AAU - Business Mathematics I

Lecture #13, May 13, 2010

13 Financial Mathematics, Simple and Compound Interest

The central theme of these notes is embodied in the question, What is the value today of a sum of money which will be paid at a certain time in the future? Since the value of a sum of money depends on the point in time at which the funds are available, a method of comparing the value of sums of money which become available at different points of time is needed. This methodology is provided by the theory of interest.

A typical part of most insurance contracts is that the insured pays the insurer a fixed premium on a periodic (usually annual or semiannual) basis. Money has time value, that is, \$1 in hand today is more valuable than \$1 to be received one year hence. A careful analysis of insurance problems must take this effect into account. The purpose of this section is to examine the basic aspects of the theory of interest. A thorough understanding of the concepts discussed here is essential.

In this last context the interest rate i is called the nominal annual rate of interest. The effective annual rate of interest is the amount of money that one unit invested at the beginning of the year will earn during the year, when the amount earned is paid at the end of the year.

13.1 Compound Interest

Interest

Interest is a fee paid on borrowed capital. The fee is compensation to the lender for foregoing other useful investments that could have been made with the loaned money. Instead of the lender using the assets directly, they are advanced to the borrower. The borrower then enjoys the benefit of the use of the assets ahead of the effort required to obtain them, while the lender enjoys the benefit of the fee paid by the borrower for the privilege. The amount lent, or the value of the assets lent, is called the principal. This principal value is held by the borrower on credit. Interest is therefore the price of credit, not the price of money as it is commonly - and mistakenly - believed to be. The percentage of the principal that is paid as a fee (the interest), over a certain period of time, is called the interest rate. (wikipedia.org)

Compound interest

The principal changes with every time period, as any interest incurred over the period is added to the principal. Put another way, the lender is charging interest on the interest.

$$A = P(1+i)^n$$

where

A is the amount of money to be paid back

P is the principal

i is the interest rate (expressed as decimal number)

n the number of time periods elapsed since the loan was taken

Example: For example, imagine that Jim borrows $23,000 to buy a car and that the compound interest is charged at a rate of 5.5% per annum. After five years, and assuming none of the loan has been paid off, Jim owes:

$$A = 23000(1 + 0.055)^5 = 30060$$

In this case Jim would owe $30,060.

13.2 Savings, Loans, Project Evaluations

Time value of money

The time value of money represents the fact that, loosely speaking, it is better to have money today than tomorrow. Investor prefers to receive a payment today rather than an equal amount in the future, all else being equal. This is because the money received today can be deposited in a bank account and an interest is received.

Present value of a future sum

$$PV = \frac{FV}{(1+i)^n}$$

where:

PV is the value at time 0

FV is the value at time n

i is the rate at which the amount will be compounded each period

n is the number of periods

Present value of an annuity

The term annuity is used in finance theory to refer to any terminating stream of fixed payments over a specified period of time. Payments are made at the end of each period.

$$PV(A) = A\frac{1}{(1+i)} + A\frac{1}{(1+i)^2} + \ldots + A\frac{1}{(1+i)^n} = A\frac{1}{(1+i)}\frac{1-\frac{1}{(1+i)^n}}{1-\frac{1}{1+i}} = A\frac{1-\frac{1}{(1+i)^n}}{i}$$

where:

$PV(A)$ is the value of the annuity at time 0

A is the value of the individual payments in each compounding period

i is the interest rate that would be compounded for each period of time

n is the number of payment periods

Present value of a perpetuity

A perpetuity is an annuity in which the periodic payments begin on a fixed date and continue indefinitely. It is sometimes referred to as a "perpetual annuity" (UK government bonds).

The value of the perpetuity is finite because receipts that are anticipated far in the future have extremely low present value (present value of the future cash flows). Unlike a typical bond, because the principal is never repaid, there is no present value for the principal. The price of a perpetuity is simply the coupon amount over the appropriate discount rate or yield, that is

$$PV(P) = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \ldots = \frac{\frac{A}{1+i}}{1 - \frac{1}{1+i}} = \frac{A}{i}$$

Future value of a present sum

$$FV = PV(1+i)^n$$

Future value of an annuity

$$FV(A) = A(1+i)^{(n-1)} + A(1+i)^{(n-2)} + \ldots + A = A\frac{1-(1+i)^n}{1-(1+i)} = A\frac{(1+i)^n - 1}{i}$$

Example: One hundred euros to be paid 1 year from now, where the expected rate of return is 5% per year, is worth in today's money:

$$PV = \frac{FV}{(1+i)^n} = \frac{100}{1.05} = 95.23.$$

So the present value of 100 euro one year from now at 5% is 95.23.

