AAC - Business Mathematics I

Lecture #1, October 6, 2007

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1 Numbers and Sets

Set: collection of distinct objects which are called elements (numbers, people, letters of alphabet)

Ways of defining sets:

• list each member of the set (e.g. {4,2,15,6}, {red, blue, white}, ...)

• rule (e.g. A = set of even numbers, $B = \{n^2, n \in N, 0 \leqq n \leqq 5\}$, ...)

Membership:

• $4 \in A$, $15 \in \{4, 2, 15, 6\}$, $16 \in B$

• $5 \notin A$, $5 \notin B$, green $\notin$ {red, blue, white}

Cardinality: the number of member of a set

• $|A| = \infty$

• $|B| = 6$

• $|C| = 0$, where C={three sided squares}

Subsets:

• $A \subseteq B$ if every member of A is in B as well

• if $A \subseteq B$ but $A \neq B$, then A is a proper subset of B, $A \subset B$

• $\{1, 2\} \subseteq \{1, 2, 3, 4\}$ and also $\{1, 2\} \subset \{1, 2, 3, 4\}$

• $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$ but it is not true that $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$

• set of men is a proper subset of the set of all people

Venn diagram:

Note: $A \subseteq A$, $\varnothing \subseteq A$ for every set A

Special Sets:

P - primes, N - natural numbers, Z - integers, $Q = \{\frac{a}{b}, a, b \in Z, b \neq 0\}$ - rational, R - real, I - irrational

$P \subset N \subset Z \subset Q \subset R$

BASIC OPERATIONS

• **Union:** $A \cup B$ elements that belong to A or B.

Example: $\{1, 2\} \cup \{\text{blue,red}\} = \{1, 2, \text{blue,red}\}$

Properties:

- $A \cup B = B \cup A$
- $A \subseteq A \cup B$
- $A \cup A = A$
- $A \cup \varnothing = A$

• **Intersection:** $A \cap B$ elements that belong to A and B at the same time.

Example: $\{1, 2\} \cap \{\text{blue,red}\} = \varnothing$

$\{1, 2\} \cap \{1, 2, 4, 7\} = \{2\}$

Properties:

- $A \cap B = B \cap A$
- $A \cap B \subseteq A$
- $A \cap A = A$
- $A \cap \varnothing = \varnothing$

• **Difference and Complement:** $B \setminus A$ or $B - A$: set of elements which belong to B, but not to A

In certain settings all sets under discussion are considered to be subsets of a given universal set U. Then, $U \setminus A$ is called complement of A and is denoted A' or A^C

Example: $\{1, 2, \text{green}\} \setminus \{\text{red,white,green}\} = \{1, 2\}$

$\{1, 2\} \setminus \{1, 2\} = \varnothing$

Integers $\setminus$ Even numbers = Odd numbers

Properties:

- $A \cup A^C = U$
- $A \cap A^C = \varnothing$
- $(A^C)^C = A$
- $A \setminus A = \varnothing$
- $A \setminus B = A \cap B^C$

• **Cartesian product:** $A \times B$ combining every element from A with every element from B; set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

Example: $\{1, 2\} \times \{\text{red, white,blue}\} = \{(1, \text{red}), (1, \text{white}), (1, \text{blue}), (2, \text{red}), (2, \text{white}), (2, \text{blue})\}$

Properties:

- $A \times \varnothing = \varnothing$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$

Some identities:

- $A \setminus B = A \cap B^C$
- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$

Numbers

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complex (C)  — imaginary
             ⟍ real (R)   — irrational (I)
                          ⟍ rational (Q)  — non-integer
                                          ⟍ integer (Z)  — natural (N)
                                                         ⟍ 0
                                                         ⟍ negatives of natural
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Real numbers (R) - represented on real line with origin 0

Intervals - subsets of a real line

closed - e.g. [2,5] - 2 and 5 belong to the interval

open - e.g. (3,9) - 3 and 9 do not belong to the interval

Intersection - $[-4, 1] \cap [0, 2) = [0, 1]$

Union - $[-4, 1] \cup [0, 2) = [-4, 2)$

Logic

A simple statement - one that does not contain any other statement as a part (p, q, can be true or false; if p is true then "NOT p" or " $\sim p$" or "$\rceil p$" is false)

A compound statement - one with two or more simple statements as parts

An operator - joins simple statements into compounds

Compound statements:

statement	symbol	how we read it
Conjunction	$p \wedge q$	both p and q are true
Disjunction	$p \vee q$	either p or q is true or both
Implication	$p \Rightarrow q$	if p is true then q is true
Equivalence	$p \Leftrightarrow q$	p and q are either both true or both false

Every statement has its truth value, i.e. every statement is either true or false. Truth value of a compound statement can be derived based on truth values of its parts (simple statements)

Truth table - complete list of the possible truth values of a statement:

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Examples: Look at the 3^{rd} column and 3^{rd} row in the table above. Interpretation: if p is true and q is false, than conjunction is false; e.g.: A day has 24 hours and an hour has 70 minutes. Here, p is "A day has 24 hours" and q is "an hour has 70 minutes". p is true, q is false and the compound statement - conjunction - is false, because conjunction requires both simple statements to be true ("p and q" means p is true and at the same time q is true).

Now, let's look at the 5^{th} column and 4^{th} row in the table above. Interpretation: if p is false and q is true, than implication is true; e.g.: If it doesn't rain, I'll go out with you. Here, p is "it doesn't rain" and q is "I'll go out with you". I make a promise to go out only if it does not rain, I don't say a word about what I'll do if it does rain. So $false \Rightarrow true$ is a true statement as well as $false \Rightarrow false$. The only case when implication is $false$ is $true \Rightarrow false$ - it does not rain, but I will not go out. This is the only case when the original statement was a lie.

Examples: Statements and their negations:

- Today it is Sunday $\not\Leftrightarrow$ Today it is not Sunday
- All people have black hair $\not\Leftrightarrow$ At least one person does not have black hair
- At least one student is a girl $\not\Leftrightarrow$ None of students is a girl
- John and Susan are sick $\not\Leftrightarrow$ Either John or Susan is not sick
- If a firm has smart CEO then it makes a profit $\not\Leftrightarrow$ Firm has smart CEO and it does not make a profit
- Profit of Microsoft is either \$1000 or \$5000 $\not\Leftrightarrow$ Profit of Microsoft is neither \$1000 nor \$5000

2 Algebraic Expressions and Polynomials

Algebraic expressions are formed using constants, variables and operators

e.g. $\sqrt{x^3+5}$, $x+y-7$, $(2x-y)^2$, ...

Polynomials are special algebraic expressions which include only addition, subtraction, multiplication and raising to a natural number powers

e.g. $4x^3-2x+7$ (polynomial of 3^{rd} degree), $x^3-3x^2y+xy^2+2y^7$ (7^{th} degree), $2x^3y^2-5x-2y^2$ (5^{th} degree), ...

BASIC OPERATIONS

Addition: $(3x^3+2x+1)+(7x^2-x+3) = 3x^3+7x^2-x+4$

Subtraction: $(3x^3+2x+1)-(7x^2-x+3) = 3x^3+7x^2+3x-2$

Multiplication: $(2x-3)(3x^2-2x+3) = 2x(3x^2-2x+3)-3(3x^2-2x+3) = 6x^3-4x^2+6x-9x^2+6x-9 = 6x^3-13x^2+12x-9$

Special products: $(a+b)^2 = a^2+2ab+b^2$ NOT a^2+b^2!!!

$(a-b)^2 = a^2-2ab+b^2$ NOT a^2-b^2!!!

$a^2-b^2 = (a+b)(a-b)$

Factoring: Factor of an algebraic expression is one of two or more algebraic expressions whose product is the given algebraic expression; e.g. $x^2-4=(x+2)(x-2)$. $(x+2)$ and $(x-2)$ are factors.

RATIONAL EXPRESSIONS: BASIC OPERATIONS

Rational expressions are fractional expressions whose numerator and denominator are polynomials

Simplify:

$$\frac{x^2-6x+9}{x^2-9} = \frac{(x-3)^2}{(x+3)(x-3)} = \frac{x-3}{x+3} \quad \text{for all} \quad x \neq \pm 3$$

Reduce to the lowest terms:

$$\frac{6x^4(x^2+1)^2-3x^2(x^2+1)^3}{x^6} = \frac{(x^2+1)^2[6x^4-3x^2(x^2+1)]}{x^6} =$$

$$= \frac{(x^2+1)^2 3x^2[2x^2-x^2-1]}{x^6} = \frac{(x^2+1)^2(x^2-1)}{x^4} \quad \text{for all} \quad x \neq 0$$

Least common denominator: is found as follows: Factor each denominator completely; identify each different prime factor from all the denominators; form a product using each different factor to the highest power that occurs in any one denominator. This product is the LCD.

Example:
$$\frac{x^2}{x^2+2x+1}+\frac{x-1}{3x+3}-\frac{1}{6}=\frac{x^2}{(x+1)^2}+\frac{x-1}{3(x+1)}-\frac{1}{6}=$$
$$=\frac{6x^2+2(x+1)(x-1)-(x+1)^2}{6(x+1)^2}=\frac{7x^2+2x-3}{6(x+1)^2}$$

More problems:

$$x^a x^b = x^{a+b} \qquad x^2x^4 = x^6 \qquad 2^2 2^3 = 4.8 = 32 = 2^5$$
$$(x^a)^b = x^{ab} \qquad (x^2)^3 = x^6 \qquad (2^2)^3 = 4^3 = 64 = 2^6$$
$$x^{-a} = \frac{1}{x^a} \qquad x^{-2} = \frac{1}{x^2} \qquad 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$x^{1/2} = \sqrt{x} \qquad 9^{1/2} = \sqrt{9} = 3$$

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3 Equations: linear, quadratic, rational

Equation: mathematical statement that relates two algebraic expressions involving at least one variable.

- $5x + 3 = 2 - x$
- $x^3 + 3x^2 - 1 = 7 + x - x^2$
- $\frac{3}{x^2 - x + 1} = x + 2$

Domain: the set of numbers that are permitted to replace the variable (no "0" in the denominator, no negative number under the square root).

- $\frac{3}{x-1} = \frac{x+2}{x}$

Here, the domain is the set of all real numbers except 0 and 1 for which we would have 0 in the denominator.

Properties of equality:

1. if $a = b$ then $a + c = b + c$ — addition
2. if $a = b$ then $a - c = b - c$ — subtraction
3. if $a = b$ then $ca = cb$, $c \neq 0$ — multiplication
4. if $a = b$ then $\frac{a}{c} = \frac{b}{c}$, $c \neq 0$ — division
5. if $a = b$ then they can be used interchangeably — substitution

LINEAR EQUATIONS - $ax + b = 0$

To solve linear equations in one variable we use the properties of equality. Remember, that whatever you do with one side of the equation has to be done with the other side as well.

$7x - 4 = 3$	add 4 to both sides of equation
$7x - 4 + 4 = 3 + 4$	
$7x = 7$	dived both sides of equation by 7
$\frac{7x}{7} = \frac{7}{7}$	
$x = 1$	

Example: Solve the following equation and check.

$$6x + 2 = 2x + 14$$
$$6x - 2x + 2 = 14$$
$$4x + 2 = 14$$
$$4x = 14 - 2$$
$$4x = 12$$
$$x = 3$$

Check: we substitute 3 for x in the original equation in order to check that our solution is correct:

$$6x + 2 =^? 2x + 14$$
$$6 \times 3 + 2 =^? 2 \times 3 + 14$$
$$20 =^{\checkmark} 20$$

So indeed, $x = 3$ is a solution to our equation.

Problem: Find 5 consecutive natural numbers such that their sum is 50.

Solution: Let's denote the first number x. Then the four remaining numbers are $x+1, x+2, x+3$ and $x + 4$. Their sum is supposed to be equal to 50. So we have the following equation:

$$x + (x + 1) + (x + 2) + (x + 3) + (x + 4) = 50$$
$$5x + 10 = 50$$
$$5x = 40$$
$$x = 8$$

Hence the numbers are 8, 9, 10, 11 and 12.

Problem: Find 4 consecutive odd integers such that the sum of the last two is equal to 2 times the sum of the first two numbers.

Solution: Let's denote the first number x. Then the three remaining numbers are $x+2, x+4$ and $x+6$. Sum of the first two numbers is $x+(x+2)$ and the sum of two last numbers is $(x+4)+(x+6)$. Sum of the last two is 2 times the sum of the first two numbers. Therefore, to have an equality we have to multiply the sum of the first two numbers by 2:

$$2[x + (x + 2)] = (x + 4) + (x + 6)$$
$$4x + 4 = 2x + 10$$
$$2x = 6$$
$$x = 3$$

Hence the numbers are 3, 5, 7 and 9.

SYSTEM OF TWO EQUATIONS IN TWO VARIABLES

$$3x + 2y = 12$$
$$4x - y = 5$$

SOLVING BY SUBSTITUTION:

Eliminate one of the variables by replacement when solving a system of equations. Think of it as ”grabbing” what one variable equals from one equation and ”plugging” it into the other equation.

Solution:

$$3x + 2y = 12$$
$$4x - y = 5 \qquad \Rightarrow y = 4x - 5$$

Now, plug $4x - 5$ for y in the first equation:

$$3x + 2(4x - 5) = 12$$
$$3x + 8x - 10 = 12$$
$$11x = 22$$
$$x = 2$$

Now we get back to $y = 4x - 5$ and therefore $y = 4 \times 2 - 5 = 3$.

Problem: Solve the system of linear equations given below using substitution.

Suppose there is a piggybank that contains 57 coins, which are only quarters and dimes. The total number of coins in the bank is 57, and the total value of these coins is \$9.45. This information can be represented by the following system of equations:

$$D + Q = 57$$
$$00.10D + 0.25Q = 9.45$$

Determine how many of the coins are quarters and how many are dimes.

