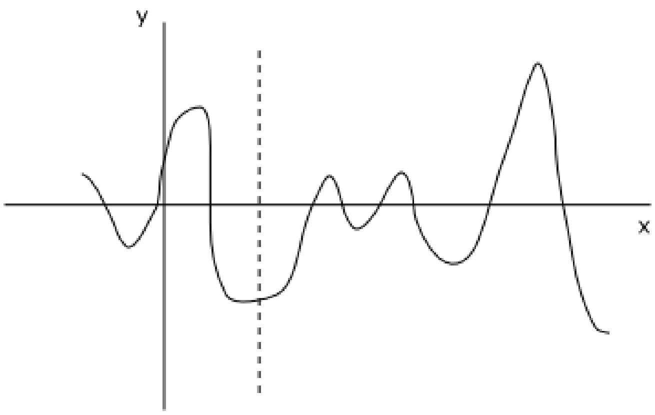
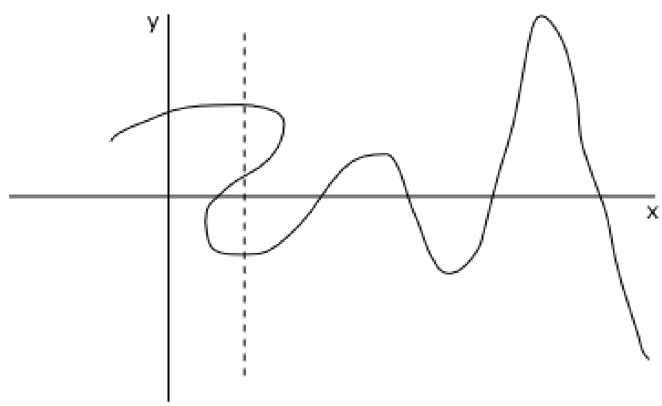
AAU - Business Mathematics I

Lecture #8, April 1, 2010

10 Functions

In mathematics, a function is a relation between a given set of elements (the domain) and another set of elements (the range), which associates each element in the domain with **exactly one** element in the range. The elements so related can be any kind of things (words, objects, qualities) but are typically mathematical quantities, such as real numbers. To decide if a relation is also a function we can apply so called **vertical line test** - a test which takes a vertical line at any horizontal position. If at any position there is a crossing at more than one point, then the relation fails the vertical line test, and the relation is not a function. Below we have an example of a function (picture on the left) where any vertical line crosses the graph of the function at exactly one point and an example of relation that is not a function (picture on the right) where there exist a vertical line (at least one) for which there is more than one crossing (there are three of them).





There are many ways to represent or visualize functions: a function may be described by a formula, by a plot or graph, by an algorithm that computes it, or by a description of its properties. Sometimes, a function is described through its relationship to other functions (for example, inverse functions). In applied disciplines, functions are frequently specified by tables of values or by formulas.

The symbol for the input to a function is often called the independent variable or argument and is often represented by the letter x or, if the input is a particular time, by the letter t. The symbol for the output is called the dependent variable or value and is often represented by the letter y. The function itself is most often called f, and thus the notation $y = f(x)$ indicates that a function named f has an input named x and an output named y.

Functions with multiple inputs and outputs

The concept of function can be extended to an object that takes a combination of two (or more) argument values to a single result. This intuitive concept is formalized by a function whose domain is the Cartesian product of two or more sets. For example, consider the function that associates two integers to their product: $f(x, y) = x \times y$.

10.1 Domain, Image, Range

In general we use term *function* to describe relationship between independent variable x and dependent variable y and we write $y = f(x)$.

Domain: the set of numbers that are permitted to replace the independent variable x (no "0" in the denominator, no negative number under the square root). *Note:* Always use the original expression to determine the domain of the expression, not the one after simplification!

Image: value of y into which an x value is mapped.

Range: set of all values that the y variable can take.

When no specification is given, it is to be understood that the domain will only include number for which a function makes an economic sense.

Example: The total cost C of a firm per day is a function of its daily output Q: $C = 150 + 7Q$. The firm has a capacity limit of 100 units of output per day. What are the domain and the range of the cost function?

Q can vary only between 0 and 100, hence the domain is:

$$\{Q|0 \leq Q \leq 100\}$$

The minimum value of C is 150 (for $Q = 0$) and maximum value of C is 850 (for $Q = 100$). Thus the range is

$$\{C|150 \leq C \leq 850\}$$

Note: extreme values of the range may not always occur where the extreme values of the domain are attained. It is so in this example because of the linearity of relationship between Q and C.