Example: Consider a 10 year mortgage where the principal amount P is \$200,000 and the annual interest rate is 6%. What will be a monthly payment?

The number of monthly payments is

$$n = 10 \text{ years} \times 12 \text{ months} = 120 \text{ months}$$

The monthly interest rate is

$$i = \frac{6\% \text{ per year}}{12 \text{ monhs per year}} = 0.5\% \text{ per month}$$

$$PV(A) = A\frac{1-\frac{1}{(1+i)^n}}{i} \quad \Rightarrow A = PV(A)\frac{i}{1-\frac{1}{(1+i)^n}} = PV(A)\frac{i(1+i)^n}{(1+i)^n-1}$$

$$A = 200000\frac{0.005(1+0.005)^{120}}{(1+0.005)^{120}-1} = \$2220.41 \text{ per month.}$$

Example: Consider a deposit of \$ 100 placed at 10% annually. How many years are needed for the value of the deposit to double?

$$\begin{aligned}
&FV = PV(1+i)^n \\
&200 = 100(1+0.1)^n \\
&1.1^n = \frac{200}{100} = 2 \\
&\ln 1.1^n = \ln 2 \\
&n \ln 1.1 = \ln 2 \\
&n = \frac{\ln 2}{\ln 1.1} = 7.27 \text{ years}
\end{aligned}$$

Example: Similarly, the present value formula can be rearranged to determine what rate of return is needed to accumulate a given amount from an investment. For example, \$100 is invested today and \$200 return is expected in five years; what rate of return (interest rate) does this represent?

$$\begin{aligned}
&FV = PV(1+i)^n \\
&200 = 100(1+i)^5 \\
&(1+i)^5 = \frac{200}{100} = 2 \\
&(1+i) = 2^{1/5} \\
&i = 2^{1/5} - 1 = 0.15 = 15\%
\end{aligned}$$

Example: A manager of a company has to choose one of two possible projects. Project A requires immediate investment \$500 and yields returns \$200, \$300, and \$400 in the following three years. For project B it is necessary to invest \$400 now and the expected returns in the next three years are \$400, \$100 and \$50. Supposed that an interest rate is 10%. Which project should the manager choose?

Having time value of money in mind, manager should choose project with a higher present value.

$$PV_A = -500 + \frac{200}{1+i} + \frac{300}{(1+i)^2} + \frac{400}{(1+i)^3} = -500 + \frac{200}{1.1} + \frac{300}{1.1^2} + \frac{400}{1.1^3} =$$

$$= -500 + 182 + 248 + 300 = 230$$

$$PV_B = -400 + \frac{400}{1+i} + \frac{100}{(1+i)^2} + \frac{50}{(1+i)^3} = -400 + \frac{400}{1.1} + \frac{100}{1.1^2} + \frac{50}{1.1^3} =$$

$$= -400 + 364 + 83 + 38 = 85$$

Project A has a higher present value and hence should be chosen.

Problem: Instead of making payments of 300, 400, and 700 at the end of years 1, 2, and 3, the borrower prefers to make a single payment of 1400. At what time should this payment be made if the interest rate is 6% compounded annually?

Solution: Computing all of the present values at time 0 shows that the required time t satisfies the equation of value:

$$\frac{300}{1.06} + \frac{400}{1.06^2} + \frac{700}{1.06^3} = \frac{1400}{1.06^t}$$
$$1.06^t = \frac{1400}{283 + 356 + 588} = \frac{1400}{1227} = 1.141$$
$$\log 1.06^t = \log 1.141$$
$$t = \frac{\log 1.141}{\log 1.06} \approx \frac{0.0573}{0.0253} \approx 2.26$$

Problem: An investor purchases an investment which will pay 2000 at the end of one year and 5000 at the end of four years. The investor pays 1000 now and agrees to pay X at the end of the third year. If the investor uses an interest rate of 7% compounded annually, what is X?

Solution: The equation of value today is:

$$\frac{2000}{1.07} + \frac{5000}{1.07^4} = 1000 + \frac{X}{1.07^3}$$
$$\frac{2000 \times 1.07^3 + 5000}{1.07^4} = \frac{1000 \times 1.07^4 + X \times 1.07}{1.07^4}$$
$$X = \frac{2000 \times 1.07^3 + 5000 - 1000 \times 1.07^4}{1.07} = \frac{2450 + 5000 - 1311}{1.07} = 5737$$

Thus $X = 5737$.