Solution:

$$D + Q = 57 \qquad \Rightarrow D = 57 - Q$$
$$00.10D + 0.25Q = 9.45$$

Plug $57 - Q$ for D in the second equation

$$\begin{aligned} 00.10(57 - Q) + 0.25Q &= 9.45 \\ 5.7 - 0.1Q + 0.25Q &= 9.45 \\ 0.15Q &= 3.75 \\ Q &= 25 \qquad D = 57 - Q = 57 - 25 = 32 \end{aligned}$$

SOLVING BY ADDITION (ELIMINATION) METHOD:

The addition method says we can just add everything on the left hand side and add everything on the right hand side and keep the equal sign in between.

$$3x + y = 14$$
$$4x - y = 14$$

Solution: Add the two equations; i.e sum left hand sides, sum right hand sides and keep equal sign in between. This way, we eliminate variable y and get only one equation in one variable x:

$$3x + 4x + y - y = 14 + 14$$
$$7x = 28$$
$$x = 4$$

Now we plug 4 for x and use any of two equations to determine y:

$$3x + y = 14$$
$$3 \times 4 + y = 14$$
$$y = 2$$

Check:

$$3x + y = 14 \ldots 3 \times 4 + 2 =^{?} 14 \ldots 14 =^{\checkmark} 14$$
$$4x - y = 14 \ldots 4 \times 4 - 2 =^{?} 14 \ldots 14 =^{\checkmark} 14$$

Problem:

$$2x + 2y = 12$$
$$3x - y = 14$$

Solution: First multiply the second equation by 2 so that we can use the addition method.

$$2x + 2y = 12$$
$$6x - 2y = 28$$

Adding the two equations we get:

$$8x = 40$$
$$x = 5$$

Now we plug 5 for x and use any of two equations to determine y:

$$
\begin{aligned}
&2x + 2y = 12 \\
&2 \times 5 + 2y = 12 \\
&2y = 2 \\
&y = 1
\end{aligned}
$$

Check:

$$
\begin{aligned}
&2x + 2y = 12 \ldots 2 \times 5 + 2 \times 1 =^{?} 12 \ldots 12 =^{\checkmark} 12 \\
&3x - y = 14 \ldots 3 \times 5 - 1 =^{?} 14 \ldots 14 =^{\checkmark} 14
\end{aligned}
$$

Problem: Find the equilibrium price of apple and equilibrium quantity consumed if demand and supply equations are as follows:

$$
\begin{aligned}
&p = -q + 20 \quad \text{Demand equation (consumer)} \\
&p = 4q - 55 \quad \text{Supply equation (supplier)}
\end{aligned}
$$

Solution:

$$
\begin{aligned}
&p = -q + 20 \\
&p = 4q - 55 \quad \Rightarrow \quad -q + 20 = 4q - 55 \quad \Rightarrow \quad 5q = 75 \quad \Rightarrow \quad q = 15 \\
&p = -q + 20 = -15 + 20 = 5
\end{aligned}
$$

QUADRATIC EQUATIONS - $ax^2 + bx + c = 0$

Equations with the second power of a variable; e.g.

$x^2 - 6x + 9 = 0$

$y^2 + 3y - 1 = 2y^2 - 4y - 3$

SOLVING BY SQUARE ROOT:

$$
\begin{aligned}
&3x^2 - 27 = 0 \\
&3x^2 = 27 \\
&x^2 = 9 \\
&x = \pm\sqrt{9} \\
&x_{1,2} = \pm 3
\end{aligned}
$$

Note: if $a^2 = b$, then $a \neq \sqrt{b}$!!!, but $a = \pm\sqrt{b}$

SOLVING BY FACTORING:

$$x^2 - x - 6 = 0$$
$$(x+2)(x-3) = 0$$
$$x_1 = -2 \quad x_2 = 3$$

SOLVING BY QUADRATIC FORMULA:

$$ax^2 + bx + c = 0$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{where} \quad D = b^2 - 4ac$$

$$x^2 - x - 6 = 0$$
$$D = b^2 - 4ac = 1 - 4 \times 1 \times (-6) = 25$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 5}{2} = -2, 3$$

EQUATIONS WITH ABSOLUTE VALUE:

- if a is some number, than absolute value of a, $|a|$, is the distance of a from 0.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples: $|4| = 4$, $|-5| = 5$, $|1 - \sqrt{2}| = \sqrt{2} - 1$, ...

- if a and b are some numbers, than absolute value of $a - b$, $|a - b|$, is the distance of a from b, or the distance between a and b. It holds, that $|a - b| = |b - a|$.

Examples: $|9 - 4| = |5| = 5$, $|4 - 9| = |-5| = 5$, $|5| = |5 - 0| = 5$, ...

Problem: Solve $|x - 1| = 2$

Solution: We are looking for such number(s) x that the distance of x from 1 is equal to 2. It's clear that there are 2 such numbers: -1 and 3.

Formally: $|x - a| = b \Rightarrow x - a = b$ or $x - a = -b$, i.e. $x - a = \pm b$. Then it follows that $x = a \pm b$.

Problem: Solve $|x + 4| = 1$

Solution: Note that $|x + 4| = 1$ can be written as $|x - (-4)| = 1$. We are looking for such number(s) x that the distance of x from -4 is equal to 1. It's clear that there are 2 such numbers: -5 and -3.

Note: In the problem $|x+4|=1$ we are looking for number(s) such that their distance from -4 is equal to 1. Not distance from 4 is equal to 1 !!!

Problem: Solve $|3x-7|=2$

Solution:

$$\begin{aligned}
&3x-7=\pm 2\\
&3x=7\pm 2\\
&x=\frac{7\pm 2}{3}\\
&x=3,\frac{5}{3}
\end{aligned}$$

Problem: Solve $|2x+5|=3$

Solution:

$$\begin{aligned}
&|2x+5|=3\\
&2x+5=\pm 3\\
&2x=-5\pm 3\\
&x=\frac{-5\pm 3}{2}\\
&x=-4,-1
\end{aligned}$$

If we have variable x on both sides of the equation, solution is not that easy any more:

Problem: Solve $|x+4|=3x-8$

Solution: We distinguish two cases:

- $x+4\geq 0 \Leftrightarrow x\geq -4 \ldots x+4=3x-8 \Rightarrow x=6$
- $x+4<0 \Leftrightarrow x<-4 \ldots -x-4=3x-8 \Rightarrow x=1$

In the first case, the initial condition is $x+4\geq 0$ and $x=6$ satisfies this condition. Hence, $x=6$ is a solution to our equation.

In the second case, the initial condition is $x+4<0$. But corresponding solution $x=1$ does not satisfy this condition. Hence, $x=1$ is **not** a solution to our equation.

Problem: Solve $|x+4|=|2x-6|$

Solution: Absolute value keeps the expression the same if it is positive, in changes the sign of the expression if it is negative. Generally, any equation with any number of absolute values can be solved by getting rid of absolute values. To do so, we need to divide the problem into subcases for which we can eliminate the absolute value:

- $x+4$ is positive for $x\geq -4$ and negative for $x<-4$
- $2x-6$ is positive for $x\geq 3$ and negative for $x<3$

Put the two together:

if $x \in (-\infty, -4)$, both expressions are negative

if $x \in [-4, 3)$, $x + 4$ is positive and $2x - 6$ is negative

if $x \in [3, \infty)$, both expressions are positive

Now we solve following three problems:

$$\begin{array}{lll} x \in (-\infty, -4) & -x - 4 = -2x + 6 & \Rightarrow x = 10 \\ x \in [-4, 3) & x + 4 = -2x + 6 & \Rightarrow x = \dfrac{2}{3} \\ x \in [3, \infty) & x + 4 = 2x - 6 & \Rightarrow x = 10 \end{array}$$

In the firs equality, $x = 10$ does not satisfy the initial condition $x \in (-\infty, -4)$ and therefore this is not a solution. So we have two solutions of our problem $x = 2/3$ and 10.

4 Inequalities: linear, quadratic, rational

$3(x-5) \geq 5(x+7)$, $-4 \leq 3-2x < 7$, ...

Properties of inequality:

1. if $a < b$ then $a + c < b + c$ — addition
2. if $a < b$ then $a - c < b - c$ — subtraction
3. if $a < b$ then $ca < cb$ for $c > 0$
   $ca > cb$ for $c < 0$ — multiplication
4. if $a < b$ then $a/c < b/c$ for $c > 0$
   $a/c > b/c$ for $c < 0$ — division
5. if $a < b$ and $b < c$ then $a < c$ — transitivity

Problem: Solve $2(2x+3) - 10 < 6(x-2)$

Solution: Solving linear inequalities is almost the same as solving linear equalities. Only if you multiply or divide by a negative number, change the sign!!!

$$\begin{aligned} 2(2x+3) - 10 &< 6(x-2) \\ 4x + 6 - 10 &< 6x - 12 \\ -2x &< -8 \qquad /(-2) \qquad \text{Change the sign of the inequality!} \\ x &> 4 \end{aligned}$$

The inequality holds for all $x \in (4, \infty)$.

Problem: Solve $-6 < 2x + 3 \leq 5x - 3$

Solution: We divide this problem into two parts and solve simultaneously these two inequalities: $-6 < 2x + 3$ and $2x + 3 \leq 5x - 3$

$$\begin{array}{rl} -6 < 2x + 3 & 2x + 3 \leq 5x - 3 \\ -9 < 2x & -3x \leq -6 \\ -9/2 < x & x \geq 2 \end{array}$$

The inequality holds for all $x \in (-9/2, \infty)$ and at the same time $x \in [2, \infty)$. So the solution is $x \in [2, \infty)$.

Problem: Apple Inc. produces 100% apple juice. Its production function is $J \leq 12A - 4$, where J is quantity of juice in liters and A is quantity of apples in kilograms. For what quantity does Apple Inc. produce at most 20 liters of juice?

Solution:

$$\begin{aligned} J &\leq 20 \\ 12A - 4 &\leq 20 \\ 12A &\leq 24 \\ A &\leq 2 \end{aligned}$$

In order to produce at most 20 liters of juice, Apple Inc. can use at most 2 kilograms of apples.

INEQUALITIES WITH ABSOLUTE VALUE:

Again, solving inequalities with absolute value is almost the same as solving equations with absolute value. Only multiplying or dividing by a negative number changes the sign of inequality.

Problem: Solve $|x - 5| < 1$

Solution: We are looking for all x such that the difference of x from 5 is less than 1. It's clear that this is true for all $x \in (4, 6)$.

$$|x - 5| < 1$$
$$-1 < x - 5 < 1$$
$$4 < x < 6$$

Indeed, the inequality holds for all $x \in (4, 6)$.

Problem: Solve $|3x - 2| < 7$

Solution:

$$|3x - 2| < 7$$
$$-7 < 3x - 2 < 7$$
$$-5 < 3x < 9$$
$$-5/3 < x < 3$$

The inequality holds for all $x \in (-5/3, 3)$.

QUADRATIC INEQUALITIES - $ax^2 + bx + c > 0$

We always start solving quadratic inequality with solving a corresponding quadratic equality. This allows for sketching a graph of out quadratic function (parabola). From the graph we can determine the solution easily.

Problem: Solve $x^2 + 5x + 6 > 0$

Solution:

$$\begin{aligned}&x^2 + 5x + 6 = 0\\&(x+2)(x+3) = 0\\&x = -2, -3\end{aligned}$$

Therefore, $x^2 + 5x + 6 > 0$ holds for all $x \in (-\infty, -2) \cup (-3, \infty)$.

Problem: Solve $x^2 - 5x + 4 > 0$

Solution:

$$\begin{aligned}&x^2 - 5x + 4 = 0\\&(x-1)(x-4) = 0\\&x = 1, 4\end{aligned}$$

Therefore, $x^2 - 5x + 4 > 0$ holds for all $x \in (1, 4)$.

RATIONAL INEQUALITIES

$$\frac{x+1}{x-3} > 1, \quad \frac{x+1}{x^2-3x+5} < 0, \quad \frac{x^2-x-1}{2x^2+4x-3} > 5, \ \ldots$$

Problem: Solve $\frac{2x}{x+2} > 1$

Solution: Note that we can **not** just multiply the inequality by $(x+2)$ and solve the resulting linear inequality $2x > x+2$, because we do not know whether $x+2$ is positive or negative and hence we do not know whether we should change the sign of inequality or not. So we distinguish two cases:

- $x + 2 > 0 \quad \Rightarrow \quad x > -2 \quad \ldots \quad 2x > x + 2 \quad \Rightarrow \quad x > 2$
- $x + 2 < 0 \quad \Rightarrow \quad x < -2 \quad \ldots \quad 2x < x + 2 \quad \Rightarrow \quad x < 2$

Alternative solution:

$$\begin{aligned}&\frac{2x}{x+2} > 1\\&\frac{2x}{x+2} - 1 > 0\\&\frac{x-2}{x+2} > 0\end{aligned}$$

Here, we need the ratio of two numbers be positive. This is the case if both numerator and denominator are positive or if both numerator and denominator are negative:

• $x - 2 > 0$ and $x + 2 > 0 \quad \Leftrightarrow \quad x > 2$ and $x > -2 \quad \Rightarrow x > 2$

OR

• $x - 2 < 0$ and $x + 2 < 0 \quad \Leftrightarrow \quad x < 2$ and $x < -2 \quad \Rightarrow x < -2$

Hence the solution to the problem is $x \in (-\infty, -2) \cup (2, \infty)$.

Problem: Solve and $\frac{x^2-3x-10}{1-x} \geq 2$

Solution:

$$\frac{x^2 - 3x - 10}{1 - x} \geq 2$$
$$\frac{x^2 - 3x - 10 - 2 + 2x}{1 - x} \geq 0$$
$$\frac{x^2 - x - 12}{1 - x} \geq 0$$

This fraction is greater or equal to 0 if both numerator and denominator are positive or if both are negative. In this problem, numerator can be equal to 0 as well. Denominator can never be equal to 0!

$$\frac{(x + 3)(x - 4)}{1 - x} \geq 0$$

• $(x + 3)(x - 4) \geq 0$ and $1 - x > 0 \quad \Leftrightarrow \quad x \in (-\infty, -3] \cup [4, \infty)$ and $x < 1 \quad \Rightarrow x \in (-\infty, -3]$

OR

• $(x + 3)(x - 4) \leq 0$ and $1 - x < 0 \quad \Leftrightarrow \quad x \in [-3, 4]$ and $x > 1 \quad \Rightarrow x \in (1, 4]$

Therefore, the solution to this problem is $x \in (-\infty, -3] \cup (1, 4]$.

5 Exponents and logarithms

Note that there is a difference between x^2 and 2^x. It makes a big difference whether a variable appears as a base with a constant exponent or as an exponent with a constant base.

Exponential function: $f(x) = b^x, b > 0, b \neq 0$

$f(x)$ defines an exponential function for each different constant b, called the base. The independent variable x may assume any real value.

We require the base to be positive $(b > 0)$ because if x is for example $1/2$, we have $f(x) = b^x = b^{1/2} = \sqrt{b}$ and we only can have a non negative number under the square root.