Problem: Determine if the given relation is function or not. Give its domain and range.

$\{(1, 2), (1, -2), (2, 3), (3, 4)\}$

Solution: Does every first element (or input) correspond with **exactly one** second element (or output)? In this case, the answer is no. The input value of 1 goes with two output values, 2 and -2. It only takes one input value to associate with more than one output value to be invalid as a function. So, this relation would **not** be an example of a **function**.

Domain: We need to find the set of all input values. In terms of ordered pairs, that correlates with the first component of each one. So, what do you get for the domain? $\{1, 2, 3\}$. (Note that if any value repeats, we only need to list it one time.)

Range: We need to find the set of all output values. In terms of ordered pairs, that correlates with the second component of each one. So, what do you get for the range? $\{2, -2, 3, 4\}$.

Problem: Determine if the given relation is function or not. Give its domain and range.

$\{(5,10),(10,10),(15,10)\}$

Solution: Does every first element (or input) correspond with **exactly one** second element (or output)? In this case, the answer is yes. 5 only goes with 10, 10 only goes with 10, and 15 only goes with 10. Note that a relation can still be a function if an output value associates with more than one input value as shown in this example. But again, it would be a no no the other way around, where an input value corresponds to two or more output values. So, this relation would be an example of a **function**.

Domain: We need to find the set of all input values. In terms of ordered pairs, that correlates with the first component of each one. So, what do you get for the domain? $\{5,10,15\}$.

Range: We need to find the set of all output values. In terms of ordered pairs, that correlates with the second component of each one. So the range is $\{10\}$. Note that if any value repeats, we only need to list it one time.

Problem: Decide whether y is a function of x: $3x - 2y = 4$

Solution: To check if y is a function of x, we need to solve for y first and then check to see if there is only one output for every input.

$y = -2 + \frac{3}{2}x$

Note that any value you would plug in for x would produce only one value for y. That means that y is a function of x.

Problem: Decide whether y is a function of x: $y^2 = x + 1$

Solution: To check if y is a function of x, we need to solve for y and then check to see if there is only one output for every input.

$y = \pm\sqrt{x+1}$

Would we get one value for y if we plug in any value for x? No. For example, if our input value x is 3, then our output value y could either be 2 or -2. Note that I could have picked an infinite number of examples like this one. You only need to show one example where the input value is associated with more than one output value to disqualify it from being a function. This means that at least one input value is associated with more than one output value, so by definition, y is not a function of x.

Constant function

If the function is constant, that means that the functional value never changes, it is always equal to that constant.

Example: Find the functional values $h(0)$ and $h(2)$ of the constant function $h(x) = -5$. $h(0) = -5$, $h(2) = -5$. Since there is no x to plug into, the functional value is going to be -5 no matter what x is.

10.2 Inverse function

If f is a function from X to Y then an inverse function for f, denoted by f^{-1}, is a function in the opposite direction, from Y to X, with the property that a round trip (a composition) returns each element to itself. The inverse of a function literally undoes the action of the function if $f(2) = 3$, then $f^{-1}(3) = 2$. Not every function has an inverse function. Or put differently, every function has an inverse, but not every inverse will be a function. The inverse may fail the vertical line test. Graphically, inverse function is a mirror image with respect to identity function.

As a simple example, if $f$converts a temperature in degrees Celsius C to degrees Fahrenheit F, the function converting degrees Fahrenheit to degrees Celsius would be a suitable f^{-1}.

$$F = f(C) = \frac{9}{5}C + 32$$
$$C = f^{-1}(F) = \frac{5}{9}(F - 32)$$

For example, 5C is 41F, because $\frac{9}{5}5 + 32 = 41$. And similarly, 41F is 5C, because $\frac{5}{9}(41 - 32) = 5$.

Problem: Find the inverse of the function y if:

$$(a) \quad y = 2x + 3$$
$$(b) \quad y = x^2 - 1$$

Solution: We find inverse functions in two steps: first, we switch x for y and second we rearrange terms in the equation such that again y is on the left hand side and everything else on the right hand side.

$$(a) \quad y = 2x + 3 \quad \rightarrow \quad x = 2y + 3 \quad \Rightarrow \quad y = \frac{x-3}{2}$$
$$(b) \quad y = x^2 - 1 \quad \rightarrow \quad x = y^2 - 1 \quad \Rightarrow \quad y = \pm\sqrt{x+1}$$