1 Numbers and Sets

1.1 Sets

Set: collection of distinct objects which are called elements (numbers, people, letters of alphabet)

Two ways of defining sets:

• list each member of the set (e.g. {4,2,15,6}, {red, blue, white}, ...)

The order in which the elements of a set are listed is irrelevant, as are any repetitions in the list. For example,

$$\{6, 11\} = \{11, 6\} = \{11, 11, 6, 11\}$$

are equivalent, because the specification means merely that each of the elements listed is a member of the set.

• rule (e.g. A = set of even numbers, $B = \{n^2, n \in N, 0 \leqq n \leqq 5\}$, ...)

Membership: some elements belong to a set and some do not

• $4 \in A$, $15 \in \{4, 2, 15, 6\}$, $16 \in B$ (read "$\in$" as "belongs to")

• $5 \notin A$, $5 \notin B$, green $\notin$ {red, blue, white} (read "$\notin$" as "does not belong to")

Cardinality: the number of members of a set

• $|A| = \infty$ (The set A has infinitely many elements)

• $|B| = 6$ (The set B has six elements)

• $|C| = 0$, where C={three sided squares} (The set C is an empty set)

Subsets:

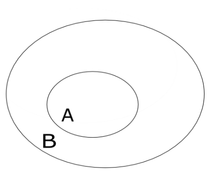
• $A \subseteq B$ if every member of A is in B as well (A is subset of B)

• if $A \subseteq B$ but $A \neq B$, then A is a proper subset of B, $A \subset B$

• $\{1, 2\} \subseteq \{1, 2, 3, 4\}$ and also $\{1, 2\} \subset \{1, 2, 3, 4\}$

• $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$ but it is not true that $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$

• set of men is a proper subset of the set of all people

Venn diagram: 

Note: $A \subseteq A$, $\varnothing \subseteq A$ for every set A

Special Sets:

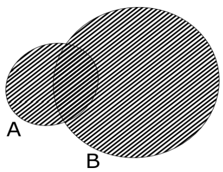
P - primes, N - natural numbers, Z - integers, $Q = \{\frac{a}{b}, a, b \in Z, b \neq 0\}$ - rational, R - real, I - irrational

$P \subset N \subset Z \subset Q \subset R$

1.2 Basic Operations on Sets

There are ways to construct new sets from existing ones. Two sets can be "added" together, "subtracted", etc.

• **Union:** $A \cup B$ elements that belong to A or B.

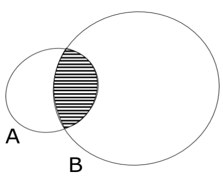


Example: $\{1, 2\} \cup \{\text{blue,red}\} = \{1, 2, \text{blue,red}\}$

Properties:

- $A \cup B = B \cup A$
- $A \subseteq A \cup B$
- $A \cup A = A$
- $A \cup \varnothing = A$

• **Intersection:** $A \cap B$ elements that belong to A and B at the same time.



Example: $\{1, 2\} \cap \{\text{blue,red}\} = \varnothing$

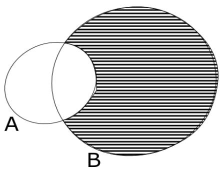
$\{1, 2\} \cap \{1, 2, 4, 7\} = \{2\}$

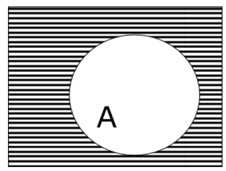
Properties:

- $A \cap B = B \cap A$
- $A \cap B \subseteq A$
- $A \cap A = A$
- $A \cap \varnothing = \varnothing$
- If $A \cap B = \varnothing$, then A and B are said to be disjoint.

• **Difference and Complement:** $B \setminus A$ or $B - A$: set of elements which belong to B, but not to A

In certain settings all sets under discussion are considered to be subsets of a given universal set U. Then, $U \setminus A$ is called complement of A and is denoted A' or A^C





Example: $\{1, 2, \text{green}\} \setminus \{\text{red,white,green}\} = \{1, 2\}$

$\{1, 2\} \setminus \{1, 2\} = \varnothing$

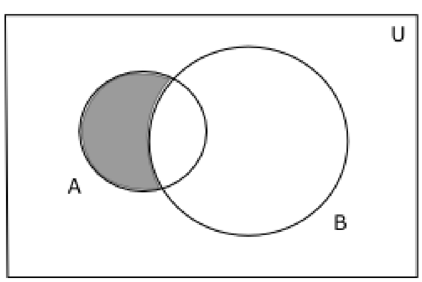
Integers $\setminus$ Even numbers = Odd numbers

Properties:

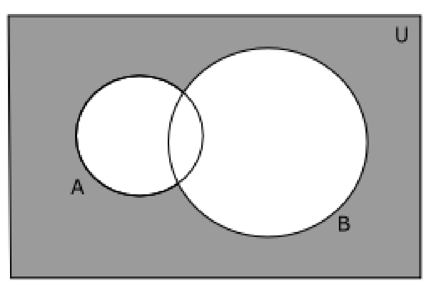
- $A \cup A^C = U$
- $A \cap A^C = \varnothing$
- $(A^C)^C = A$
- $A \setminus A = \varnothing$
- $A \setminus B = A \cap B^C$