Exponential function properties:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$

$$a^x = a^y \text{ if and only if } x = y$$

$$\text{for } x \neq 0, a^x = b^x \text{ if and only if } a = b$$

$$0^x = 0, \quad 1^x = 1, \quad x^0 = 1 \text{ for all } x$$

Example: Simplify:

(a) $\left(\frac{4}{3}\right)^2 \frac{3^3}{4}$

$$\left(\frac{4}{3}\right)^2 \frac{3^3}{4} = \frac{4^2}{3^2}\frac{3^3}{4} = 4^{(2-1)}3^{(3-2)} = 4 \times 3 = 12$$

(b) $\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3}$

$$\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3} = \frac{4a^2}{9b^2}\frac{5}{2^3} = \frac{5}{18}a^2b^{-2}$$

Example: Suppose \$4000 is invested at 10% annual rate compounded annually. How much money will be in the account in 1 year, 2 years, and in 10years?

in one year: $4000 + 0.10 \times 4000 = 4000 \times (1 + 0.1)$

in two years: $4000 \times (1 + 0.1) + 0.1 \times 4000 \times (1 + 0.1) = 4000 \times (1 + 0.1)^2$

in ten years: $4000 \times (1 + 0.1)^{10} \approx 2.6 \times 4000 = 10400$

Generally: If P is the amount of money invested (principal) at an annual rate r (expressed in decimal form), then the amount A in the account at the end of t years is given by:

$$A = P(1+r)^t$$

Example: How much do you have to invest if you want to have \$ 5000 in 3 years at 5 % compounded annually?

$$\begin{aligned} A &= P(1+r)^t \\ 5000 &= P(1+0.05)^3 \\ P &= \frac{5000}{1.05^3} = 4320 \end{aligned}$$

Example: Suppose you deposit \$1000 in a savings and loan that pays 8% compounded semiannually. How much will the savings and loan owe you at the end of 2 years? 10 years?

$$\begin{aligned} &\text{in half a year: } 1000 + \frac{0.08}{2} \times 1000 = 1000 \times (1+0.04) \\ &\text{in a year: } 1000 \times (1+0.04) + \frac{0.08}{2} \times 1000(1+0.04) = 1000 \times (1+0.04)^2 \\ &\text{in two years: } 1000 \times (1+0.04)^4 \\ &\text{in ten years: } 1000 \times (1+0.04)^{20} \end{aligned}$$

Generally: If a principal P is invested at an annual rate r (expressed in decimal form) compounded n times a year, then the amount A in the account at the end of t years is given by:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Exponential function with base e: $f(x) = e^x$, where $e = 2.7182$

Now, let's get back to the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$. What happens if n increases to infinity? In other words, what it an annual rate r is compounded continuously?

$$A = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{1}{n/r}\right)^{(n/r)rt} = P\left[\left(1+\frac{1}{m}\right)^m\right]^{rt} = Pe^{rt}$$

For a very large values of m, $m \to \infty$, $(1+1/m)^m \approx e$

Continuous compound interest formula: If a principal P is invested at an annual rate r (expressed as a decimal) compounded continuously, then the amount A in the account at the end of t years is given by:

$$A = Pe^{rt}$$

This formula is widely used in business, banking and economics.

Definition of logarithmic function: For $b > 0$ and $b \neq 1$,

logarithmic form		exponential form
$y = \log_b x$	is equivalent to	$x = b^y$

For example,

$$y = \log_{10} x \quad \text{is equivalent to} \quad x = 10^y$$
$$y = \log_e x \quad \text{is equivalent to} \quad x = e^y$$

Example: Change each logarithmic form to an equivalent exponential form:

$$\begin{aligned}
\log_2 8 = 3 &\quad \text{is equivalent to} \quad 8 = 2^3 \\
\log_{25} 5 = 1/2 &\quad \text{is equivalent to} \quad 5 = 25^{1/2} \\
\log_2 1/4 = -2 &\quad \text{is equivalent to} \quad 1/4 = 2^{-2} \\
\log_3 27 = 3 &\quad \text{is equivalent to} \quad 27 = 3^3 \\
\log_{36} 6 = 1/2 &\quad \text{is equivalent to} \quad 6 = 36^{1/2} \\
\log_3 1/9 = -2 &\quad \text{is equivalent to} \quad 1/9 = 3^{-2}
\end{aligned}$$

Example: Change each exponential form to an equivalent logarithmic form:

$$\begin{aligned}
49 = 7^2 &\quad \text{is equivalent to} \quad \log_7 49 = 2 \\
3 = \sqrt{9} &\quad \text{is equivalent to} \quad \log_9 3 = 1/2 \\
1/5 = 5^{-1} &\quad \text{is equivalent to} \quad \log_5 1/5 = -1 \\
16 = 4^2 &\quad \text{is equivalent to} \quad \log_4 16 = 2 \\
3 = 27^{1/3} &\quad \text{is equivalent to} \quad \log_{27} 3 = 1/3 \\
4 = 16^{1/2} &\quad \text{is equivalent to} \quad \log_{16} 4 = 1/2
\end{aligned}$$

Properties of logarithmic functions: If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then:

$$\begin{aligned}
\log_b 1 = 0 &\qquad \log_b MN = \log_b M + \log_b N \\
\log_b b = 1 &\qquad \log_b \frac{M}{N} = \log_b M - \log_b N \\
\log_b b^x = x &\qquad \log_b M^p = p \log_b M \\
b^{\log_b x} = x, x > 0 &\qquad \log_b M = \log_b N \ \text{ iff } \ M = N
\end{aligned}$$

6 Exponential equations and logarithmic equations

There are two basic methods of solving exponential equations:

First: if it is possible to transform the equation to one with the same base on both sides of the equation, we just compare the exponents.

Second: if it is possible to transform the equation to one with the same base on both sides of the equation, we use logarithm

Example: Solve $4^{x-3} = 16$

$$\begin{aligned}
&(2^2)^{x-3} = 16\\
&2^{2(x-3)} = 2^4\\
&2x - 6 = 4\\
&2x = 10\\
&x = 5
\end{aligned}$$

Example: Solve $27^{x+1} = 9$

$$\begin{aligned}
&(3^3)^{x+1} = 3^2\\
&3^{3(x+1)} = 3^2\\
&3x + 3 = 2\\
&3x = -1\\
&x = -1/3
\end{aligned}$$

Example: Solve $7^{x^2} = 7^{2x+3}$

$$\begin{aligned}
&7^{x^2} = 7^{2x+3}\\
&x^2 = 2x + 3\\
&x^2 - 2x - 3 = 0\\
&D = b^2 - 4ac = 4 - 4 \times 1 \times (-3) = 16\\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 4}{2} = -1, 3
\end{aligned}$$

Example: Solve $4^{5x-x^2} = 4^{-6}$

$$4^{5x-x^2} = 4^{-6}$$

$$-x^2+5x+6=0$$
$$D=b^2-4ac=25-4\times(-1)\times 6=49$$
$$x_{1,2}=\frac{-b\pm\sqrt{D}}{2a}=\frac{-5\pm 7}{-2}=-1,6$$

Example: Find y: $y=\log_3 27$

$$y=\log_3 27 \Leftrightarrow 3^y=27$$
$$y=3$$

Example: Find y: $y=\log_9 27$

$$y=\log_9 27 \Leftrightarrow 9^y=27$$
$$(3^2)^y=3^3$$
$$3^{(2y)}=3^3$$
$$2y=3$$
$$y=3/2$$

Example: Find x: $\log_2 x=-3$

$$\log_2 x=-3 \Leftrightarrow x=2^{(-3)}$$
$$x=\frac{1}{2^3}$$
$$x=1/8$$

Example: Find b: $\log_b 100=2$

$$\log_b 100=2 \Leftrightarrow b^2=100$$
$$b=\sqrt{100}$$
$$b=10$$

Examples:

$$\log_e 1=0 \qquad \log_{10} 10=1$$
$$10^{\log_{10} 7}=7 \qquad \log_e e^{2x+1}=2x+1$$
$$e^{\log_e x^2}=x^2$$

- If you know that $\log_e 3 = 1.1$ and $\log_e 7 = 1.95$, find $\log_e(\frac{7}{3})$ and $\log_e \sqrt[3]{21}$.

$$\log_e \left(\frac{7}{3}\right) = \log_e 7 - \log_e 3 = 1.95 - 1.1 = 0.85$$
$$\log_e \sqrt[3]{21} = \log_e 21^{1/3} = 1/3 \log_e 21 = 1/3 \log_e(3 \times 7) = 1/3[\log_e 3 + \log_e 7] =$$
$$= 1/3[1.1 + 1.95] = 1/3 \times 3.05 \approx 1.02$$

- Find x: $\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$

$$\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$$
$$\log_b x = \log_b 27^{2/3} + \log_b 2^2 - \log_b 3$$
$$\log_b x = \log_b[27^{2/3} \times 2^2/3]$$
$$\log_b x = \log_b[9 \times 4/3]$$
$$x = \frac{9 \times 4}{3} = 12$$

- $2 \log_5 x = \log_5(x^2 - 6x + 2)$

$$2 \log_5 x = \log_5(x^2 - 6x + 2)$$
$$\log_5 x^2 = \log_5(x^2 - 6x + 2)$$
$$x^2 = x^2 - 6x + 2$$
$$6x = 2$$
$$x = 1/3$$

- $\log_e(x + 8) - \log_e x = 3 \log_e 2$

$$\log_e(x + 8) - \log_e x = 3 \log_e 2$$
$$\log_e \frac{x + 8}{x} = \log_e 2^3$$
$$x + 8 = 8x$$
$$7x = 8$$
$$x = 8/7$$

- $(\ln x)^2 = \ln x^2$, where ln is a short notation for $\log_e$

$$(\ln x)^2 = \ln x^2$$
$$(\ln x)^2 = 2 \ln x$$
$$(\ln x)^2 - 2 \ln x = 0$$
$$\ln x(\ln x - 2) = 0$$
$$\ln x = 0 \qquad \text{OR} \quad \ln x - 2 = 0$$
$$x = e^0 = 1 \quad \text{OR} \quad x = e^2$$

• $2^{3x-2} = 5$ This equation can not be transformed to get the equation with the same base on each side, therefore we will use logarithm to solve it.

$$2^{3x-2} = 5 / \log_{10}$$
$$\log_{10} 2^{3x-2} = \log_{10} 5$$
$$(3x-2)\log_{10} 2 = \log_{10} 5$$
$$(3x-2) = \frac{\log_{10} 5}{\log_{10} 2}$$
$$x = \frac{1}{3}\left(2 + \frac{\log_{10} 5}{\log_{10} 2}\right)$$

• You have \$10000 and the annual interest rate is 10%. Imagine that you have to options. You can deposit the money for 4 years and interest rate being compounded annually or you can deposit the money for 2 years and interest rate being compounded semiannually. Decide which option is better (in terms of money).

First option: $A = P(1+r)^t = 10000 \times (1+0.10)^4$

Second option: $A = P(1+r/n)^n t = 10000 \times (1+0.10/2)^4 = 10000 \times (1+0.05)^4$

Hence, the second option gives more money that the first one.

7 Midterm Exam

1. (6b) Find intersection and union of the following sets:

(a) $A = (-1, 3),\ B = [0, 5]$

(b) $A = (-\infty, 1),\ B = [-2, 2],\ C = (-1, \infty)$

(c) $A = (1, 5),\ B = (4, 6),\ C = (5, 7)$

Solution:

(a) $A \cap B = [0, 3)$; $A \cup B = (-1, 5]$

(b) $A \cap B \cap C = (-1, 1)$; $A \cup B \cup C = (-\infty, \infty)$

(c) $A \cap B \cap C = \emptyset$; $A \cup B \cup C = (1, 7)$

2. (16b) Simplify the following algebraic expressions:

$$(a)\quad \frac{1}{x} - \frac{1}{x+1}$$

$$(b)\quad \frac{1}{x^2-1} - \frac{2}{2x^2+2x}$$

$$(c)\quad \frac{9x^6y^4z^3}{3x^4y^2z}xy^2z^3$$

$$(d)\quad \frac{x^2+6x+9}{x+3}$$

Solution:

$$(a)\quad \frac{1}{x} - \frac{1}{x+1} = \frac{x+1}{x(x+1)} - \frac{x}{x(x+1)} = \frac{x+1-x}{x(x+1)} = \frac{1}{x(x+1)}$$

$x \neq -1, 0$

$$(b)\quad \frac{1}{x^2-1} - \frac{2}{2x^2+2x} = \frac{1}{(x+1)(x-1)} - \frac{1}{x(x+1)} = \frac{x-(x-1)}{x(x+1)(x-1)} = \frac{1}{x(x^2-1)}$$

$x \neq -1, 0, 1$

$$(c) \quad \frac{9x^6y^4z^3}{3x^4y^2z}xy^2z^3 = 3x^3y^4z^5$$

$x, y, z \neq 0$

$$(d) \quad \frac{x^2+6x+9}{x+3} = \frac{(x+3)^2}{x+3} = (x+3)$$

$x \neq -3$

3. (10b) Find negations of the following statements:

(a) Sally had either tea or coffee this morning.

(b) At most 10 students came to the class.

(c) If Peter has a car then he doesn't have a bicycle.

(d) At least 3 banks went bankrupt last year.

(e) Monica has a cat and a dog.

Solution:

(a) Sally did not have tea nor coffee this morning.

(b) At least 11 students came to the class.

(c) Peter has a car and he has a bicycle.

(d) At most 2 banks went bankrupt last year.

(e) Monica does not have a cat or she does not have a dog.

4. (10b) There are 30 students in a class both boys and girls. Some of the students play the guitar and some of them do not. There are two more girls playing the guitar than boys playing the guitar. Moreover, there are three times as many girls who don't play the guitar than boys who don't play the guitar. Finally, the teacher tells you that there are exactly 4 boys who do not play the guitar. How many boys are in the class? How many girls play the guitar?

Solution: Let's denote number of boys who play the guitar x and number of boys who do not play the guitar y.

Teacher tells you that there are exactly 4 boys who do not play the guitar $\Rightarrow y = 4$.

There are three times as many girls who don't play the guitar than boys who don't play the guitar $\Rightarrow$ there are $3y = 4 \times 4 = 12$ girls who don't play the guitar.

There are 30 students: $x+y+(x+2)+3y = 30$ and $y = 4$. This implies $x+4+(x+2)+3\times 4 = 30$. Therefore, $2x + 18 = 30$ and finally, $x = 6$.

So there are:

- 6 boys who play the guitar
- 4 boys who don't play the guitar
- 8 girls who play the guitar
- 12 girls who don't play the guitar

5. (24b) Solve the following equations and inequalities:

$$
\begin{aligned}
&(a) \quad x^2 - \frac{3}{4}x + \frac{1}{8} = 0 \\
&(b) \quad (x-2)(x+3) = 0 \\
&(c) \quad 4x^2 - 4x + 1 = 0 \\
&(d) \quad 3x - 1 < x + 2 < 8 \\
&(e) \quad 2x - 2 < 3x + 4 \\
&(f) \quad \frac{2x+2}{x-3} > 0
\end{aligned}
$$

Solution:

$$
\begin{aligned}
&(a) \quad x^2 - \frac{3}{4}x + \frac{1}{8} = 0 \\
& \quad 8x^2 - 6x + 1 = 0 \\
& \quad D = b^2 - 4ac = 36 - 32 = 4 \\
& \quad x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{6 \pm 2}{16} = \frac{1}{2}, \frac{1}{4} \\
&(b) \quad (x-2)(x+3) = 0 \\
& \quad \text{either} \;\; (x-2) = 0 \;\; \text{or} \;\; (x+3) = 0 \quad \Rightarrow x = -3, 2 \\
&(c) \quad 4x^2 - 4x + 1 = 0 \\
& \quad D = b^2 - 4ac = 16 - 16 = 0 \\
& \quad x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{4 \pm 0}{8} = \frac{1}{2} \\
&(d) \quad 3x - 1 < x + 2 < 8 \\
& \quad 3x - 1 < x + 2 \;\; \text{and} \;\; x + 2 < 8 \\
& \quad 2x < 3 \;\; \text{and} \;\; x < 6 \\
& \quad x < 3/2 \;\; \text{and} \;\; x < 6 \\
& \quad x \in (-\infty, 3/2)
\end{aligned}
$$

(e) $2x - 2 < 3x + 4$

$-x < 6$

$x > -6; \quad x \in (-6, \infty)$

(f) $\dfrac{2x+2}{x-3} > 0$

(i) $2x + 2 > 0$ and $x - 3 > 0$

$x > -1$ and $x > 3 \Rightarrow x \in (3, \infty)$

(ii) $2x + 2 < 0$ and $x - 3 < 0$

$x < -1$ and $x < 3 \Rightarrow x \in (-\infty, -1)$

$x \in (-\infty, -1) \cup (3, \infty)$

6. (10b) A groups of workers in a certain company produces 40 to 60 cars per month. If the total production is 240 cars what is the minimum and maximum number of months that the group of workers worked?

