AAU - Business Mathematics I

Problem set #4, Due May 13, 2010 - Suggested Solution

1. Find inverse matrix to the following matrices and check that your answer is correct (their product is identity matrix):

(a) $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$

Solution:

(a)

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right) \ / \div 2 \ \sim \left(\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ -3 & 2 & 0 & 1 \end{array}\right) \ \begin{matrix} / \times 3 \\ \swarrow \end{matrix} \ \sim \left(\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 0 & 1/2 & 3/2 & 1 \end{array}\right) \begin{matrix} \nwarrow \\ \nearrow \end{matrix} \sim$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1/2 & 3/2 & 1 \end{array}\right) \ / \times 2 \ \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{array}\right)$$

Check:

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} (2,-1)\begin{pmatrix} 2 \\ 3 \end{pmatrix} & (2,-1)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ (-3,2)\begin{pmatrix} 2 \\ 3 \end{pmatrix} & (-3,2)\begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b)

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{array}\right) \ \begin{matrix} / \times (-4) \\ \swarrow \end{matrix} \ \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{array}\right) \ / \times (-1) \ \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 4 & -1 \end{array}\right) \ \begin{matrix} \nwarrow \\ / \times (-2) \end{matrix} \ \sim$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right)$$

Check:

$$\begin{pmatrix} 1 & 2 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} -7 & 2 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} (1,2)\begin{pmatrix} -7 \\ 4 \end{pmatrix} & (1,2)\begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ (4,7)\begin{pmatrix} -7 \\ 4 \end{pmatrix} & (4,7)\begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(c)

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{cc} /\times(-1) & /\times(-2) \\ \swarrow & \downarrow \\ & \swarrow \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right) \begin{array}{c} \\ \nwarrow \\ /\times(-1) \end{array} \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

Check:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2. Find the following determinants:

(a) $$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

(b) $$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

(c) $$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{vmatrix}$$

Solution:

(a)

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \times 1 - 0 \times 0 = 1$$

(b)

$$\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 3 \times 4 - 1 \times 2 = 10$$

(c)

$$\begin{vmatrix} 1 & 2 & 3 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} -2 & 1 \\ 3 & -1 \end{vmatrix} =$$

$$= 1 + 4 - 3 = 2$$

3. Solve the system in part (a) using matrix method; system in part (b) using Gauss elimination, and inverse matrix, and system in part (c) using Cramer's rule:

(a) $3x - 2y = -1$
$x + y = 3$

(b) $-2x + 3y = 2$
$x + y = 4$

(c) $x - 2y + z = 7$
$3x - y - z = -2$
$x - y + 2z = 6$

Solution:

(a)

Matrix method:

$$\left(\begin{array}{cc|c} 3 & -2 & -1 \\ 1 & 1 & 3 \end{array}\right) \begin{array}{c}\nwarrow \\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 3 & -2 & -1 \end{array}\right) \begin{array}{c}\times(-3) \\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -5 & -10 \end{array}\right) \div(-5) \sim$$

$$\sim \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right) \begin{array}{c}\nwarrow \\ \times(-1)\end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) \Rightarrow \begin{array}{c} x = 1 \\ y = 2 \end{array}$$

(b)

Gauss elimination method:

$$\left(\begin{array}{cc|c} -2 & 3 & 2 \\ 1 & 1 & 4 \end{array}\right) \begin{array}{c}\nwarrow \\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 4 \\ -2 & 3 & 2 \end{array}\right) \begin{array}{c}\times(2) \\ \swarrow\end{array} \sim \left(\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 5 & 10 \end{array}\right)$$

Second row of the matrix:

$5y = 10 \quad \Rightarrow \quad y = 2$

First row of the matrix:

$x + y = 4 \quad \Rightarrow \quad x + 2 = 4 \quad \Rightarrow \quad x = 2$

Inverse matrix:

$$A = \begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -2 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \begin{array}{c}\nwarrow \\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ -2 & 3 & 1 & 0 \end{array}\right) \begin{array}{c}\times(2) \\ \swarrow\end{array} \sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 5 & 1 & 2 \end{array}\right) \div(5) \sim$$

$$\sim \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1/5 & 2/5 \end{array}\right) \begin{array}{c}\nwarrow \\ \times(-1)\end{array} \sim \left(\begin{array}{cc|cc} 1 & 0 & -1/5 & 3/5 \\ 0 & 1 & 1/5 & 2/5 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1/5 & 3/5 \\ 1/5 & 2/5 \end{pmatrix}$$

Check:

$$AA^{-1} = \begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} \end{pmatrix} = \begin{pmatrix} -2 \times (-\frac{1}{5}) + 3 \times \frac{1}{5} & -2 \times \frac{3}{5} + 3 \times \frac{2}{5} \\ 1 \times (-\frac{1}{5}) + 1 \times \frac{1}{5} & 1 \times \frac{3}{5} + 1 \times \frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$AX = B \longrightarrow \begin{pmatrix} -2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$X = A^{-1}B \longrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1/5 & 3/5 \\ 1/5 & 2/5 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \times 2 + \frac{3}{5} \times 4 \\ \frac{1}{5} \times 2 + \frac{2}{5} \times 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \begin{matrix} x = 2 \\ y = 2 \end{matrix}$$

(c)

Cramer's rule:

$$x = \frac{\begin{vmatrix} 7 & -2 & 1 \\ -2 & -1 & -1 \\ 6 & -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 1 \\ 3 & -1 & -1 \\ 1 & -1 & 2 \end{vmatrix}} = \frac{-9}{9} = -1, \quad y = \frac{\begin{vmatrix} 1 & 7 & 1 \\ 3 & -2 & -1 \\ 1 & 6 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 1 \\ 3 & -1 & -1 \\ 1 & -1 & 2 \end{vmatrix}} = \frac{-27}{9} = -3$$

$$z = \frac{\begin{vmatrix} 1 & -2 & 7 \\ 3 & -1 & -2 \\ 1 & -1 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 1 \\ 3 & -1 & -1 \\ 1 & -1 & 2 \end{vmatrix}} = \frac{18}{9} = 2$$