Exponential function properties:

$$a^x a^y = a^{x+y} \qquad (a^x)^y = a^{xy} \qquad (ab)^x = a^x b^x$$
$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \frac{a^x}{a^y} = a^{x-y}$$
$$a^x = a^y \text{ if and only if } x = y$$
$$\text{for } x \neq 0, a^x = b^x \text{ if and only if } a = b$$
$$0^x = 0, \quad 1^x = 1, \quad x^0 = 1 \text{ for all } x$$

Example: Simplify:

(a) $\left(\frac{4}{3}\right)^2 \frac{3^3}{4}$

$$\left(\frac{4}{3}\right)^2 \frac{3^3}{4} = \frac{4^2}{3^2}\frac{3^3}{4} = 4^{(2-1)}3^{(3-2)} = 4 \times 3 = 12$$

(b) $\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3}$

$$\left(\frac{2a}{3b}\right)^2 \frac{5}{2^3} = \frac{4a^2}{9b^2}\frac{5}{2^3} = \frac{5}{18}a^2b^{-2}$$

Properties of logarithmic functions: If b, M, and N are positive real numbers, $b \neq 1$, and p and x are real numbers, then:

$$\log_b 1 = 0 \qquad \log_b MN = \log_b M + \log_b N$$
$$\log_b b = 1 \qquad \log_b \frac{M}{N} = \log_b M - \log_b N$$
$$\log_b b^x = x \qquad \log_b M^p = p \log_b M$$
$$b^{\log_b x} = x, x > 0 \qquad \log_b M = \log_b N \text{ iff } M = N$$

Example: How much do you have to invest if you want to have \$ 5000 in 3 years at 5 % compounded annually?

$$A = P(1+r)^t$$
$$5000 = P(1+0.05)^3$$
$$P = \frac{5000}{1.05^3} = 4320$$

9.2 Solving Exponential and Logarithmic Equations

There are two basic methods of solving exponential equations:

First: if it is possible to transform the equation to one with the same base on both sides of the equation, we just compare the exponents.

Second: if it is not possible to transform the equation to one with the same base on both sides of the equation, we use logarithm.

• Transformation to the Same Base on Both Sides

Example: Solve $4^{x-3} = 16$

$$(2^2)^{x-3} = 16$$
$$2^{2(x-3)} = 2^4$$
$$2x - 6 = 4$$
$$2x = 10$$
$$x = 5$$

Exercise 3: Solve $27^{x+1} = 9$

Example: Solve $7^{x^2} = 7^{2x+3}$

$$7^{x^2} = 7^{2x+3}$$
$$x^2 = 2x + 3$$
$$x^2 - 2x - 3 = 0$$
$$D = b^2 - 4ac = 4 - 4 \times 1 \times (-3) = 16$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{2 \pm 4}{2} = -1, 3$$

Exercise 4: Solve $4^{5x-x^2} = 4^{-6}$

Example: Find y: $y = \log_3 27$

$$y = \log_3 27 \Leftrightarrow 3^y = 27$$
$$y = 3$$

Example: Find y: $y = \log_9 27$

$$y = \log_9 27 \Leftrightarrow 9^y = 27$$
$$(3^2)^y = 3^3$$
$$3^{(2y)} = 3^3$$
$$2y = 3$$
$$y = 3/2$$

Example: Find x: $\log_2 x = -3$

$$\log_2 x = -3 \Leftrightarrow x = 2^{(-3)}$$
$$x = \frac{1}{2^3}$$
$$x = 1/8$$

Example: Find b: $\log_b 100 = 2$

$$\log_b 100 = 2 \Leftrightarrow b^2 = 100$$
$$b = \sqrt{100}$$
$$b = 10$$

Note: In general, $b^2 = 100$ implies that $b = \pm\sqrt{100} = \pm 10$. But base of logarithm can only be positive number so we can ignore negative solution -10.

