Review

5.3 Graphical Representation

Problem: Solve the following system numerically and graphically:

$x + y = 5$
$2x - y = 1$

Numerical solution to this system is $x = 2$ and $y = 3$.

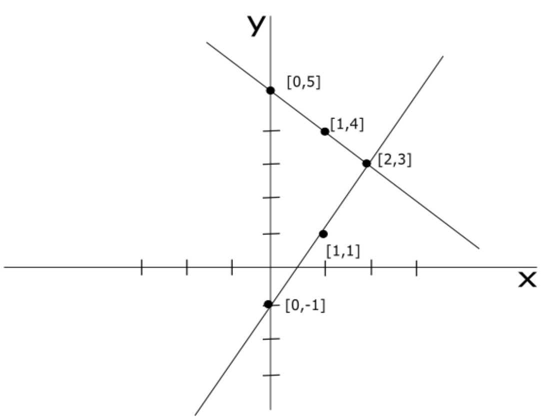
To find graphical solution we first need to draw both lines:

$x + y = 5$ or alternatively $y = 5 - x$

x	0	1
y	5	4

$2x - y = 1$ or alternatively $y = 2x - 1$

x	0	1
y	-1	1



The two lines intercept in point [2,3].

Generally, the system of two equations and two variables can have no solution, exactly one solution (see the example above) or infinitely many solutions.

Problem: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -4$

Solution:

$$\begin{aligned}
&3x - y = 2 && \Rightarrow \quad y = 3x - 2\\
&-9x + 3y = -4 &&\\
\hline
&-9x + 3(3x - 2) = -4 &&\\
&-9x + 9x - 6 = -4 &&\\
&-6 = -4 &&
\end{aligned}$$

The last equality does not hold for any values of x and y. This means that this system does not have any solution.

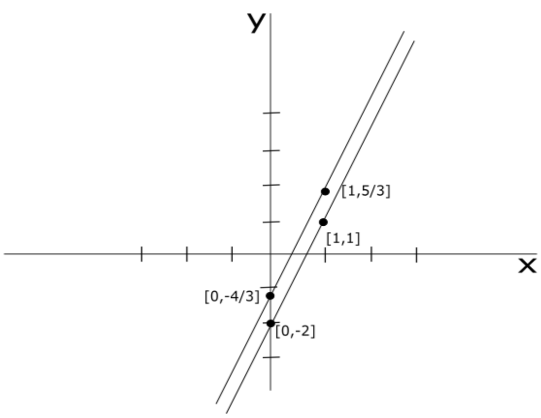
Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -4$ or alternatively $y = \frac{1}{3}(9x - 4)$

x	0	1
y	$-4/3$	$5/3$



From the picture we see that the two lines are parallel, i.e. they do not intercept in any point. That is the reason why the system does not have any solution.

Problem: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -6$

Solution:

$$3x - y = 2 \qquad \Rightarrow \quad y = 3x - 2$$

$-9x + 3y = -6$

$-9x + 3(3x - 2) = -6$
$-9x + 9x - 6 = -6$
$-6 = -6$

The last equality holds for all values of x and y (-6 = -6 no matter what are the values of x and y). This means that this system has infinitely many solutions.

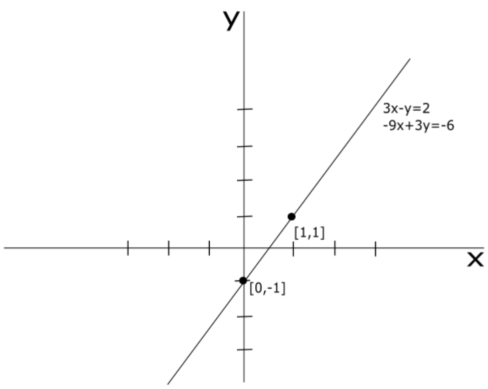
Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -6$ or alternatively $y = \frac{1}{3}(9x - 6) = 3x - 2$

Note that both lines are represented by the same equation. This means that the two lines coincide and therefore there are infinitely mane points where these two lines intercept and hence the system has infinitely many solutions.



5.4 Changes in Systems of Linear Equations

In this section we look at what happens with the solution to the system of equations if one of them changes.

Problem: Consider the following supply and demand function for beef.

Supply: $P = 4Q$

Demand: $P = 150 - Q$

(a) Find the equilibrium

(b) Suppose now that the government impose taxes \$5 per unit sold. Find new equilibrium

(c) Illustrate the situation in (b) graphically. Does it matter whether the tax is imposed on the producers or the consumers? Explain.