Solution: Let's denote the number of months x. Then $240/60 \leq x \leq 240/40$. This means that the group of workers worked at least 4 and at most 6 months.

7. (10b) Solve the following systems of equations for both x and y:

(a) $4x + 3y = 7$
$2x + y = 3$

(b) $x - y = 1$
$x + y = 7$

Solution:

(a) $4x + 3y = 7$

$2x + y = 3 \Rightarrow y = 3 - 2x$

$4x + 3(3 - 2x) = 7 \Rightarrow -2x = -2 \Rightarrow x = 1$

$y = 3 - 2x = 1$

(b) $x - y = 1$

$x + y = 7$ add two equations

$2x = 8 \Rightarrow x = 4$

$y = 7 - x = 3$

8. (20b) Solve the following equations and inequalities with absolute value:

$$
\begin{aligned}
&(a)\quad |x-4| = 2\\
&(b)\quad |2x-2| + |x+3| = 4\\
&(c)\quad |x-3| < 1\\
&(d)\quad |x-2| > 1
\end{aligned}
$$

Solution:

$$
\begin{aligned}
&(a)\quad |x-4| = 2\\
&\qquad x-4 = \pm 2\\
&\qquad x = 2, 6\\
&(b)\quad |2x-2| + |x+3| = 4\\
&\qquad (i)\ \ 2x-2 \geq 0 \quad \Rightarrow \quad x \geq 1\\
&\qquad\qquad x+3 \geq 0 \quad \Rightarrow \quad x \geq -3\\
&\qquad\qquad 2x-2+x+3 = 4 \quad \Rightarrow \quad x = 1 \in [1, \infty) \ldots \ \ \text{solution}\\
&\qquad (ii)\ \ 2x-2 \leq 0 \quad \Rightarrow \quad x \leq 1\\
&\qquad\qquad x+3 \geq 0 \quad \Rightarrow \quad x \geq -3\\
&\qquad\qquad -2x+2+x+3 = 4 \quad \Rightarrow \quad x = 1 \in [-3, 1] \ldots \ \ \text{solution}\\
&\qquad (iii)\ \ 2x-2 \leq 0 \quad \Rightarrow \quad x \leq 1\\
&\qquad\qquad x+3 \leq 0 \quad \Rightarrow \quad x \leq -3\\
&\qquad\qquad -2x+2-x-3 = 4 \quad \Rightarrow \quad x = -\frac{5}{3} \notin (-\infty, -3) \ldots \ \ \text{not solution}\\
&(c)\quad |x-3| < 1\\
&\qquad -1 < x-3 < 1\\
&\qquad 2 < x < 4; \quad x \in (2, 4)\\
&(d)\quad |x-2| > 1\\
&\qquad -1 > x-2 > 1\\
&\qquad 1 > x > 3; \quad x \in (-\infty, 1) \cup (3, \infty)
\end{aligned}
$$

9. (24b) Solve the following exponential and logarithmic equations:

$$
\begin{aligned}
&(a)\quad \log_7(2x+1) = 1\\
&(b)\quad 2^{x-3} = 4^x\\
&(c)\quad 5^{\log_5(x+2)} = 3\\
&(d)\quad 2^{x^2-7x+10} = 2^{2x-10}
\end{aligned}
$$

Solution:

$$
\begin{aligned}
(a)\quad & \log_7(2x+1) = 1 \\
& \log_7(2x+1) = \log_7 7 \\
& 2x+1 = 7 \\
& x = 3 \\
(b)\quad & 2^{x-3} = 4^x \\
& 2^{x-3} = 2^{2x} \\
& x-3 = 2x \\
& x = -3 \\
(c)\quad & 5^{\log_5(x+2)} = 3 \\
& x+2 = 3 \\
& x = 1 \\
(d)\quad & 2^{x^2-7x+10} = 2^{2x-10} \\
& x^2-7x+10 = 2x-10 \\
& x^2-9x+20 = 0 \\
& D = b^2-4ac = 81-80 = 1 \\
& x_{1,2} = \frac{-b\pm\sqrt{D}}{2a} = \frac{9\pm 1}{2} = 4, 5
\end{aligned}
$$

10. (10b) Your grandfather told you that 8 years ago he deposited 10 000 CZK into the bank in your name. First 3 years the annual rate 6% was compounded annually. Then for the remaining 5 years annual rate 5% was compounded quarterly. How much money is in your bank account today?

Solution:

$$
\begin{aligned}
& A = P(1+r/n)^{tn} \\
& A = 10000(1+0.06)^3 \ldots \text{the sum on the bank account after three years} \\
& [10000(1+0.06)^3]\left(1+\frac{0.05}{4}\right)^{5\times 4} = 15270 \ldots \text{the sum of money after eight years}
\end{aligned}
$$

11. (10b) Suppose that you need 600 000 CZK to buy a new car at the end of year 2015. How much money do you have to deposit at the beginning of the next year if an annual rate is 9%?

Solution:

$$
\begin{aligned}
& A = P(1+r)^t \\
& 600000 = P(1+0.09)^8 \\
& P = \frac{600000}{(1+0.09)^8}
\end{aligned}
$$

BONUS QUESTIONS: (Solve only after you solved all you could from questions 1-11)

12. (15b) An oil-drilling rig in the Gulf of Mexico stands so that one-fifth of it is sand, 20 feet of it is in water, and two-thirds of it is in the air. What is the total height of the rig?

Solution:

$$\begin{array}{rcl} 1/5 & \ldots & \text{send} \\ 20\text{feet} & \ldots & \text{water} \\ 2/3 & \ldots & \text{air} \end{array}$$

We know, that 1/5 is in the send and 2/3 is in the air. This means, that 1-1/5-2/3 is in the water. At the same time we know, that 20 feet is in the water:

$$20\text{feet} = 1 - \frac{1}{5} - \frac{2}{3} = \frac{15-3-10}{15} = \frac{2}{15} \text{ of the total height}$$
$$20\text{feet} \times \frac{15}{2} = 1 = \text{ total height}$$
$$\text{total height} = 20 \times \frac{15}{2} = 150 \text{ feet}$$

13. (15b) Suppose you have \$ 12000 to invest. If part is invested at 10% and the rest at 15%, how much should be invested at each rate to yield 12% on the total amount invested?

Solution: Let's denote the part to be invested at rate 10% as x. Than the part invested at rate 15% is $(12000 - x)$.

$$0.1x + 0.15(12000 - x) = 0.12 \times 12000$$
$$-0.05x + 0.15 \times 12000 = 0.12 \times 12000$$
$$0.05x = 0.03 \times 12000$$
$$x = \frac{0.03 \times 12000}{0.05} = \frac{3}{5}12000$$

So we have to invest 3/5 of 12 000 at rate 10% and 2/5 of 12 000 at rate 15% to yield 12% on the total amount invested.

14. (15b) An investor instructs a broker to purchase a certain stock whenever the price per share p of the stock is within 10\$ of 200\$. Express this instruction as an absolute value inequality.

Solution: $|p - 200| < 10$

8 Matrices and operations on matrices

Matrices: In mathematics, a matrix (plural matrices) is a rectangular table of elements (or entries), which may be numbers or, more generally, any abstract quantities that can be added and multiplied. Matrices are mostly used to describe linear equations and solve systems of equations in a more efficient way. Matrices can be added, multiplied, and decomposed in various ways, making them a key concept in linear algebra and matrix theory.

The horizontal lines in a matrix are called rows and the vertical lines are called columns. A matrix with m rows and n columns is called an $m-by-n$ matrix (written $m \times n$) and m and n are called its dimensions. The dimensions of a matrix are always given with the number of rows first, then the number of columns.

Almost always capital letters denote matrices with the corresponding lower-case letters with two indices representing the entries. For example, the entry of a matrix A that lies in the i-th row and the j-th column is written as $a_{i,j}$ and called the i, j entry or (i, j)-th entry of A.

Example:

$$A = \begin{pmatrix} 8 & 9 & 6 \\ 1 & 2 & 7 \\ 9 & 2 & 4 \\ 6 & 0 & 5 \end{pmatrix}$$

is a 4×3 matrix. The element $a_{2,3}$ is 7.

$a_{1,1} = 8, a_{1,2} = 9, a_{1,3} = 6$
$a_{2,1} = 1, a_{2,2} = 2, a_{2,3} = 7$
$a_{3,1} = 9, a_{3,2} = 2, a_{3,3} = 4$
$a_{4,1} = 6, a_{4,2} = 0, a_{4,3} = 5$

More examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 5 & 7 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Square matrix 3×3	Column matrix 4×1	Row matrix 1×4	Zero matrix 2×3	Identity matrix 3×3

Relationship between system of equations and matrix:

$$2x - 3y = 5$$
$$x + 2y = -3$$

In matrix notation:

$$\left(\begin{array}{cc|c} 2 & -3 & 5 \\ 1 & 2 & -3 \end{array}\right)$$

Elementary Row Operations producing Row-Equivalent Matrices:

1. Two rows are interchanged
2. A row is multiplied by a non-zero constant
3. A constant multiple of one row is added to another row.

Example: Solve the following system by using matrix:

$$3x + 4y = 1$$
$$x - 2y = 7$$

Solution: We start by writing corresponding matrix form:

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right)$$

Our objective is to use row operations as described above to transform matrix into the following form (which is called *reduced form*):

$$\left(\begin{array}{cc|c} 1 & 0 & \mathrm{m} \\ 0 & 1 & \mathrm{n} \end{array}\right)$$

where m and n are some real numbers. The solution to our system is then obvious because if we rewrite the matrix form into the system form we get:

$$1x + 0y = m$$
$$0x + 1y = n$$

or equivalently

$$x = m$$
$$y = n$$

which is the solution that we were looking for. So the only problem to be solved is to transform

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right) \text{ into } \left(\begin{array}{cc|c} 1 & 0 & \mathrm{m} \\ 0 & 1 & \mathrm{n} \end{array}\right)$$

Step 1: To get 1 in the upper left corner, we interchange rows 1 and 2:

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right)$$

Step 2: To get 0 in the lower left corner, we multiply row 1 by (-3) and add to row 2:

$$\left(\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right) \begin{array}{c} (-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right)$$

Step 3: To get 1 in the second row, second column, we multiply row 2 by 1/10:

$$\left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right) \begin{array}{c} \\ 1/10 \end{array} \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right)$$

Step 4: To get 0 in the first row, second column, we multiply row 2 by 2 and add the result to row 1:

$$\left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right) \begin{array}{c} \nwarrow \\ 2 \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right)$$

The last matrix is the matrix for:

$x = 3$

$y = -2$

Example: Solve the following system by using matrix method:

$$\begin{aligned} 2x + 3y &= 11 \\ x - 2y &= 2 \end{aligned}$$

Solution:

$$\left(\begin{array}{cc|c} 2 & 3 & 11 \\ 1 & -2 & 2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 2 & 3 & 11 \end{array}\right) \begin{array}{c} (-2) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 7 & 7 \end{array}\right) \begin{array}{c} \\ (1/7) \end{array} \sim \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 1 \end{array}\right) \begin{array}{c} \nwarrow \\ 2 \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right) \Longrightarrow \begin{array}{l} x = 4 \\ y = 1 \end{array}$$

Example: Solve the following system by using matrix method:

$$\begin{aligned} x+y+z&=6\\ 2x+y-z&=1\\ 3x+y+z&=8 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 1&1&1&6\\ 2&1&-1&1\\ 3&1&1&8 \end{array}\right) \begin{array}{cc} (-2) & (-3)\\ \swarrow & \\ & \swarrow \end{array} \sim \left(\begin{array}{ccc|c} 1&1&1&6\\ 0&-1&-3&-11\\ 0&-2&-2&-10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1&1&1&6\\ 0&1&3&11\\ 0&2&2&10 \end{array}\right) \begin{array}{c} \\ (-2)\\ \swarrow \end{array}$$

$$\left(\begin{array}{ccc|c} 1&1&1&6\\ 0&1&3&11\\ 0&0&-4&-12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1&1&1&6\\ 0&1&3&11\\ 0&0&1&3 \end{array}\right) \begin{array}{c} \\ \nwarrow\\ (-3) \end{array} \sim \left(\begin{array}{ccc|c} 1&1&1&6\\ 0&1&0&2\\ 0&0&1&3 \end{array}\right) \begin{array}{c} \nwarrow\\ (-1)\\ (-1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1&0&0&1\\ 0&1&0&2\\ 0&0&1&3 \end{array}\right) \Longrightarrow \begin{array}{c} x=1\\ y=2\\ z=3 \end{array}$$

MATRICES: BASIC OPERATIONS

1. Addition and Subtraction

The sum of two matrices of the same size is a matrix, with elements that are the sums of the corresponding elements of the two given matrices.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \pm \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} (a \pm w) & (b \pm x) \\ (c \pm y) & (d \pm z) \end{pmatrix}$$

Example:

$$\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 8 & 7 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} (2+8) & (1+7) \\ (3+0) & (5+4) \end{pmatrix} = \begin{pmatrix} 10 & 8 \\ 3 & 9 \end{pmatrix}$$

2. Multiplication of a Matrix by a Number

The product of a number k and a matrix M, denoted by kM, is a matrix formed by multiplying each element of M by k.

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

Example:

$$2\begin{pmatrix} 1 & 3 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 12 & 14 \end{pmatrix}$$

3. Matrix Product

Product of a row and a column matrix: is given by:

$$(a_1, a_2, \ldots a_n)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (a_1b_1 + a_2b_2 + \cdots + a_nb_n)$$

Example:

$$(2, -3, 0)\begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix} = (2(-5) + (-3)2 + 0(-2)) = -10 - 6 = -16$$

Example: A factory produces a slalom water ski that requires 4 labor-hours in the fabricating department and 1 labor-hour in the finishing department. Fabricating personnel receive \$10 per hour, and finishing personnel receive \$8 per hour. Total labor cost per ski is given by the product:

$$(4, 1)\begin{pmatrix} 10 \\ 8 \end{pmatrix} = (4 \times 10 + 1 \times 8) = 40 + 8 = \$48 \text{ per ski.}$$

Now the factory also produces a trick water ski that requires 6 labor-hours in the fabricating department and 1.5 labor-hours in the finishing department. Compute the cost for the trick water ski and the total cost.

$$(6, 1.5)\begin{pmatrix} 10 \\ 8 \end{pmatrix} = (6 \times 10 + 1.5 \times 8) = 60 + 12 = \$72 \text{ per trick ski.}$$

The total cost is 48+72 = \$120.

Matrix product: If A is an $m \times p$ matrix and B is a $p \times n$ matrix, then the matric product of A and B, denoted AB, is an $m \times n$ matrix whose element in the ith row and jth column is the real number obtained from the product of the ith wor of A and the jth column of B. If the number of columns in A does not equal the number of rows in B, then the matrix product AB is not defined.