Some identities:

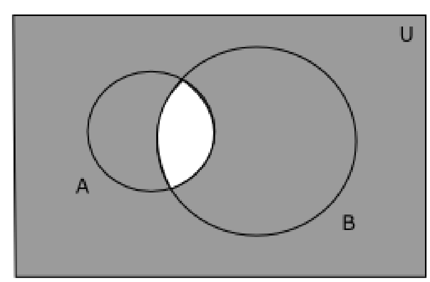
- $A \setminus B = A \cap B^C$
- $(A \cup B)^C = A^C \cap B^C$
- $(A \cap B)^C = A^C \cup B^C$



$A \setminus B$



$(A \cup B)^C$



$(A \cap B)^C$

Exercise 1: Do the following identities hold? You can find the answers to exercises at the end of this chapter.

(a) $(A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

(b) $(A \cap B^C) \cup C = A \cap B$

1.3 Numbers

complex (C) — imaginary
＼ real (R) — irrational (I)
＼ rational (Q) — non-integer
＼ integer (Z) — natural (N)
＼ 0
＼ negatives of natural

Real numbers (R): represented on real line with origin 0

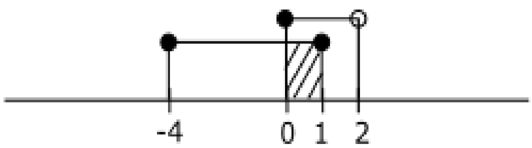
Intervals: subsets of a real line

closed - e.g. [2,5] - 2 and 5 belong to the interval

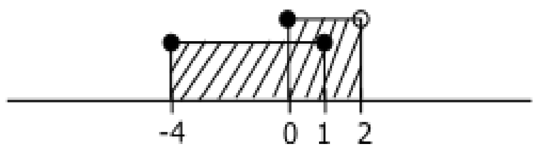
open - e.g. (3,9) - 3 and 9 do not belong to the interval

Intersection: $[-4, 1] \cap [0, 2) = [0, 1]$

Union: $[-4, 1] \cup [0, 2) = [-4, 2)$



Intersection



Union

Example:

- $A = [-5, 3], B = [1, 10] \;\rightarrow\; A \cap B = [1, 3],\; A \cup B = [-5, 10]$
- $A = (-\infty, 2), B = (0, 4] \;\rightarrow\; A \cap B = (0, 2),\; A \cup B = (-\infty, 4]$
- $A = [-2, 8], B = (3, 10), C = (9, 15) \;\rightarrow\; A \cap B \cap C = \emptyset,\; A \cup B \cup C = [-2, 15)$

Exercise 2: Find intercept and union of the following intervals:

- $A = (0, 7], B = [1, 6]$
- $A = [-3, 4], B = (3, 10), C = (-1, 7)$

1.4 Answers

Exercise 1: The identity (a) holds and the identity (b) does not hold (to see this draw Venn diagrams).

Exercise 2:

- $A = (0, 7], B = [1, 6] \;\rightarrow\; A \cap B = [1, 6],\; A \cup B = (0, 7]$
- $A = [-3, 4], B = (3, 10), C = (-1, 7) \;\rightarrow\; A \cap B \cap C = (3, 4],\; A \cup B \cup C = [-3, 10)$

2 Logic

2.1 Simple Statement, Negation

A statement - a declarative sentence that is either true or false.

A simple statement - one that does not contain any other statement as a part.

Examples of sentences that are (or make) statements:

- "Socrates is a man."
- "A triangle has three sides."
- "Paris is the capital of England."

The first two (make statements that) are true, the third is (or makes a statement that is) false.

Examples of sentences that are not (or do not make) statements:

- "Who are you?"
- "Run!"

Negation - In logic and mathematics, negation or "not" is an operation on logical values, which changes true to false and false to true. Intuitively, the negation of a proposition holds exactly when that proposition does not hold.

Notation: statement - p, can be true or false; if p is true then "NOT p" or " $\sim p$" or "$\rceil p$" is false.

Examples of negations of previous statements:

- "Socrates is not a man." or "Socrates is a woman."
- "A triangle does not have three sides." or "A triangle has at most two or at least four sides."
- "Paris is not the capital of England."

Notation: $\forall$ - for all/every/each; $\exists$ - there exists/at least one

Examples: A is a set of all natural numbers larger than 10: $A = \{\forall x \in N; x > 10\}$

There exists at least one number for which the square root is negative: $\exists x; \sqrt{x} < 0$

Examples: Decide if the following statements are true or false and find negations:

- Math teacher is nice. True. Math teacher is not nice.
- There are **no** unemployed people in the CR. False. There is **at least** one unemployed person in the CR.
- **At least four** students in class are women. True (most of the time). **At most three/less than four** students in class are women.
- Today is Tuesday. False (depends on when you read this material). Today is not Tuesday.

Exercise 1: Decide if the following statements are true or false and find negations:

- **At most three** students have blue eyes.
- Every day has 25 hours.
- **All** cars are red.
- The semester begins today.