Solution:

(a) To solve for market equilibrium we need to find solution to the following system:

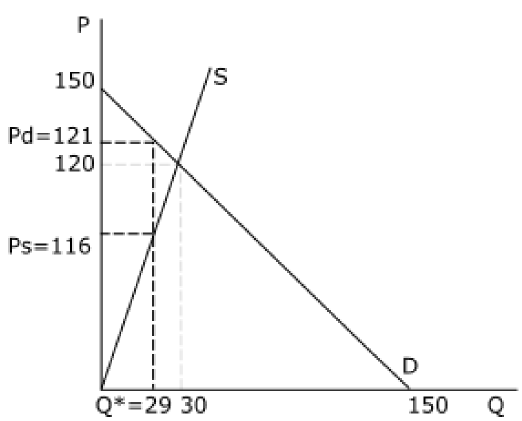
$$P = 4Q$$
$$P = 150 - Q$$
$$4Q = 150 - Q \quad \Rightarrow \quad Q^* = 30, P^* = 120$$

(b) We need to find price paid by buyers P_d, price received by sellers P_s, and quantity Q. Irrespective of who pays the tax (producer or consumer) the difference between P_d and P_s is \$5. So $P_s = P_d - 5$. If the tax is officially paid by producer, the system of two equations is as follows:

$$P_d - 5 = 4Q$$
$$P_d = 150 - Q$$
$$4Q + 5 = 150 - Q \quad \Rightarrow \quad Q^* = 29, \quad P_d^* = 121, \quad P_s^* = 116$$

(c) No, it does not matter. Look at the graph of the market, and put the tax on the graph. The tax puts a wedge between the price paid by buyers and the price received by sellers. If producer pays the tax we leave the demand function the same and shift supply function upwards. On the other hand, if consumer pays the tax we leave supply function unchanged and shift demand function downwards. One way or another the result is the same.

Economic explanation: No matter who formally pays the tax, the costs of the tax are borne by both sides of the transaction, and who pays what share depends on the relative elasticities (slopes of demand and supply curve). If the demand is relative inelastic in comparison to supply (the reaction of consumers on change in price is subtle) most of the tax will be paid by consumers. If on the other hand consumers are very sensitive to changes in price and producers are not, most of the tax will be paid by producers.



5.5 Answers

Exercise 1: $x = 5$ and $y = 1$.

Exercise 2: Infinitely many solutions. All combinations of x and y such that $x + y = 1$.

6 Linear and Rational Inequalities

6.1 Linear Inequalities in One Variable

$3(x-5) \geq 5(x+7)$, $-4 \leq 3-2x < 7$, ...

Properties of inequality:

1. if $a < b$ then $a+c < b+c$ — addition
2. if $a < b$ then $a-c < b-c$ — subtraction
3. if $a < b$ then $ca < cb$ for $c > 0$
   $ca > cb$ for $c < 0$ — multiplication
4. if $a < b$ then $a/c < b/c$ for $c > 0$
   $a/c > b/c$ for $c < 0$ — division
5. if $a < b$ and $b < c$ then $a < c$ — transitivity

Problem: Solve $2(2x+3) - 10 < 6(x-2)$

Solution: Solving linear inequalities is almost the same as solving linear equalities. Only if you multiply or divide by a negative number, change the sign!!!

$$\begin{aligned} 2(2x+3)-10 &< 6(x-2) \\ 4x+6-10 &< 6x-12 \\ -2x &< -8 \qquad /(-2) \qquad \text{Change the sign of the inequality!} \\ x &> 4 \end{aligned}$$

The inequality holds for all $x \in (4, \infty)$.

Problem: Solve $-6 < 2x+3 \leq 5x-3$

Solution: We divide this problem into two parts and solve simultaneously these two inequalities: $-6 < 2x+3$ and $2x+3 \leq 5x-3$

$$\begin{array}{rl} -6 < 2x+3 & 2x+3 \leq 5x-3 \\ -9 < 2x & -3x \leq -6 \\ -9/2 < x & x \geq 2 \end{array}$$

The inequality holds for all $x \in (-9/2, \infty)$ and at the same time $x \in [2, \infty)$. So the solution is $x \in [2, \infty)$.

Exercise 1: Solve $1 < 3x-5 \leq 2x+5$

Problem: Apple Inc. produces 100% apple juice. Its production function is $J \leq 12A - 4$, where J is quantity of juice in liters and A is quantity of apples in kilograms. For what quantity does Apple Inc. produce at most 20 liters of juice?