Example:

$$\begin{pmatrix} 2 & 3 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & \begin{pmatrix} -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 9 & 4 \\ -2 & -2 \end{pmatrix}$$

9 Matrices, determinants, inverse matrix, Cramer's Rule

Basic properties of matrices:

Addition properties:

- Associative: $(A + B) + C = A + (B + C)$
- Commutative: $A + B = B + A$
- Additive identity: $A + 0 = 0 + A = A$
- Additive inverse: $A + (-A) = 0$

Multiplication properties:

- Associative: $(AB)C = A(BC)$
- Multiplicative identity: $AI = IA = A$
- Multiplicative inverse: If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$
- Note: $AB \neq BA$ (see the following example)

Example:

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 11 & 5 \\ 24 & -10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & -1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 13 \\ -10 & 0 \end{pmatrix}$$

Equality:

- Addition: If $A = B$, then $A + C = B + C$
- Left Multiplication: If $A = B$, then $CA = CB$
- Right multiplication: If $A = B$, then $AC = BC$

Inverse of a square matrix: If M is a square matrix of order n and if there exists a matrix M^{-1} such that

$$MM^{-1} = M^{-1}M = 1$$

then M^{-1} is called the inverse of M.

How to find an inverse matrix:

$$M = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

We are looking for

$$M^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

such that $MM^{-1} = M^{-1}M = 1$

So we have:

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2a+3b & 2c+3d \\ a+2b & c+2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These matrices represent the following system:

$$\begin{aligned} 2a+3b &= 1 \qquad & 2c+3d &= 0 \\ a+2b &= 0 \qquad & c+2d &= 1 \end{aligned}$$

Solving these two systems we find that $a = 2, b = -1, c = -3, d = 2$ and therefore

$$M^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

We can check if M^{-1} is really inverse matrix to M:

$$MM^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 2+3\times(-1) & 2\times(-3)+3\times 2 \\ 1\times 2+2\times(-1) & 1\times(-3)+2\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This method to find an inverse matrix gets more complicated for larger matrices, but we can use an alternative method:

Example: Find the inverse of the following matrix:

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}$$

We start as before:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This holds if

$$\begin{array}{rrr} a-b+c=1 & d-e+f=0 & g-h+i=0 \\ 2b-c=0 & 2e-f=1 & 2h-i=0 \\ 2a+3b=0 & 2d+3e=0 & 2g+3h=1 \end{array}$$

with corresponding matrices:

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \\ 2 & 3 & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)$$

Since the left side of all matrices is the same we would use the same operations to transform them into identity matrices in order to get the solution. This process can be facilitated by combining all three matrices into the single one:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

Now, we transform matrix to the left from the vertical line into the identity matrix and the new matrix to the right from the vertical line is the inverse matrix that we are looking for.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} (-2) \\ | \\ \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{c} \\ /2 \\ \\ \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 5 & -2 & -2 & 0 & 1 \end{array}\right) \begin{array}{c} \\ (-5) \\ \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & -2 & -5/2 & 1 \end{array}\right) \begin{array}{cc} \nwarrow & \\ \nearrow & \nwarrow \\ & \nearrow \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1/2 & -2 & -5/2 & 1 \end{array}\right) \begin{array}{cc} & \nwarrow \\ & \\ (-1) \nearrow & \times 2 \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 3 & -1 \\ 0 & 1 & 0 & -2 & -2 & 1 \\ 0 & 0 & 1 & -4 & -5 & 2 \end{array}\right)$$

Hence,

$$M^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

Again, you can check that $MM^{-1} = I$.

Solving a Matrix Equation

Problem: Given an $n \times n$ matrix A and $n \times 1$ column matrices B and X, solve $AX = B$ for X.

Solution:

$$\begin{aligned}
&AX = B \\
&A^{-1}(AX) = A^{-1}B \\
&(A^{-1})AX = A^{-1}B \\
&IX = A^{-1}B \\
&X = A^{-1}B
\end{aligned}$$

Using inverses to solve systems of equations:

Solve the following system of linear equations:

$$\begin{aligned}
x - y + z &= 1 \\
2y - z &= 1 \\
2x + 3y &= 1
\end{aligned}$$

Solution: We start by rewriting this system into the matrix form:

$$\underbrace{\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{B}$$

From the previous example we know, that the inverse matrix to matrix A is:

$$A^{-1} = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

Thus we have:

$$\begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ -7 \end{pmatrix}$$

Therefore the solution to our system is $x = 5, y = -3$ and $z = -7$.

Determinants: Determinant is a real number associated with each square matrix. If A is a square matrix, then the determinant of A is denoted by **det A** of by writing the array of elements in A using vertical lines in place of square brackets. For example, if

$$A = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

then the determinant is denoted

$$det \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} = \begin{vmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{vmatrix}$$

Value of a second-order determinant:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Examples:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 2 = -2$$

$$\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = (-1) \times (-4) - (-3) \times 2 = 10$$

Value of a third-order determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

Note: you do not need to remember this formula, there are two options how to calculate a third-order determinant:

Option 1: Copy the first two lines of the matrix below it:

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

Now the determinant is just a sum of products of elements on main diagonals with positive sign and elements on secondary diagonals with negative signs:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

Option 2: Using minors and cofactors: The ***minor*** of an element in a third-order determinant is a second-order determinant obtained by deleting the row and column that contains that element. E.g.:

$$\text{Minor of } \ a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{31} \end{vmatrix}, \ \text{Minor of } \ a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

The **cofactor** of an element a_{ij} is a product of the minor of a_{ij} and $(-1)^{i+j}$. E.g.:

$$\text{Cofactor of } \ a_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{31} \end{vmatrix}, \ \text{Cofactor of } \ a_{32} = (-1)^{3+2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Value of a third-order determinant is the sum of three products obtained by multiplying each element of any one row (or any one column) by its cofactor.

Example: Find determinant by using cofactors:

$$\begin{vmatrix} 2 & -2 & 0 \\ -3 & 1 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 2(-1)^{1+1} \begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} + (-2)(-1)^{1+2} \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} + 0(-1)^{1+2} \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} =$$

$$= 2 \times 5 + 2 \times 1 + 0 = 12$$

Properties of determinants:

• **Multiplying a row or column by a constant:** If each element of any row (or column) of a determinant is multiplied by a constant k, the new determinant is k times the original.

Example:

$$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 2 \times 3 - 2 \times 1 = 4$$

$$\begin{vmatrix} 2 \times 2 & 2 \times 1 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = 4 \times 3 - 2 \times 2 = 8 = 2 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

• **Row or column of zeros:** If every element in a row (or column) is 0, then the value of the determinant is 0.

Example:

$$\begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} = 2 \times 0 - 0 \times 1 = 0$$

• **Interchanging rows or columns:** If two rows (or columns) are interchanged, the new determinant is the negative of the original.

Example:

$$\begin{vmatrix} 1 & 0 & 9 \\ -2 & 1 & 5 \\ 3 & 0 & 7 \end{vmatrix} = - \begin{vmatrix} 1 & 9 & 0 \\ -2 & 5 & 1 \\ 3 & 7 & 0 \end{vmatrix}$$

• **Equal rows or columns:** If the corresponding elements are equal in two rows (or columns), the value of the determinant is 0.

• **Addition of rows or columns:** If a multiple of any row (or column) of a determinant is added to any other row (or column), the value of the determinant is not changed.

Cramer's Rule:

Given the system:

$$\begin{matrix} a_{11}x + a_{12}y = k_1 \\ a_{21}x + a_{22}y = k_2 \end{matrix} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

Example: Solve the following system using: 1. matrix method, 2. inverse matrix, 3. Cramer's rule.

$-2x + y = 6$
$x - y = -5$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} -2 & 1 & 6 \\ 1 & -1 & -5 \end{array}\right) \begin{matrix} \nwarrow \\ \swarrow \end{matrix} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ -2 & 1 & 6 \end{array}\right) \begin{matrix} /2 \\ \swarrow \end{matrix} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & -1 & -4 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & 1 & 4 \end{array}\right) \begin{matrix} \nwarrow \\ \nearrow \end{matrix} \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right) \quad \Rightarrow \quad \begin{matrix} x = -1 \\ y = 4 \end{matrix}$$

2. using inverse matrix: First we find the inverse matrix:

$$\left(\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ -2 & 1 & 1 & 0 \end{array}\right) \begin{array}{c}/2\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right) \begin{array}{c}\nwarrow\\ \nearrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

Now, using the inverse matrix we get x and y:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} x = -1 \\ y = 4 \end{array}$$

3. Cramer's rule:

$$x = \frac{\begin{vmatrix} 6 & 1 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-1}{1} = -1$$

$$y = \frac{\begin{vmatrix} -2 & 6 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{4}{1} = 4$$

Example: Solve the following system using: 1. matrix method, 2. inverse matrix, 3. Cramer's rule.

$3x - 2y = 0$
$x + 2y = 8$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} 3 & -2 & 0 \\ 1 & 2 & 8 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -2 & 0 \end{array}\right) \begin{array}{c}/(-3)\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -8 & -24 \end{array}\right) / \div 3 \sim$$

$$\left(\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & 1 & 3 \end{array}\right) \begin{array}{c}\nwarrow\\ /(-2)\end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \end{array}\right) \quad \Rightarrow \quad \begin{array}{c} x = 2 \\ y = 3 \end{array}$$

2. using inverse matrix: First we find the inverse matrix:

$$\left(\begin{array}{cc|cc} 3 & -2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & -2 & 1 & 0 \end{array}\right) \begin{array}{c}/(-3)\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -8 & 1 & -3 \end{array}\right) / \div (-8) \sim$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1/8 & 3/8 \end{array}\right) \begin{array}{c}\nwarrow\\ /(-2)\end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/4 & 1/4 \\ 0 & 1 & -1/8 & 3/8 \end{array}\right)$$

Now, using the inverse matrix we get x and y:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/4 & 1/4 \\ -1/8 & 3/8 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \begin{array}{c} x = 2 \\ y = 3 \end{array}$$

3. Cramer's rule:

$$x = \frac{\begin{vmatrix} 0 & -2 \\ 8 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{16}{8} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 0 \\ 1 & 8 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{24}{8} = 3$$

10 Cartesian coordinate system, point, line

Cartesian coordinate system is formed by two real lines, one horizontal and one vertical, which cross through their origins. These two lines are called the horizontal axis and vertical axis.

Point: Every point is represented by two numbers - coordinates. The first number represents the value on axis x and the second number represents the value on axis y.

Linear function - Straight line:

Generally, linear function has the following form: $y = ax + b$. This can be graphically represented by a straight line. Any straight line can be represented by two points. If we find two points lying on the line, we can draw the whole line. Coefficient a is called *slope*. The bigger (smaller) a the steeper (flatter) the line.

Example: $y = 3x + 1$.

To find two points lying on this line we use 0 and 1 for x and find corresponding values of y from the equation:

x	0	1
y	$3 \times 0 + 1 = 1$	$3 \times 1 + 1 = 4$

In economics, we often deal with the budget constraint. We can draw the budget line or alternatively budget set in the following way:

Example: Assume that there are only two goods: apples and bananas. The price of apples is \$2 and the price of bananas is \$4. You have \$12. If you spend all the money on apples, you can afford to buy 6 of them. If you spend all the money on bananas, you can buy 3. So the budget line goes through points [6,0] and [0,3]. The budget line can be represented by the following equation $2a + 4b = 12$ and graphically:

Budget line represents all combinations of apples and bananas that we can buy spending *exactly* \$12.

The budget set represents all combinations of apples and bananas that we can afford, i.e. that we can buy spending *at most* \$12. This can be represented by inequality $2a + 4b \leq 12$ or graphically it is the triangle below the budget line.

We know already that an equation represents a straight line. Intuitively, the system of equations represents the system of lines. Solving system of equation means looking for the intercept of lines. See the following example:

Example: Solve the following system numerically and graphically:

$x + y = 5$
$2x - y = 1$

Numerical solution to this system is $x = 2$ and $y = 3$.

To find graphical solution we first need to draw both lines:

$x + y = 5$ or alternatively $y = 5 - x$

x	0	1
y	5	4

$2x - y = 1$ or alternatively $y = 2x - 1$

x	0	1
y	-1	1

The two lines intercept in point [2,3].

Generally, the system of two equations and two variables can have no solution, exactly one solution (see the example above) or infinitely many solutions.

Example: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -4$

Solution:

$$
\begin{aligned}
&3x - y = 2 && \Rightarrow \quad y = 3x - 2 \\
&-9x + 3y = -4 &&
\end{aligned}
$$

$$
\begin{aligned}
&-9x + 3(3x - 2) = -4 \\
&-9x + 9x - 6 = -4 \\
&-6 = -4
\end{aligned}
$$

The last equality does not hold for any values of x and y. This means that this system does not have any solution.

Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -4$ or alternatively $y = \frac{1}{3}(9x - 4)$

x	0	1
y	$-4/3$	$5/3$

From the picture we see that the two lines are parallel, i.e. they do not intercept in any point. That is the reason why the system does not have any solution.

Example: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -6$

Solution:

$$3x - y = 2 \qquad \Rightarrow \quad y = 3x - 2$$
$$-9x + 3y = -6$$

$$-9x + 3(3x - 2) = -6$$
$$-9x + 9x - 6 = -6$$
$$-6 = -6$$

The last equality holds for all values of x and y (-6 = -6 no matter what are the values of x and y). This means that this system does not have any solution.

Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -6$ or alternatively $y = \frac{1}{3}(9x - 6) = 3x - 2$

Note that both lines are represented by the same equation. This means that the two lines coincide and therefore there are infinitely mane points where these two lines intercept and hence the system has infinitely many solutions.

Example: Assume that there are only two goods apples and bananas. Some company produces apple-banana juice. The budget of the company is $200. The price of apples is $5 and the price of bananas is $40. Further, the company has a limited capacity and can only store 15 pieces of fruit at the time. Sketch the budget set, the production possibilities set and find on the graph all combinations of apples and bananas which are feasible in terms of money and capacity.

Solution:

Budget set: The budget set is defined by the following inequality: $10a + 20b \leq 200$. If the company buys only apples, it can buy 40 kilograms. If the company spends all the money on bananas only, it can afford 10 kilograms. Therefore, the budget line goes through points [40,0] and [0,5].

Production set: The production set is defined by the inequality: $a + b \leq 15$. If the company buys only apples, it can buy 15 kilograms of apples. Similarly for bananas.

The budget set and production set are depicted on the following figure. Two lines correspond to budget line and production line. Budget (production) set is the area below the budget (production) line.

Combinations of apples and bananas which are feasible in terms of money and capacity are combinations which belong to both sets at the same time. In other words, we find the intercept of two triangles. This intercept is represented be the shaded area on the picture below.

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11 Combinatorial mathematics

Loosely speaking, combinatorics is a branch of mathematics dealing with counting objects satisfying certain criteria. Since this theory relies on factorials and combinatorial symbols we start with their definitions.

Factorial: For n a natural number, n factorial denoted by $n!$ is the product of the first n natural numbers. Zero factorial is defined to be 1.