Examples:

$$\log_e 1 = 0 \qquad \log_{10} 10 = 1$$
$$10^{\log_{10} 7} = 7 \qquad \log_e e^{2x+1} = 2x + 1$$
$$e^{\log_e x^2} = x^2$$

- If you know that $\log_e 3 = 1.1$ and $\log_e 7 = 1.95$, find $\log_e(\frac{7}{3})$ and $\log_e \sqrt[3]{21}$.

$$\log_e\left(\frac{7}{3}\right) = \log_e 7 - \log_e 3 = 1.95 - 1.1 = 0.85$$
$$\log_e \sqrt[3]{21} = \log_e 21^{1/3} = 1/3 \log_e 21 = 1/3 \log_e(3 \times 7) = 1/3[\log_e 3 + \log_e 7] =$$
$$= 1/3[1.1 + 1.95] = 1/3 \times 3.05 \approx 1.02$$

- Find x: $\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$

$$\log_b x = \frac{2}{3} \log_b 27 + 2 \log_b 2 - \log_b 3$$
$$\log_b x = \log_b 27^{2/3} + \log_b 2^2 - \log_b 3$$
$$\log_b x = \log_b[27^{2/3} \times 2^2/3]$$
$$\log_b x = \log_b[9 \times 4/3]$$
$$x = \frac{9 \times 4}{3} = 12$$

Exercise 5: Find x: $2 \log_5 x = \log_5(x^2 - 6x + 2)$; $\log_e(x + 8) - \log_e x = 3 \log_e 2$; $(\ln x)^2 = \ln x^2$, where ln is a short notation for $\log_e$

• Using Logarithm

• $2^{3x-2} = 5$ This equation can not be transformed to get the equation with the same base on each side, therefore we will use logarithm to solve it.

$$2^{3x-2} = 5 / \log_{10}$$
$$\log_{10} 2^{3x-2} = \log_{10} 5$$
$$(3x-2)\log_{10} 2 = \log_{10} 5$$
$$(3x-2) = \frac{\log_{10} 5}{\log_{10} 2}$$
$$x = \frac{1}{3}\left(2 + \frac{\log_{10} 5}{\log_{10} 2}\right)$$

Example: You deposit 20000 CZK on a bank account with an interest rate of 10%. How many years does it take to get 29282 CZK back?

Solution:

$$20000(1+0.1)^t = 29282$$
$$1.1^t = \frac{29282}{20000} = 1.4641$$
$$\log 1.1^t = \log 1.4641$$
$$t \log 1.1 = \log 1.4641$$
$$t = \frac{\log 1.4641}{\log 1.1} = \frac{0.16557}{0.0414} \approx 4$$

It takes 4 years to get 29282 back.

Example: There are two options for investing money. If you invest 10000 Bank A offers to give you back 13382 after 5 years. Bank B offers 8% interest rate compounded annually if you leave your money there for 5 years. Which option should you choose?

Solution: We have different types of information about these two options. To be able to compare them we need to know either what is the interest rate in Bank A or what will be the amount of money that we get in Bank B. We will find both.

$$10000(1+r)^5 = 13382$$
$$(1+r)^5 = \frac{13382}{10000} = 1.3382$$
$$5\log(1+r) = \log 1.3382$$
$$\log(1+r) = \frac{1}{5}\log 1.3382$$
$$\log(1+r) = \log 1.3382^{\frac{1}{5}}$$
$$r = 1.3382^{\frac{1}{5}} - 1 = 0.06 = 6\%$$

This means that Bank A offers lower interest rate and hence we should choose Bank B.

If we deposit 10000 in Bank B after 5 years we will get:

$$10000(1 + r)^5 = 10000(1 + 0.08)^5 = 14693$$

This means that in Bank B we get more money back compared to Bank A and hence again we should choose bank B.

9.3 Answers

Exercise 1: $27 = 3^3$; $6 = 36^{1/2}$; $1/9 = 3^{-2}$

Exercise 2: $\log_4 16 = 2$; $\log_{27} 3 = 1/3$; $\log_{16} 4 = 1/2$

Exercise 3: $x = -1/3$

Exercise 4: $x = -1, 6$

Exercise 5: $x = 1/3$; $x = 8/7$; $x = 1, e^2$