b)^n$ appears frequently in probabilistic theory and statistics

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$

$\vdots$

$$(a+b)^n = \left(\begin{array}{c} n \\ 0 \end{array}\right) a^n + \left(\begin{array}{c} n \\ 1 \end{array}\right) a^{n-1}b + \left(\begin{array}{c} n \\ 2 \end{array}\right) a^{n-2}b^2 + \cdots + \left(\begin{array}{c} n \\ n \end{array}\right) b^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) a^{n-k}b^k$$

where

$$C_{n,r} =_n C_r = C(n,r) = \left(\begin{array}{c} n \\ r \end{array}\right) = \frac{n!}{r!(n-r)!} \quad \text{is called combinatorial symbol}$$

Example: Use binomial formula to expand $(x+y)^6$

$$\begin{aligned} (a+b)^6 &= \binom{6}{0} a^6 + \binom{6}{1} a^5b + \binom{6}{2} a^4b^2 + \binom{6}{3} a^3b^3 + \binom{6}{4} a^2b^4 + \binom{6}{5} ab^5 + \binom{6}{6} b^6 = \\ &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \end{aligned}$$

Example: Use binomial formula to expand $(3p-2q)^4$

$$(3p-2q)^4 = \binom{4}{0}(3p)^4 + \binom{4}{1}(3p)^3(-2q) + \binom{4}{2}(3p)^2(-2q)^2 + \binom{4}{3} 3p\,(-2q)^3 + \binom{4}{4}(-2q)^4 =$$

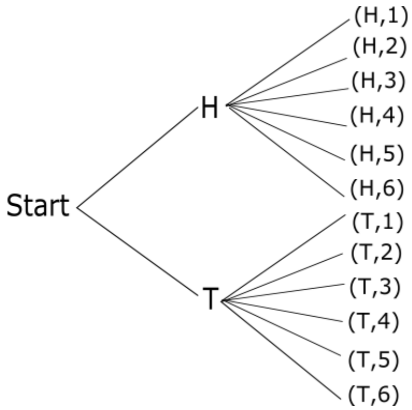
$$= (3p)^4 + 4(3p)^3(-2q) + 6(3p)^2(-2q)^2 + 4(3p)(-2q)^3 + (-2q)^4$$

Example: Use binomial formula to find the fourth and sixteenth term in the expansion of $(x-2)^{20}$

$$4^{th}: \binom{20}{3} x^{17}(-2)^3 \qquad 16^{th}: \binom{20}{15} x^5(-2)^{15}$$

Multiplication principle

Suppose we flip a coin and then throw a single die. What are the possible combined outcomes?



There are 12 possible combined outcomes - two ways in which the coin can come up followed by six ways in which the die can come up.

If the problem is more complicated than this, drawing a diagram is not possible. In such a case, we use the following **multiplication principle:**

1. If two operations O_1 and O_2 are performed in order, with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$N_1 \times N_2$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2 \ldots O_n$ are performed in order, with possible number of outcomes $N_1, N_2 \ldots N_n$ respectively, then there are

$N_1 \times N_2 \times \ldots \times N_n$

possible combined outcomes of the operations performed in the given order.

Example: From the 26 letters in the alphabet, how many ways can 3 letters appear in a row on a license plate if no letter is repeated?

Solution: There are 26 ways the first letter can be chosen. After a first letter is chosen, 25 letters remain; hence there are 25 ways a second letter can be chosen. And after 2 letters are chosen, there are 24 ways a third letter can be chosen. Hence, using the multiplication principle, there are $26 \times 25 \times 24 = 15600$ possible ways 3 letters can be chosen from the alphabet without allowing any letter to repeat.

Example: Many universities and colleges are now using computer-assisted testing procedures. Suppose a screening test is to consist of 5 questions, and computer stores 5 equivalent questions for the firs test question, 8 equivalent questions for the second, 6 for the third, 5 for the forth, and 10 for the fifth. How many different 5-question tests can the computer select? Two test are considered different if they differ in one or more questions.

Solution:

O_1 :	Select the first question	N_1 :	5 ways
O_2 :	Select the second question	N_2 :	8 ways
O_3 :	Select the third question	N_3 :	6 ways
O_4 :	Select the fourth question	N_4 :	5 ways
O_5 :	Select the fifth question	N_5 :	10 ways

Thus the computer can generate $5 \times 8 \times 6 \times 5 \times 10 = 12000$ different tests.

Example: How many 3-letter code words are possible using the first 8 letters of the alphabet if:

(A) No letter can be repeated?

(B) Letters can be repeated?

(C) Adjacent letters cannot be alike?

Solution:

(A) No letter can be repeated.

O_1 :	Select first letter	N_1 :	8 ways	
O_2 :	Select second letter	N_2 :	7 ways	because 1 letter has been used already
O_3 :	Select third letter	N_3 :	6 ways	because 2 letters have been used already

Thus, there are $8 \times 7 \times 6 = 336$ possible code words.

(B) Letters can be repeated.

O_1 :	Select first letter	N_1 :	8 ways	
O_2 :	Select second letter	N_2 :	8 ways	repeats are allowed
O_3 :	Select third letter	N_3 :	8 ways	repeats are allowed

Thus, there are $8 \times 8 \times 8 = 512$ possible code words.

(C) Adjacent letters cannot be alike.