For n a natural number:

$$n! = n \times (n-1) \times (n-2) \ldots 2 \times 1$$

Note that:

$$n! = n \times (n-1)!$$

Example:

$$4! = 4 \times 3! = 4 \times 3 \times 2! = 4 \times 3 \times 2 \times 1! = 4 \times 3 \times 2 \times 1 = 24$$

$$\frac{7!}{6!} = \frac{7 \times 6!}{6!} = 7$$

$$\frac{6!}{3!} \neq 2! \qquad \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 120$$

Note:

$$\left(\begin{array}{c} n \\ 0 \end{array} \right) = \left(\begin{array}{c} n \\ n \end{array} \right) = 1 \quad \text{for all } n$$

$$\frac{m!}{n!} \neq \left(\frac{m}{n} \right)!$$

$$m!n! \neq (m \times n)!$$

Binomial formula:

$(a+b)^n$ appears frequently in probabilistic theory and statistics

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$
$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$
$$\vdots$$
$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

where

$$C_{n,r} =_n C_r = C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{is called combinatorial symbol}$$

Example: Use binomial formula to expand $(x+y)^6$

$$\begin{aligned}(a+b)^6 &= \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6 = \\ &= x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6\end{aligned}$$

Example: Use binomial formula to expand $(3p-2q)^4$

$$(3p-2q)^4 = \binom{4}{0}(3p)^4 + \binom{4}{1}(3p)^3(-2q) + \binom{4}{2}(3p)^2(-2q)^2 + \binom{4}{3}3p\,(-2q)^3 + \binom{4}{4}(-2q)^4 =$$
$$= (3p)^4 + 4(3p)^3(-2q) + 6(3p)^2(-2q)^2 + 4(3p)(-2q)^3 + (-2q)^4$$

Example: Use binomial formula to find the fourth and sixteenth term in the expansion of $(x-2)^{20}$

$$4^{th}: \binom{20}{3}x^{17}(-2)^3 \qquad 16^{th}: \binom{20}{15}x^5(-2)^{15}$$

Multiplication principle

Suppose we flip a coin and then throw a single die. What are the possible combined outcomes?

There are 12 possible combined outcomes - two ways in which the coin can come up followed by six ways in which the die can come up.

If the problem is more complicated than this, drawing a diagram is not possible. In such a case, we use the following **multiplication principle:**

1. If two operations O_1 and O_2 are performed in order, with N_1 possible outcomes for the first operation and N_2 possible outcomes for the second operation, then there are

$N_1 \times N_2$

possible combined outcomes of the first operation followed by the second.

2. In general, if n operations $O_1, O_2 \ldots O_n$ are performed in order, with possible number of outcomes $N_1, N_2 \ldots N_n$ respectively, then there are

$N_1 \times N_2 \times \ldots \times N_n$

possible combined outcomes of the operations performed in the given order.

Example: From the 26 letters in the alphabet, how many ways can 3 letters appear in a row on a license plate if no letter is repeated?

Solution: There are 26 ways the first letter can be chosen. After a first letter is chosen, 25 letters remain; hence there are 25 ways a second letter can be chosen. And after 2 letters are chosen, there are 24 ways a third letter can be chosen. Hence, using the multiplication principle, there are $26 \times 25 \times 24 = 15600$ possible ways 3 letters can be chosen from the alphabet without allowing any letter to repeat.

Example: Many universities and colleges are now using computer-assisted testing procedures. Suppose a screening test is to consist of 5 questions, and computer stores 5 equivalent questions for the firs test question, 8 equivalent questions for the second, 6 for the third, 5 for the forth, and 10 for the fifth. How many different 5-question tests can the computer select? Two test are considered different if they differ in one or more questions.

Solution:

O_1 :	Select the first question	N_1 :	5 ways
O_2 :	Select the second question	N_2 :	8 ways
O_3 :	Select the third question	N_3 :	6 ways
O_4 :	Select the fourth question	N_4 :	5 ways
O_5 :	Select the fifth question	N_5 :	10 ways

Thus the computer can generate $5 \times 8 \times 6 \times 5 \times 10 = 12000$ different tests.

Example: How many 3-letter code words are possible using the first 8 letters of the alphabet if:

(A) No letter can be repeated?

(B) Letters can be repeated?

(C) Adjacent letters cannot be alike?

Solution:

(A) No letter can be repeated.

O_1 :	Select first letter	N_1 :	8 ways	
O_2 :	Select second letter	N_2 :	7 ways	because 1 letter has been used already
O_3 :	Select third letter	N_3 :	6 ways	because 2 letters have been used already

Thus, there are $8 \times 7 \times 6 = 336$ possible code words.

(B) Letters can be repeated.

O_1 :	Select first letter	N_1 :	8 ways	
O_2 :	Select second letter	N_2 :	8 ways	repeats are allowed
O_3 :	Select third letter	N_3 :	8 ways	repeats are allowed

Thus, there are $8 \times 8 \times 8 = 512$ possible code words.

(C) Adjacent letters cannot be alike.

O_1 :	Select first letter	N_1 :	8 ways	
O_2 :	Select second letter	N_2 :	7 ways	cannot be the same as the first
O_3 :	Select third letter	N_3 :	7 ways	cannot be the same as the second, but can be the same as the first

Thus, there are $8 \times 7 \times 7 = 392$ possible code words.

Permutations of n Objects:

Example: Suppose 4 pictures are to be arranged from left to right on one wall of an art gallery. How many arrangements are possible? Using the multiplication principle, there are 4 ways of selecting the first picture, 3 ways of selecting the second picture, 2 ways of selecting the third one and only 1 way to select the fourth. Thus the number of arrangements is

$$4 \times 3 \times 2 \times 1 = 4! = 24$$

The number of **permutations of n objects**, denoted by $P_{n,n}$, is given by:

$$P_{n,n} = n \times (n-1) \times \ldots \times 1 = n!$$

Example: Now suppose that the director of the art gallery decides to use only 2 out of 4 available pictures on the wall, arranged from left to right. How many arrangements of 2 pictures can be formed out of 4? There are 4 ways the first picture can be selected. After selecting the first picture, there are 3 ways the second picture can be selected. Thus the number of arrangements of 2 pictures from 4 pictures, denoted be $P_{4,2}$ is given by

$$P_{4,2} = 4 \times 3 = \frac{4 \times 3 \times 2!}{2!} = \frac{4!}{2!} = \frac{4!}{(4-2)!} = 12$$

The number of **permutations of n objects taken r at a time** $(0 \leq r \leq n)$, denoted by $P_{n,r}$, is given by:

$$P_{n,r} = n \times (n-1) \times \ldots \times (n-3+1) = \frac{n!}{(n-r)!}$$

Note that if $r = n$, then the number of permutations of n objects taken n at a time is:

$$P_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Example: From a committee of 8 people, in how many ways can we choose a chair and a vice-chair, assuming one person cannot hold more than one position?

Solution:

$$P_{8,2} = \frac{8!}{(8-2)!} = \frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56$$

Example: From a committee of 10 people, in how many ways can we choose a chair, a vice-chair and secretary, assuming one person cannot hold more than one position?

Solution:

$$P_{10,3} = \frac{10!}{(10-3)!} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

Combinations:

Now suppose that an art museum owns 8 paintings by a given artist and another art museum wishes to borrow 3 of these paintings for a special show. How many ways can 3 paintings be selected for shipment out of the 8 available? Here, the order of the items selected doesn't matter. What we are actually interested in is how many subsets of 3 objects can be formed from a set of 8 objects. We call such a subset a combination of 8 objects taken 3 at a time. The total number of combinations is denoted by the symbol

$$C_{8,3} \quad \text{or} \quad \binom{8}{3}$$

To find the number of combinations of 8 objects taken 3 at a time, $C_{8,3}$, we make use of the formula for $P_{n,r}$ and the multiplication principle. We know that the number of permutations of 8 objects taken 3 at a time is given by $P_{8,3}$, and we have a formula for computing this quantity. Now, think of $P_{8,3}$ in terms of two operations:

O_1 : Select a subset of 3 objects
N_1 : $C_{8,3}$ ways
O_2 : Arrange the subset in a given order
N_2 : 3! ways

The combined operation O_1 followed by O_2 produces a permutation of 8 objects taken 3 at a time. Thus,

$$P_{8,3} = C_{8,3} \cdot 3!$$
$$\frac{8!}{(8-3)!} = C_{8,3} \cdot 3!$$
$$C_{8,3} = \frac{8!}{3!(8-3)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

Thus the museum can make 56 different selections of 3 paintings from the 8 available.

The number of **combinations of n objects taken r at a time** $(0 \le r \le n)$, denoted by $C_{n,r}$, is given by:

$$C_{n,r} = \binom{n}{r} = \frac{P_{n,r}}{r!} = \frac{n!}{r!(n-r)!}$$

Example: From a committee of 8 people, in how many ways can we choose a subcommittee of 2 people?

Solution: In this example, the ordering does not matter:

$$C_{8,2} = \binom{8}{2} = \frac{8!}{2!(8-2)!} = \frac{8 \cdot 7 \cdot 6!}{2! \cdot 6!} = \frac{8 \cdot 7}{2} = 28$$

Example: Out of standard 52-card deck (13 cards in each out of 4 suits - hearts, spades, diamonds, clubs), how many 5-card hands will have 3 aces and 2 kings?

Solution:

O_1 : Choose 3 aces out of 4 possible Order is not important
N_1 : $C_{4,3}$
O_2 : Choose 2 kings out of 4 possible Order is not important
N_2 : $C_{4,2}$

Using multiplication principle we have that the number of hands is:

$$C_{4,3} \cdot C_{4,2} = 4 \cdot 6 = 24$$

Example: In a horse race, how many different finishes among the first 3 places are possible for a 10-horse race?

Solution: In this example the ordering matters, so we use permutations:

$$P_{10,3} = \frac{10!}{(10-3)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

Example: From a standard 52-cards deck, how many 5-card hands will have all hearts?

Solution: Here, the order does not matter, we are choosing 5-all-heart-card hands out of 13 heart cards:

$$C_{13,5} = \binom{13}{5} = \frac{13!}{5!(13-5)!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8!}{5!8!} = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 13 \cdot 11 \cdot 9 = 1287$$

Example: How many ways can 2 people be seated in a row of 5 chairs? 3 people? 4 people? 5 people?

Solution: In this example, the order matters:

$$P_{5,2} = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{3!} = 20$$

$$P_{5,3} = \frac{5!}{(5-3)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2!}{2!} = 60$$

$$P_{5,4} = \frac{5!}{(5-4)!} = \frac{5!}{1} = 5! = 120$$

$$P_{5,5} = \frac{5!}{(5-5)!} = \frac{5!}{1} = 5! = 120$$

12 Financial mathematics, simple and compound interest

Arithmetic sequence: is a sequence $a_1, a_2, \ldots a_n$ such that $a_n - a_{n-1} = d$ for all n. So the distance between the two following elements of the sequence is constant.

For example: 1,2,3, … ($d = 1$); 2,4,6, … 16 ($d = 2$); 0,3,6, … 18 ($d = 3$)

Geometric sequence: is a sequence $a_1, a_2, \ldots a_n$ such that $\frac{a_n}{a_{n-1}} = r$ for all n. So the ratio between the two following elements is constant.

For example: 2,4,8, … ($r = 2$); 1,3,9,27,51 ($r = 3$)

Arithmetic series: is a sum of elements of arithmetic sequence. The sum is given by:

$$S_n = \frac{(a_1 + a_n) \cdot n}{2}$$

Geometric series: is a sum of elements of geometric sequence. The sum is given by:

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

Interest

Interest is a fee paid on borrowed capital. The fee is compensation to the lender for foregoing other useful investments that could have been made with the loaned money. Instead of the lender using the assets directly, they are advanced to the borrower. The borrower then enjoys the benefit of the use of the assets ahead of the effort required to obtain them, while the lender enjoys the benefit of the fee paid by the borrower for the privilege. The amount lent, or the value of the assets lent, is called the principal. This principal value is held by the borrower on credit. Interest is therefore the price of credit, not the price of money as it is commonly - and mistakenly - believed to be. The percentage of the principal that is paid as a fee (the interest), over a certain period of time, is called the interest rate. (wikipedia.org)

Simple interest

Simple Interest is calculated only on the principal, or on that portion of the principal which remains unpaid. The amount of simple interest is calculated according to the following formula:

$$A = P(1 + in)$$

where

A is the amount of money to be paid back

P is the principal

i is the interest rate (expressed as decimal number)

n the number of time periods elapsed since the loan was taken

For example, imagine Jim borrows $23,000 to buy a car and that the simple interest is charged at a rate of 5.5% per annum. After five years, and assuming none of the loan has been paid off, Jim owes:

$$A = 23000(1 + 0.055 \times 5) = 29325$$

At this point, Jim owes a total of $29,325 (principal plus interest).

Compound interest

In the short run, compound Interest is very similar to Simple Interest, however, as time goes on difference becomes considerably larger. The conceptual difference is that the principal changes with every time period, as any interest incurred over the period is added to the principal. Put another way, the lender is charging interest on the interest.

$$A = P(1 + i)^n$$

In this case Jim would owe principal of $30,060.

Savings, loans, project evaluations

Time value of money

The time value of money represents the fact that, loosely speaking, it is better to have money today than tomorrow. Investor prefers to receive a payment today rather than an equal amount in the future, all else being equal. This is because the money received today can be deposited in a bank account and an interest is received.

Present value of a future sum

$$PV = \frac{FV}{(1+i)^n}$$

where:

PV is the value at time 0

FV is the value at time n

i is the rate at which the amount will be compounded each period

n is the number of periods

Present value of an annuity

The term annuity is used in finance theory to refer to any terminating stream of fixed payments over a specified period of time. Payments are made at the end of each period.

$$PV(A) = A\frac{1}{(1+i)} + A\frac{1}{(1+i)^2} + \ldots + A\frac{1}{(1+i)^n} = A\frac{1}{(1+i)}\frac{1-\frac{1}{(1+i)^n}}{1-\frac{1}{1+i}} = A\frac{1-\frac{1}{(1+i)^n}}{i}$$

where:

$PV(A)$ is the value of the annuity at time 0

A is the value of the individual payments in each compounding period

i is the interest rate that would be compounded for each period of time

n is the number of payment periods

Present value of a perpetuity

A perpetuity is an annuity in which the periodic payments begin on a fixed date and continue indefinitely. It is sometimes referred to as a ”perpetual annuity” (UK government bonds).

The value of the perpetuity is finite because receipts that are anticipated far in the future have extremely low present value (present value of the future cash flows). Unlike a typical bond, because the principal is never repaid, there is no present value for the principal. The price of a perpetuity is simply the coupon amount over the appropriate discount rate or yield, that is

$$PV(P) = \frac{A}{i}$$

Future value of a present sum

$$FV = PV(1+i)^n$$

Future value of an annuity

$$FV(A) = A\frac{(1+i)^n - 1}{i}$$

Example: One hundred euros to be paid 1 year from now, where the expected rate of return is 5% per year, is worth in today's money:

$$PV = \frac{FV}{(1+i)^n} = \frac{100}{1.05} = 95.23.$$

So the present value of 100 euro one year from now at 5% is 95.23.