Solution:

$$\begin{aligned} J &\leq 20 \\ 12A - 4 &\leq 20 \\ 12A &\leq 24 \\ A &\leq 2 \end{aligned}$$

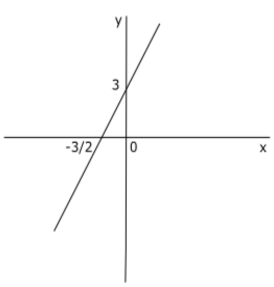
In order to produce at most 20 liters of juice, Apple Inc. can use at most 2 kilograms of apples.

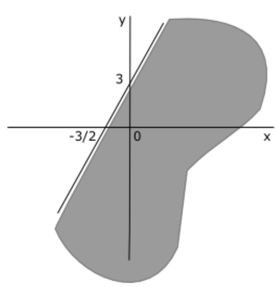
6.2 Linear Inequalities in Two Variables

Graphing linear inequalities on the number line: For instance, graph $x > 2$. First, draw the number line, find the "equals" part (in this case, $x = 2$), mark this point with the appropriate notation (an open dot, indicating that the point $x = 2$ wasn't included in the solution), and then you'd shade everything to the right, because "greater than" meant "everything off to the right". The steps for graphing two-variable linear inequalities are very much the same.

Problem: Graph the solution to $y \leq 2x + 3$.

Solution: Just as for number-line inequalities, first find the "equals" part. For two-variable linear inequalities, the "equals" part is the graph of the straight line; in this case, that means the "equals" part is the line $y = 2x + 3$ which is depicted on the left hand picture below:





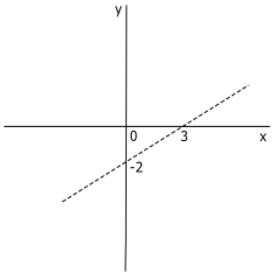
We have the graph of the "or equal to" part (it's just the line); now we need "y less than" part. In other words, we need to shade one side of the line or the other. If we need y LESS THAN the line, we want to shade below the line as it is depicted on the right hand picture above.

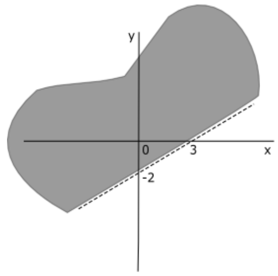
Problem: Graph the solution to $2x - 3y < 6$.

Solution: First, solve for y:

$$\begin{aligned} &2x - 3y < 6 \\ &-3y < -2x + 6 \\ &y > \frac{2}{3}x - 2 \end{aligned}$$

Now we need to find the "equals" part, which is the line $y = \frac{2}{3}x - 2$. Note, that here we have strict inequality therefore the line itself does not belong to the set of solutions and hence is graphed as a dashed line. It looks like the left hand picture below.





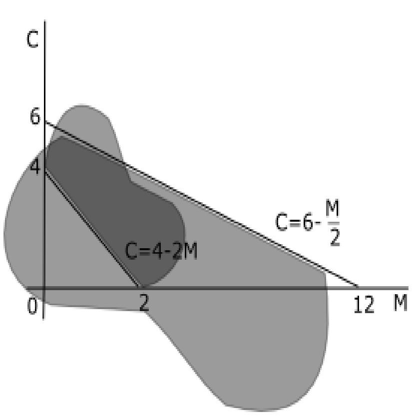
By using a dashed line, we know where the border is, but we also know that the border isn't included in the solution. Since this is a "y greater than" inequality, we need to shade above the line, so the solution looks like the right hand picture above.

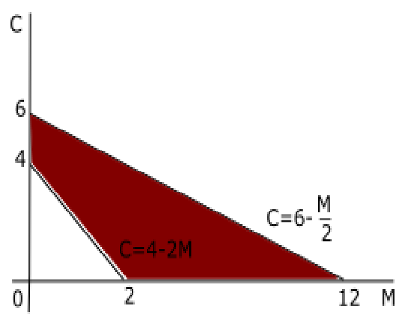
Problem: A milk company faces the following problem. It's production function is $M + 2C \leq 12$ where M is the amount of milk produced and C is the amount of cheese. Price of the cheese is 5 and the price of milk is 10. Company wants to reach a level of revenue of at least 20. Draw the set of all possible combinations of milk and cheese.

Solution: This problem is about solving two inequalities, graphing them and finding their intercept.

$$M + 2C \leq 12 \quad \Rightarrow \quad C \leq 6 - \frac{M}{2} \quad \text{Production function}$$
$$P_M M + P_C C \geq 20 \quad \Rightarrow \quad 10M + 5C \geq 20 \quad \Rightarrow \quad C \geq 4 - 2M \quad \text{Revenue requirement}$$

The set of all possible combinations of Milk and Cheese is depicted in red on the graph below.