O_1 :	Select first letter	N_1 :	8 ways	
O_2 :	Select second letter	N_2 :	7 ways	cannot be the same as the first
O_3 :	Select third letter	N_3 :	7 ways	cannot be the same as the second, but can be the same as the first

Thus, there are $8 \times 7 \times 7 = 392$ possible code words.

Permutations of n Objects:

Example: Suppose 4 pictures are to be arranged from left to right on one wall of an art gallery. How many arrangements are possible? Using the multiplication principle, there are 4 ways of selecting the first picture, 3 ways of selecting the second picture, 2 ways of selecting the third one and only 1 way to select the fourth. Thus the number of arrangements is

$$4 \times 3 \times 2 \times 1 = 4! = 24$$

The number of **permutations of n objects**, denoted by $P_{n,n}$, is given by:

$$P_{n,n} = n \times (n-1) \times \ldots \times 1 = n!$$

Example: Now suppose that the director of the art gallery decides to use only 2 out of 4 available pictures on the wall, arranged from left to right. How many arrangements of 2 pictures can be formed out of 4? There are 4 ways the first picture can be selected. After selecting the first picture, there are 3 ways the second picture can be selected. Thus the number of arrangements of 2 pictures from 4 pictures, denoted be $P_{4,2}$ is given by

$$P_{4,2} = 4 \times 3 = \frac{4 \times 3 \times 2!}{2!} = \frac{4!}{2!} = \frac{4!}{(4-2)!} = 12$$

The number of **permutations of n objects taken r at a time** $(0 \leq r \leq n)$, denoted by $P_{n,r}$, is given by:

$$P_{n,r} = n \times (n-1) \times \ldots \times (n-3+1) = \frac{n!}{(n-r)!}$$

Note that if $r = n$, then the number of permutations of n objects taken n at a time is:

$$P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Example: From a committee of 8 people, in how many ways can we choose a chair and a vice-chair, assuming one person cannot hold more than one position?

Solution:

$$P_{8,2} = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56$$

Example: From a committee of 10 people, in how many ways can we choose a chair, a vice-chair and secretary, assuming one person cannot hold more than one position?

Solution:

$$P_{10,3} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

Combinations:

Now suppose that an art museum owns 8 paintings by a given artist and another art museum wishes to borrow 3 of these paintings for a special show. How many ways can 3 paintings be selected for shipment out of the 8 available? Here, the order of the items selected doesn't matter. What we are actually interested in is how many subsets of 3 objects can be formed from a set of 8 objects. We call such a subset a combination of 8 objects taken 3 at a time. The total number of combinations is denoted by the symbol

$$C_{8,3} \quad \text{or} \quad \binom{8}{3}$$

To find the number of combinations of 8 objects taken 3 at a time, $C_{8,3}$, we make use of the formula for $P_{n,r}$ and the multiplication principle. We know that the number of permutations of 8 objects taken 3 at a time is given by $P_{8,3}$, and we have a formula for computing this quantity. Now, think of $P_{8,3}$ in terms of two operations:

O_1 : Select a subset of 3 objects
N_1 : $C_{8,3}$ ways
O_2 : Arrange the subset in a given order
N_2 : 3! ways

The combined operation O_1 followed by O_2 produces a permutation of 8 objects taken 3 at a time. Thus,

$$P_{8,3} = C_{8,3} \cdot 3!$$
$$\frac{8!}{(8-3)!} = C_{8,3} \cdot 3!$$
$$C_{8,3} = \frac{8!}{3!(8-3)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

Thus the museum can make 56 different selections of 3 paintings from the 8 available.

The number of **combinations of n objects taken r at a time** $(0 \leq r \leq n)$, denoted by $C_{n,r}$, is given by:

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!}$$

Example: From a committee of 8 people, in how many ways can we choose a subcommittee of 2 people?

Solution: In this example, the ordering does not matter:

$$C_{8,2} = \binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8 \cdot 7 \cdot 6!}{2! \cdot 6!} = \frac{8 \cdot 7}{2} = 28$$

Example: Out of standard 52-card deck (13 cards in each out of 4 suits - hearts, spades, diamonds, clubs), how many 5-card hands will have 3 aces and 2 kings?

Solution:

O_1 : Choose 3 aces out of 4 possible   Order is not important
N_1 : $C_{4,3}$
O_2 : Choose 2 kings out of 4 possible   Order is not important
N_2 : $C_{4,2}$

Using multiplication principle we have that the number of hands is:

$$C_{4,3} \cdot C_{4,2} = 4 \cdot 6 = 24$$

Example: In a horse race, how many different finishes among the first 3 places are possible for a 10-horse race?

Solution: In this example the ordering matters, so we use permutations:

$$P_{10,3} = \frac{10!}{(10-3)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

Example: From a standard 52-cards deck, how many 5-card hands will have all hearts?

Solution: Here, the order does not matter, we are choosing 5-all-heart-card hands out of 13 heart cards:

$$C_{13,5} = \binom{13}{5} = \frac{13!}{5!(13-5)!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{5!8!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 9 = 1287$$

Example: How many ways can 2 people be seated in a row of 5 chairs? 3 people? 4 people? 5 people?

Solution: In this example, the order matters:

$$P_{5,2} = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{3!} = 20$$

$$P_{5,3} = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 60$$

$$P_{5,4} = \frac{5!}{(5-4)!} = \frac{5!}{1} = 5! = 120$$

$$P_{5,5} = \frac{5!}{(5-5)!} = \frac{5!}{1} = 5! = 120$$