Example: Consider a 10 year mortgage where the principal amount P is \$200,000 and the annual interest rate is 6%.

The number of monthly payments is

$$n = 10 \text{ years} \times 12 \text{ months} = 120 \text{ months}$$

The monthly interest rate is

$$i = \frac{6\% \text{ per year}}{12 \text{ monhs per year}} = 0.5\% \text{ per month}$$

$$PV(A) = A\frac{1 - \frac{1}{(1+i)^n}}{i} \quad \Rightarrow A = PV(A)\frac{i}{1 - \frac{1}{(1+i)^n}} = PV(A)\frac{i(1+i)^n}{(1+i)^n - 1}$$

$$A = 200000\frac{0.005(1+0.005)^{120}}{(1+0.005)^{120} - 1} = \$2220.41 \text{ per month.}$$

Example: Consider a deposit of \$ 100 placed at 10% annually. How many years are needed for the value of the deposit to double?

$$FV = PV(1+i)^n$$
$$200 = 100(1+0.1)^n$$
$$1.1^n = \frac{200}{100} = 2$$
$$\ln 1.1^n = \ln 2$$
$$n \ln 1.1 = \ln 2$$
$$n = \frac{\ln 2}{\ln 1.1} = 7.27 \text{ years}$$

Example: Similarly, the present value formula can be rearranged to determine what rate of return is needed to accumulate a given amount from an investment. For example, \$100 is invested today and \$200 return is expected in five years; what rate of return (interest rate) does this represent?

$$FV = PV(1+i)^n$$
$$200 = 100(1+i)^5$$
$$(1+i)^5 = \frac{200}{100} = 2$$
$$(1+i) = 2^{1/5}$$
$$i = 2^{1/5} - 1 = 0.15 = 15\%$$

Example: A manager of a company has to choose one of two possible projects. Project A requires immediate investment \$500 and yields returns \$200, \$300, and \$400 in the following three years. For project B it is necessary to invest \$400 now and the expected returns in the next three years are \$400, \$100 and \$50. Supposed that an interest rate is 10%. Which project should the manager choose?

Having time value of money in mind, manager should choose project with a higher present value.

$$PV_A = -500 + \frac{200}{1+i} + \frac{300}{(1+i)^2} + \frac{400}{(1+i)^3} = -500 + \frac{200}{1.1} + \frac{300}{1.1^2} + \frac{400}{1.1^3} =$$

$$= -500 + 182 + 248 + 300 = 230$$

$$PV_B = -400 + \frac{400}{1+i} + \frac{100}{(1+i)^2} + \frac{50}{(1+i)^3} = -400 + \frac{400}{1.1} + \frac{100}{1.1^2} + \frac{50}{1.1^3} =$$

$$= -400 + 364 + 83 + 38 = 85$$

Project A has a higher present value and hence should be chosen.

13 Review Lecture

Equations and Inequalities

Problem 1: Solve the absolute value inequality. Write the solution set using interval notation: $|4x+7|<5$

[A] $\left(-3,-\frac{1}{2}\right)$ [B] $\left(-\infty,-\frac{1}{2}\right)$ [C] $(-\infty,-3)$ [D] $(-\infty,-3)\cup\left(-\frac{1}{2},\infty\right)$

Solution:

$$\begin{aligned}&|4x+7|<5\\&-5<4x+7<5 \quad /-7\\&-12<4x<-2 \quad /\div 4\\&-3<x<-1/2 \quad \Rightarrow \quad x\in(-3,-1/2) \quad \rightarrow \quad [A]\end{aligned}$$

Problem 2: Write as a single interval, using interval notation: $(-\infty,1)\cap(-10,5)$

[A] $(-\infty,-5)$ [B] $(1,5)$ [C] $(-10,1)$ [D] $(-\infty,-10)$

Solution: $(-\infty,1)\cap(-10,5)=(-\infty,5) \quad \rightarrow \quad [A]$

Problem 3: Solve the following inequality for x. Express the solution set using interval notation: $\frac{1}{2}x-4<\frac{1}{3}x+5$

[A] $(-\infty,9)$ [B] $(-\infty,54)$ [C] $(-\infty,6)$ [D] $\left(-\infty,\frac{54}{4}\right)$

Solution:

$$\begin{aligned}&\frac{1}{2}x-4<\frac{1}{3}x+5\\&\frac{1}{2}x-\frac{1}{3}x<5+4\\&\frac{1}{6}x<9\\&x<54 \quad \Rightarrow \quad x\in(-\infty,54) \quad \rightarrow \quad [B]\end{aligned}$$

Problem 4: Solve the compound inequality for x. Express the solution set using interval notation: $8 \leq 5 - x$ or $3x - 2 > 10$

[A] $\emptyset$ [B] $[-3, 4)$ [C] $(-\infty, -3] \cup (4, \infty)$ [D] $[-3, \infty)$

Solution:

$$\begin{aligned}
&8 \leq 5 - x \quad \text{or} \quad 3x - 2 > 10 \\
&-3 \leq -x \quad \text{or} \quad 3x > 12 \\
&x \leq 3 \quad \text{or} \quad x > 4 \quad \Rightarrow \quad x \in (-\infty, 3] \quad \text{or} \quad x \in (4, \infty) \quad \rightarrow \quad [C]
\end{aligned}$$

Exponents and Logarithms

Problem 5: Solve for x: $\log(3x - 9) = \log(2x - 6)$

[A] 3 [B] all real numbers [C] -3 [D] No solution

Solution:

$$\begin{aligned}
&\log(3x - 9) = \log(2x - 6) \\
&3x - 9 = 2x - 6 \\
&3x - 2x = -6 + 9 \\
&x = 3 \quad \rightarrow \quad [A]
\end{aligned}$$

Problem 6: Find the exact solution for x: $(1.3)^{2x} = 4$

$[A]\ \frac{1}{2}\ln\left(\frac{4}{1.3}\right)$ $[B]\ \frac{20}{13}$ $[C]\ \frac{1}{2}[\ln(4) - \ln(1.3)]$ $[D]\ \frac{\ln(4)}{2\ln(1.3)}$

Solution:

$$\begin{aligned}
&(1.3)^{2x} = 4 \\
&\ln\left[(1.3)^{2x}\right] = \ln 4 \\
&(2x)\ln[(1.3)] = \ln 4 \\
&2x = \frac{\ln 4}{\ln[(1.3)]} \\
&x = \frac{\ln 4}{2\ln[(1.3)]} \quad \rightarrow \quad [D]
\end{aligned}$$

Problem 7: Solve the equation for x: $\left(\frac{2}{3}\right)^{x+1} = \frac{8}{27}$

[A] 1 [B] 2 [C] 3 [D] 4

Solution:

$$\begin{aligned}
&\left(\frac{2}{3}\right)^{x+1} = \frac{8}{27} \\
&\left(\frac{2}{3}\right)^{x+1} = \left(\frac{2}{3}\right)^{3} \\
&x + 1 = 3 \\
&x = 2 \quad \rightarrow \quad [B]
\end{aligned}$$

Matrices and Determinants

Problem 8: When the system of linear equations is solved, what is the value of x and y?
$x - 2y = 4$, $3x - y = -3$

Solution:

$$\begin{aligned}
&x - 2y = 4 \quad \Rightarrow \quad x = 4 + 2y \quad \text{plug in the second equation:} \\
&3(4 + 2y) - y = -3 \\
&12 + 6y - y = -3 \\
&5y = -15 \\
&y = -3 \quad \text{plug in the first equation to find the value of } \; x \\
&x = 4 + 2y = 4 + 2(-3) = 4 - 6 = -2 \\
&\Rightarrow \quad x = -2, \; y = -3
\end{aligned}$$

Problem 9: Find product of the following matrices:

$$A = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix}$$

Solution:

$$AB = \begin{pmatrix} 1 & 4 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 3 + 4\cdot 2 & 1\cdot(-1) + 4\cdot 2 \\ -2\cdot 3 + 0\cdot 2 & -2\cdot(-1) + 0\cdot 2 \end{pmatrix} = \begin{pmatrix} 11 & 7 \\ -6 & 2 \end{pmatrix}$$

Problem 10: Find the following determinants:

(a) $$\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}$$

(b) $$\begin{vmatrix} 2 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{vmatrix}$$

Solution:

$$(a) \quad \begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = 1\cdot 1 - (-2)(-3) = 1 - 6 = -5$$

$$(b) \quad \begin{vmatrix} 2 & 2 & 3 \\ 3 & -1 & -1 \\ -2 & 1 & 0 \end{vmatrix} = 2(-1)^{1+1}[(-1)\cdot 0 - 1\cdot(-1)] + 2(-1)^{1+2}[3\cdot 0 - (-1)(-2)] +$$

$$+3(-1)^{1+3}[3\cdot 1 - (-1)(-2)] = 2 + 4 + 3 = 9$$

Problem 11: Solve the following systems using (i) matrix method, (ii) inverse matrix, and (iii) Cramer's rule:

$x + y = 1,\ \ 3x - 4y = -18$

Solution: Matrix method:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & -4 & -18 \end{array}\right) \begin{array}{c} \times(-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -7 & -21 \end{array}\right) \div(-7) \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-1) \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3 \end{array}\right) \Rightarrow \begin{array}{c} x = -2 \\ y = 3 \end{array}$$

Inverse matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{array}\right) \begin{array}{c} \times(-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -7 & -3 & 1 \end{array}\right) \div(-7) \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 3/7 & -1/7 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-1) \end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 4/7 & 1/7 \\ 0 & 1 & 3/7 & -1/7 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 4/7 & 1/7 \\ 3/7 & -1/7 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 4/7 & 1/7 \\ 3/7 & -1/7 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4/7 + 1 \cdot 3/7 & 1 \cdot 1/7 + 1 \cdot (-1/7) \\ 3 \cdot 4/7 + (-4) \cdot 3/7 & 3 \cdot 1/7 - 4 \cdot (-1/7) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AX = B \longrightarrow \begin{pmatrix} 1 & 1 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -18 \end{pmatrix}$$

$$X = A^{-1}B \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4/7 & 1/7 \\ 3/7 & -1/7 \end{pmatrix} \begin{pmatrix} 1 \\ -18 \end{pmatrix} = \begin{pmatrix} \frac{4}{7} \cdot 1 + \frac{1}{7} \cdot (-18) \\ \frac{3}{7} \cdot 1 - \frac{1}{7} \cdot (-18) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \Rightarrow \begin{array}{c} x = -2 \\ y = 3 \end{array}$$

Cramer's rule:

$$x = \frac{\begin{vmatrix} 1 & 1 \\ -18 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}} = \frac{14}{-7} = -2$$

$$y = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -18 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -4 \end{vmatrix}} = \frac{-21}{-7} = 3$$

Combinatorial Mathematics

Problem 12: Evaluate:

$(a)\ \dfrac{14!}{12!}$

$(b)\ \dfrac{7!}{7!(7-7)!}$

$(c)\ \dfrac{8!}{3!(8-3)!}$

$(d)\ \dfrac{5!}{2!3!}$

Solution:

$(a)\ \dfrac{14!}{12!} = \dfrac{14 \cdot 13 \cdot 12!}{12!} = 14 \cdot 13 = 182$

$(b)\ \dfrac{7!}{7!(7-7)!} = \dfrac{7!}{7!0!} = \dfrac{7!}{7!1} = 1$

$(c)\ \dfrac{8!}{3!(8-3)!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5!}{3!5!} = \dfrac{8 \cdot 7 \cdot 6}{3!} = \dfrac{8 \cdot 7 \cdot 6}{6} = 8 \cdot 7 = 56$

$(d)\ \dfrac{5!}{2!3!} = \dfrac{5 \cdot 4 \cdot 3!}{2!3!} = \dfrac{5 \cdot 4}{2} = 10$

Problem 13: A deli serves sandwiches with the following options: 3 kinds of bread, 5 kinds of meat, and lettuce or sprouts. How many different sandwiches are possible, assuming one item is used out of each category?

Solution:

O_1 : Choose kind of brad
N_1 : 3
O_2 : Choose kind of meat
N_2 : 5
O_3 : Choose vegetable
N_3 : 2

Using multiplication principle we have that the number of sandwiches is: $3 \cdot 5 \cdot 2 = 30$

Problem 14: How many 5-digit zip codes are possible? How many of these codes contain no repeated digits?

Solution: There are ten numbers available (0, 1, 2, ... 9). If repeated digits are allowed, then there are 10 digits possible for each digit in the zip code and hence there are $10 \times 10 \times 10 \times 10 \times 10 = 10^5$ zip codes. If codes contain no repeated digits, then there are 10 digits possible for the first digit in the zip code, 9 possible numbers for the second digit in the zip code, ... and 6 possible numbers for the last digit in the zip code. Therefore there are $10 \times 9 \times 8 \times 7 \times 6$ zip codes with no repeated digits.

Financial Mathematics

Problem 15: If \$2500 is invested in the account that pays interest compounded continuously at a rate of 4%, how long will it take to double the investment? Hint: $A = Pe^{rn}$

[A] 2.8 years [B] 13.0 years [C] 14.2 years [D] 17.3 years

Solution:

$$
\begin{aligned}
&A = Pe^{rn} \\
&5000 = 2500e^{0.04n} \\
&2 = e^{0.04n} \\
&\ln 2 = \ln e^{0.04n} \\
&\ln 2 = (0.04n)\ln e = 0.04n \\
&n = \frac{\ln 2}{0.04} = 17.3 \text{ years} \quad \rightarrow \quad [D]
\end{aligned}
$$

Problem 16: If \$4500 is deposited in a bank account paying 8% compounded quarterly, then how much interest will be earned at the end of 6 years? Hint: $A = P\left(1 + \frac{i}{t}\right)^{nt}$

[A] \$353,235.81 [B] \$24,035.31 [C] \$7237.97 [D] \$2737.97

Solution:

$$
\begin{aligned}
&A = P\left(1 + \frac{i}{t}\right)^{nt} \\
&A = 4500\left(1 + \frac{0.08}{4}\right)^{4\cdot 6} = 4500 \cdot 1.02^{24} = 7237.97
\end{aligned}
$$

The amount of money in a bank account will be \$7237.97 what means that the interest earned is $\$7237.97 - \$4500 = \$2737.97 \rightarrow [D]$

Problem 17: Celia has invested \$2500 at 11% yearly interest. How much must she invest at 12% so that the interest from both investments totals \$695 after a year?

[A] \$50.40 [B] \$2928.75 [C] \$1596.67 [D] \$3500.00

Solution: The interest from the first investment is:

$$A = P(1+i)^n$$
$$A = 2500(1+0.11)^1 = 2500 \cdot 1.11 = 2775$$

Hence the interest from the first investment is $\$2775 - \$2500 = \$275$.

The total investment is supposed to be \$695 what means that the interest from the second investment has to be $\$695 - \$275 = \$420$.

$$P + 420 = P(1+i)^n$$
$$P + 420 = P(1+0.12)^n$$
$$P + 420 = P \cdot 1.12^1$$
$$420 = P \cdot 1.12 - P = 0.12P$$
$$P = \frac{420}{0.12} = 3500 \quad \rightarrow \quad [D]$$

Problem 18: After a 7% increase in salary, Laurie makes \$1016.50 per month. How much did she earn per month before the increase?