6.3 Rational Inequalities:

$$\frac{x+1}{x-3} > 1, \quad \frac{x+1}{x^2 - 3x + 5} < 0, \quad \frac{x^2 - x - 1}{2x^2 + 4x - 3} > 5, \quad \ldots$$

Problem: Solve $\frac{2x}{x+2} > 1$

Solution: Note that we can **not** just multiply the inequality by $(x+2)$ and solve the resulting linear inequality $2x > x+2$, because we do not know whether $x+2$ is positive or negative and hence we do not know whether we should change the sign of inequality or not. So we distinguish two cases:

• $x+2 > 0 \quad \Rightarrow \quad x > -2 \quad \ldots \quad 2x > x+2 \quad \Rightarrow \quad x > 2$

• $x+2 < 0 \quad \Rightarrow \quad x < -2 \quad \ldots \quad 2x < x+2 \quad \Rightarrow \quad x < 2$

Alternative solution:

$$\frac{2x}{x+2} > 1$$
$$\frac{2x}{x+2} - 1 > 0$$
$$\frac{x-2}{x+2} > 0$$

Here, we need the ratio of two numbers be positive. This is the case if both numerator and denominator are positive or if both numerator and denominator are negative:

• $x-2 > 0$ and $x+2 > 0 \quad \Leftrightarrow \quad x > 2$ and $x > -2 \quad \Rightarrow x > 2$

OR

• $x-2 < 0$ and $x+2 < 0 \quad \Leftrightarrow \quad x < 2$ and $x < -2 \quad \Rightarrow x < -2$

Hence the solution to the problem is $x \in (-\infty, -2) \cup (2, \infty)$.

Exercise 2: Solve $\frac{x^2-3x-10}{1-x} \geq 2$.

6.4 Answers

Exercise 1: $x \in (2, 10]$

Exercise 2: $x \in (-\infty, -3] \cup (1, 4]$

7 Quadratic Equations, Inequalities

7.1 Quadratic Equations

Quadratic equation has the following form: $ax^2 + bx + c = 0$

Equations with the second power of a variable; e.g.

$x^2 - 6x + 9 = 0$

$y^2 + 3y - 1 = 2y^2 - 4y - 3$

7.1.1 Solving by Square Root

Problem: Solve $3x^2 - 27 = 0$.

Solution:

$$\begin{aligned}
&3x^2 - 27 = 0 \\
&3x^2 = 27 \\
&x^2 = 9 \\
&x = \pm\sqrt{9} \\
&x_{1,2} = \pm 3
\end{aligned}$$

Note: if $a^2 = b$, then $a \neq \sqrt{b}$!!!, but $a = \pm\sqrt{b}$

7.1.2 Solving by Quadratic Formula

Problem: Solve $ax^2 + bx + c = 0$.

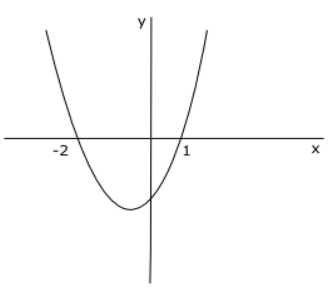
Solution:

$$\begin{aligned}
&ax^2 + bx + c = 0 \\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{where} \quad D = b^2 - 4ac \\
&x^2 - x - 6 = 0 \\
&D = b^2 - 4ac = 1 - 4 \times 1 \times (-6) = 25 \\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 5}{2} = -2, 3
\end{aligned}$$

Problem: Solve the following equation: $x^2 + x - 2 = 0$.

Solution:

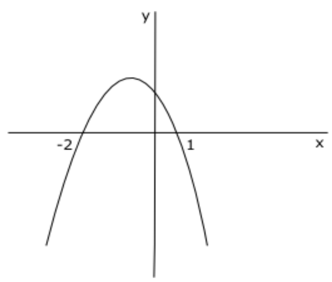
$$\begin{aligned}
&x^2 + x - 2 = 0 \\
&D = b^2 - 4ac = 1 - 4 \times 1 \times (-2) = 9 \\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm 3}{2} = 1, -2
\end{aligned}$$



Problem: Solve the following equation: $-x^2 - x + 2 = 0$.