[A] \$950 [B] \$1087.66 [C] \$945.35 [D] \$871.29

Solution: Let's denote the original salary S. The according to the setup we have that:

$$S + 0.07S = 1016.50$$
$$1.07S = 1016.50$$
$$S = \frac{1016.50}{1.07} = 950 \quad \rightarrow \quad [A]$$

Sample Final Exam

1. Solve the following equations and inequalities for x:

(i) $5 - 3(x-6) = 2(x-6)$

(a) 7 (b) $-\frac{1}{5}$ (c) -7 (d) 1

(ii) $\frac{3}{x} - \frac{1}{4} = \frac{1}{3}$

(a) $\frac{21}{2}$ (b) $\frac{36}{7}$ (c) 10 (d) $\frac{4}{7}$

(iii) $3x^2 - 5x - 2 = 0$

(a) $-3, 2$ (b) $-\frac{1}{3}, 2$ (c) $-2, \frac{1}{3}$ (d) $-2, 3$

(iv) $\log_x 32 = 5$

(a) $\frac{5}{32}$ (b) 2 (c) 5 (d) 32

(v) $3x - 2 < -x + 22$

(a) $(-\infty, 6)$ (b) $(-\infty, 6]$ (c) $(-6, \infty)$ (d) $(-6, \infty]$

(vi) $|x+2| < 7$

(a) $(-5, 12)$ (b) $[-5, 12]$ (c) $(-9, 5)$ (d) $[-9, 5]$

(vii) $\log_3(x-1) = 1$

(a) 2 (b) 1 (c) 4 (d) 3

Solution:

$$
\begin{aligned}
(i) \quad & 5 - 3(x-6) = 2(x-6) \\
& 5 - 3x + 18 = 2x - 12 \\
& 5x = 35 \\
& x = 7 \quad \rightarrow (a) \\
(ii) \quad & \frac{3}{x} - \frac{1}{4} = \frac{1}{3} \\
& \frac{3}{x} = \frac{1}{4} + \frac{1}{3} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12} \\
& 3 \cdot 12 = 7x \\
& x = \frac{36}{7} \quad \rightarrow (b)
\end{aligned}
$$

(iii) $3x^2 - 5x - 2 = 0$

$D = b^2 - 4ac = (-5)^2 - 4 \cdot 3 \cdot (-2) = 49$

$x = \dfrac{-b \pm \sqrt{D}}{2a} = \dfrac{5 \pm 7}{6} = 2, -\dfrac{1}{3} \quad \rightarrow (b)$

(iv) $\log_x 32 = 5$

$x^5 = 32$

$x = \sqrt[5]{32} = 2 \quad \rightarrow (b)$

(v) $3x - 2 < -x + 22$

$4x < 24$

$x < 6 \quad \rightarrow (a)$

(vi) $|x + 2| < 7$

$-7 < x + 2 < 7$

$-9 < x < 5 \quad \rightarrow (c)$

(vii) $\log_3(x - 1) = 1$

$\log_3(x - 1) = \log_3 3$

$x - 1 = 3$

$x = 4 \quad \rightarrow (c)$

2. Solve the following systems of equations for both x and y:

(a) $2x + 3y = 2$, $x - y = \dfrac{1}{6}$ $\qquad$ (b) $x - y = 3$, $x + y = 7$

Solution:

(a) $2x + 3y = 2$

$x - y = \dfrac{1}{6} \quad \rightarrow \quad x = y + \dfrac{1}{6}$

$2x + 3y = 2 \quad \rightarrow \quad 2\left(y + \dfrac{1}{6}\right) + 3y = 2$

$2y + \dfrac{1}{3} + 3y = 2$

$5y = \dfrac{5}{3}$

$y = \dfrac{1}{3} \quad \Rightarrow \quad x = y + \dfrac{1}{6} = \dfrac{1}{3} + \dfrac{1}{6} = \dfrac{1}{2}$

$$(b)\quad \begin{aligned} x - y &= 3 \\ x + y &= 7 \end{aligned}$$

$$\begin{aligned} 2x &= 10 \\ x &= 5, \quad y = 2 \end{aligned}$$

3. Find inverse matrix to the following matrices and check that your answer is correct:

(a) $$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Solution:

(a)

$$\left(\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \div 4 \sim \left(\begin{array}{cc|cc} 1 & -1/4 & 1/4 & 0 \\ -1 & 2 & 0 & 1 \end{array}\right) \begin{matrix}\searrow \\ \swarrow\end{matrix} \sim \left(\begin{array}{cc|cc} 1 & -1/4 & 1/4 & 0 \\ 0 & 7/4 & 1/4 & 1 \end{array}\right) \begin{matrix}\nwarrow \\ \times 4/7\end{matrix} \sim$$

$$\left(\begin{array}{cc|cc} 1 & -1/4 & 1/4 & 0 \\ 0 & 1 & 1/7 & 4/7 \end{array}\right) \times 1/4 \sim \left(\begin{array}{cc|cc} 1 & 0 & 2/7 & 1/7 \\ 0 & 1 & 1/7 & 4/7 \end{array}\right)$$

Check:

$$\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2/7 & 1/7 \\ 1/7 & 4/7 \end{pmatrix} = \begin{pmatrix} (4,-1)\begin{pmatrix} 2/7 \\ 1/7 \end{pmatrix} & (4,-1)\begin{pmatrix} 1/7 \\ 4/7 \end{pmatrix} \\ (-1,2)\begin{pmatrix} 2/7 \\ 1/7 \end{pmatrix} & (-1,2)\begin{pmatrix} 1/7 \\ 4/7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} /\times(-1) & /\times(-2) \\ \swarrow & \downarrow \\ & \swarrow \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -6 & 1 & -2 & 0 & 1 \end{array}\right) \times(-1) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & -6 & 1 & -2 & 0 & 1 \end{array}\right) \begin{matrix} \times 6 \\ \swarrow \end{matrix} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -5 & 4 & -6 & 1 \end{array}\right) \div(-5) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -4/5 & 6/5 & -1/5 \end{array}\right) \begin{array}{c} \\ \nwarrow \\ \nearrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/5 & 1/5 & -1/5 \\ 0 & 0 & 1 & -4/5 & 6/5 & -1/5 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-3) \\ \\ \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/5 & -3/5 & 3/5 \\ 0 & 1 & 0 & 1/5 & 1/5 & -1/5 \\ 0 & 0 & 1 & -4/5 & 6/5 & -1/5 \end{array}\right)$$

Check:

$$\begin{pmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2/5 & -3/5 & 3/5 \\ 1/5 & 1/5 & -1/5 \\ -4/5 & 6/5 & -1/5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

4. Find the following determinants:

(a) $\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 3 \\ -2 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix}$

Solution:

(a)

$$\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 3 \times 1 - 0 \times (-2) = 3$$

(b)

$$\begin{vmatrix} 1 & 3 & 3 \\ -2 & 0 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix} =$$

$$= -2 + 18 - 12 = 4$$

5. Use matrix method **<u>or</u>** inverse matrix **<u>or</u>** Cramer's rule to solve the following systems:

$$(a) \quad \begin{aligned} 2x + 3y &= 2 \\ x - y &= \frac{1}{6} \end{aligned} \qquad (b) \quad \begin{aligned} x - y &= 3 \\ x + y &= 7 \end{aligned}$$

Solution:

(a)

Matrix method:

$$\left(\begin{array}{cc|c} 2 & 3 & 2 \\ 1 & -1 & 1/6 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & 1/6 \\ 2 & 3 & 2 \end{array}\right) \begin{array}{c}\times(-2)\\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & 1/6 \\ 0 & 5 & 10/6 \end{array}\right) \div 5 \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & -1 & 1/6 \\ 0 & 1 & 1/3 \end{array}\right) \begin{array}{c}\nwarrow\\ \nearrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 1/3 \end{array}\right) \Rightarrow \begin{array}{l} x = 1/2 \\ y = 1/3 \end{array}$$

Inverse matrix:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{c}\nwarrow\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right) \begin{array}{c}\times(-2)\\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 5 & 1 & -2 \end{array}\right) \div 5 \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1/5 & -2/5 \end{array}\right) \begin{array}{c}\nwarrow\\ \nearrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & 3/5 \\ 0 & 1 & 1/5 & -2/5 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/5 & 3/5 \\ 1/5 & -2/5 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/5 & 3/5 \\ 1/5 & -2/5 \end{pmatrix} =$$

$$= \begin{pmatrix} 2 \times 1/5 + 3 \times 1/5 & 2 \times 3/5 + 3 \times (-2/5) \\ 1 \times 1/5 + (-1) \times 1/5 & 1 \times 3/5 + (-1) \times (-2/5) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AX = B \longrightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1/6 \end{pmatrix}$$

$$X = A^{-1}B \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 2 \\ 1/6 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \times 2 + \frac{3}{5} \times \frac{1}{6} \\ \frac{1}{5} \times 2 - \frac{2}{5} \times \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/3 \end{pmatrix} \Rightarrow \begin{array}{l} x = 1/2 \\ y = 1/3 \end{array}$$

Cramer's rule:

$$x = \frac{\begin{vmatrix} 2 & 3 \\ 1/6 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}} = \frac{-5/2}{-5} = 1/2$$

$$y = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 1/6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}} = \frac{-5/3}{-5} = 1/3$$

(b)

Matrix method:

$$\left(\begin{array}{cc|c} 1 & -1 & 3 \\ 1 & 1 & 7 \end{array}\right) \begin{array}{c} \nwarrow \\ \times(-1) \end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & -2 & -4 \end{array}\right) \div(-2) \sim \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 2 \end{array}\right) \begin{array}{c} \nwarrow \\ \nearrow \end{array} \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right) \Rightarrow \begin{array}{c} x = 5 \\ y = 2 \end{array}$$

Inverse matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{c} \times(-1) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right) \div 2 \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right) \begin{array}{c} \nwarrow \\ \nearrow \end{array} \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AX = B \longrightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$$

$$X = A^{-1}B \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times 3 + \frac{1}{2} \times 7 \\ -\frac{1}{2} \times 3 + \frac{1}{2} \times 7 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \Rightarrow \begin{array}{c} x = 5 \\ y = 2 \end{array}$$

Cramer's rule:

$$x = \frac{\begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{10}{2} = 5$$

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 1 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{4}{2} = 2$$

6. Find AB, where

$$A = \begin{pmatrix} 1 & 4 & -2 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -3 & 2 \\ 2 & 1 \end{pmatrix}$$

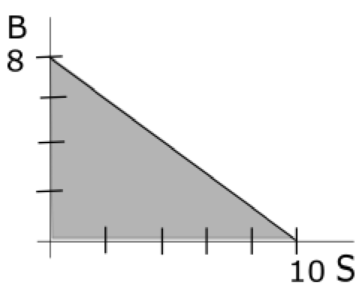
Solution:

$$AB = \begin{pmatrix} 1 & 4 & -2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} & \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} -15 & 7 \\ -3 & 6 \end{pmatrix}$$

7. You are going to buy some fruit and you currently have 340CZK with you. You like strawberries and blueberries with the price 34 and 42.50 per pack correspondingly. Draw the budget set with strawberries on horizontal and blueberries on vertical axis.

Solution: If you spend all the money on strawberries you can byu $340/34 = 10$ packs. If you only buy blueberries, you can buy $340/42.50 = 8$ packs. This means that the budget line goes through points [10,0] and [0,8] (see picture below). The shaded area is the budget set.



8. Solve the following system numerically and graphically:

$x - 2y = 1$

$x + y = 10$

Solution: Numerical solution:

$$x - 2y = 1 \quad \rightarrow \quad x = 1 + 2y$$
$$x + y = 10$$

$$(1 + 2y) + y = 10 \quad \Rightarrow \quad 1 + 3y = 10$$
$$3y = 9 \quad \Rightarrow \quad y = 3$$
$$x = 1 + 2y = 1 + 6 = 7$$

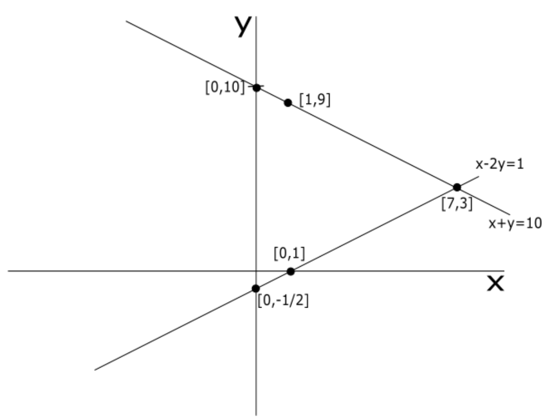
Graphical solution:

$x - 2y = 1 \Rightarrow y = \frac{1}{2}(x - 1)$

x	0	1
y	$-1/2$	0

$x + y = 10 \Rightarrow y = 10 - x$

x	0	1
y	10	9



9. You are about to choose new products for your company. There are 10 proposed products but you can finance launching of only 3 of them. In how many different ways can you choose 3 out of 10 products?

Solution: Here the order is not important, only which three particular products you choose, so we use combinations to find the number of possibilities.

$$C_{10,3} = \binom{10}{3} = \frac{10!}{3!7!} = \frac{10 \cdot 9 \cdot 8 \cdot 7!}{3!7!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 5 \cdot 3 \cdot 8 = 120$$

10. If a school has lockers with 50 numbers on each combination lock, how many possible combinations using three numbers are there?

Solution: Recognize that n, or the number of objects is 50 and that r, or the number of objects taken at one time is 3. Plug those numbers in the permutation formula.

$$P_{50,3} = \frac{50!}{(50-3)!} = \frac{50 \cdot 49 \cdot 48 \cdot 47!}{47!} = 50 \cdot 49 \cdot 48 = 117600$$

11. If \$1000 is invested at a rate 7.5% interest per year, compounded quarterly, find the amount in the account at the end of 5 years.

(a) \$1449.95 (b) \$1448.30 (c) \$31093.70 (d) \$5385.68

Solution:

$$A = P(1 + i/t)^{nt}$$
$$A = 1000(1 + 0.075/4)^{4 \cdot 5}$$
$$A = 1000 \cdot 1.01875^2 0 = \$1449.95 \quad \rightarrow \quad (a)$$

12. Consider a 15 year mortgage where the principal amount P is 500,000CZK and the annual interest rate is 8%. What is the monthly payment to be paid?

Solution: The number of monthly payments is

$$n = 15 \text{ years} \times 12 \text{ months} = 180 \text{ months}$$

The monthly interest rate is

$$i = \frac{8\% \text{ per year}}{12 \text{ monhs per year}} = 0.66\% \text{ per month}$$

$$PV(A) = A\frac{1 - \frac{1}{(1+i)^n}}{i} \Rightarrow A = PV(A)\frac{i}{1 - \frac{1}{(1+i)^n}} = PV(A)\frac{i(1+i)^n}{(1+i)^n - 1}$$

$$A = 500000\frac{0.0066(1 + 0.0066)^{180}}{(1 + 0.0066)^{180} - 1} = 4755 \text{ CZK per month.}$$