Solution:

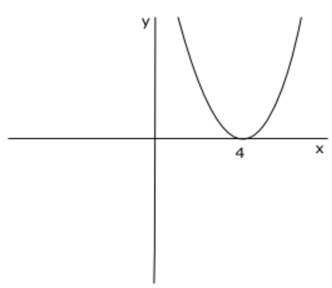
$$
\begin{aligned}
&-x^2 - x + 2 = 0 \\
&D = b^2 - 4ac = (-1)^2 - 4 \times (-1) \times 2 = 9 \\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{-2} = 1, -2
\end{aligned}
$$



Problem: Solve the following equation: $x^2 - 8x + 16 = 0$.

Solution:

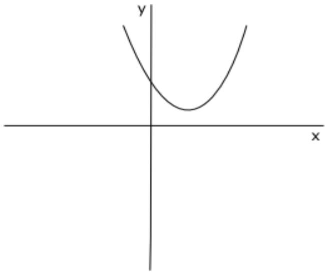
$$
\begin{aligned}
&x^2 - 8x + 16 = 0 \\
&D = b^2 - 4ac = (-8)^2 - 4 \times 1 \times 16 = 0 \\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{8 \pm 0}{2} = 4
\end{aligned}
$$



Problem: Solve the following equation: $x^2 - 4x + 10 = 0$.

Solution:

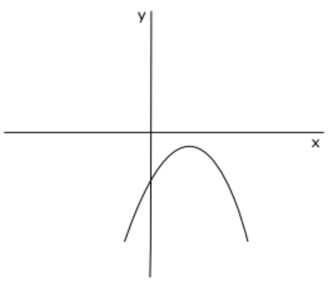
$$
\begin{aligned}
&x^2 - 4x + 10 = 0 \\
&D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 16 = 16 - 64 = -48 \\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{8 \pm \sqrt{-48}}{2} \quad \text{the equation does not have any solutions}
\end{aligned}
$$



Problem: Solve the following equation: $-2x^2 + 8x - 20 = 0$.

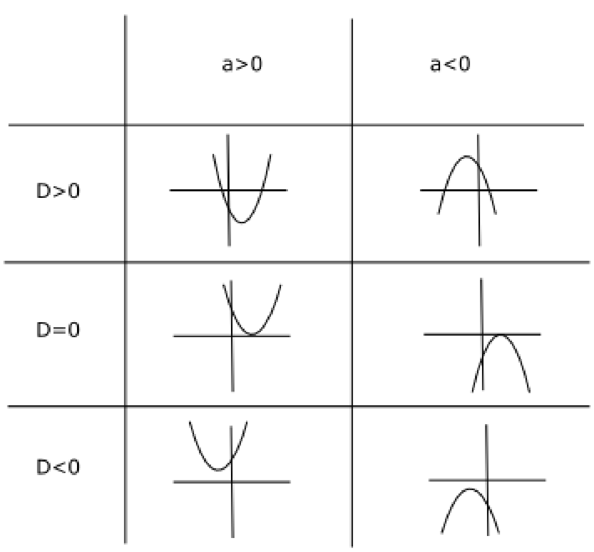
Solution:

$$-2x^2 + 8x - 20 = 0$$
$$D = b^2 - 4ac = 8^2 - 4 \times (-2) \times (-20) = 64 - 160 = -96$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-8 \pm \sqrt{-96}}{-4} \quad \text{the equation does not have any solutions}$$



Exercise 1: Solve: $x^2 - 10x + 25 = 0$, $x^2 + 2x - 8 = 0$, $x^2 - 2x + 10 = 0$.

Summary:



7.2 Quadratic Inequalities

Quadratic Inequalities have the following form: $ax^2 + bx + c > 0$

We always start solving quadratic inequality with solving a corresponding quadratic equality. This allows for sketching a graph of out quadratic function (parabola). From the graph we can determine the solution easily.

Problem: Solve $x^2 + 5x + 6 > 0$

Solution:

$$x^2 + 5x + 6 = 0$$
$$(x + 2)(x + 3) = 0$$
$$x = -2, -3$$

Therefore, $x^2 + 5x + 6 > 0$ holds for all $x \in (-\infty, -3) \cup (-2, \infty)$.

Problem: Solve $x^2 - 5x + 4 < 0$

Solution:

$$x^2 - 5x + 4 = 0$$
$$(x - 1)(x - 4) = 0$$
$$x = 1, 4$$

Therefore, $x^2 - 5x + 4 < 0$ holds for all $x \in (1, 4)$.

Exercise 2: Solve: $x^2 - 10x + 25 > 0$, $x^2 + 2x - 8 \leq 0$, $x^2 - 2x + 10 > 0$.

7.3 Answers

Exercise 1: 5; 2,-4; no solution

Exercise 2: $R - \{5\}$; [-4,2]; R (all numbers, i.e. infinitely many solutions)