9 Arithmetic and Geometric Sequence

Finite sequence: 1, 5, 9, 13, 17

Finite series: 1 + 5 + 9 + 13 +17

Infinite sequence: 1, 2, 4, 8, 16, ...

Infinite series: 1 + 2 + 4 + 8 + 16 + ...

When Gauss was a boy, the teacher ran out of stuff to teach and asked them, in the remaining time, to compute the sum of all the numbers from 1 to 40.

Gauss thought that 1+40 is 41. And 2+39 is also 41. And this is true for all the similar pairs, of which there are 20. So... the answer is 820.

One can wonder what would have happened had the teacher asked for the sum of the numbers from 1 to 39. Perhaps Gauss would have noted that 1+39 is 20, as is 2+38. This is true for all the pairs, of which there are 19, and the number 20 is left on its own. Nineteen 40's is 760 and the remaining 20 makes 780.

Example: Let's consider the series 3+5+7+9+11+13+15+17. If we add the first term to the last we get 20. If we add the second term to the second-to-last we get 20 again. Now we see that the series adds up to four 20s, or 80.

Now the question is - will this trick work for all series? If so, why? If not, which series will it work for? Answer: It will work for all **arithmetic series**. The reason that the second pair added up the same as the first pair was that we went up by two on the left, and down by two on the right. As long as you go up by the same as you go down, the sum will stay the same and this is just what happens for arithmetic series.

Arithmetic sequence: is a sequence $a_1, a_2, \dots a_n$ such that $a_n - a_{n-1} = d$ for all n. So the distance between the two following elements of the sequence is constant.

For example: 1,2,3, ... ($d = 1$); 2,4,6, ... 16 ($d = 2$); 0,3,6, ... 18 ($d = 3$)

Arithmetic series: is a sum of elements of arithmetic sequence. The sum is given by:

$$\Sigma_{i=1}^{n} a_i = S_n = \frac{(a_1 + a_n) \cdot n}{2}$$

Example: Find the sum of the following arithmetic series: 1 + 5 + 9 + 13 +17.

a_1=1, a_5=17, d=4, n=5.

$$\Sigma_{i=1}^{n} a_i = \Sigma_{i=1}^{5} a_i = \frac{(a_1 + a_n) \cdot n}{2} = \frac{(1 + 17) \cdot 5}{2} = 45$$

Example: Now consider the following sum: 2+6+18+54+162+486+1458. Clearly the "arithmetic series trick" will not work here: 2+1458 is not 6+486. We need a whole new trick. Here it comes.

$S = 2 + 6 + 18 + 54 + 162 + 486 + 1458$

where S is the sum we are looking for. Now, we multiply the whole equation by 3:

$3S = 6 + 18 + 54 + 162 + 486 + 1458 + 4374$

Now let's subtract the first equation from the second one:

$2S = 4374 - 2$ which means that $S = 2186$.

This trick will work for all **geometric series**.

Geometric sequence: is a sequence $a_1, a_2, \ldots a_n$ such that $\frac{a_n}{a_{n-1}} = r$ for all n. So the ratio between the two following elements is constant.

For example: 2,4,8, … ($r = 2$); 1,3,9,27,51 ($r = 3$)

Geometric series: is a sum of elements of geometric sequence. The sum is given by:

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

Example: Find the following sum: 1 + 2 + 4 + 8 + 16

In this example, $a_1 = 1$, $a_5 = 16$, $n = 5$, $r = 2$.

$$S_n = a_1 \frac{1 - r^n}{1 - r} = 1 \frac{1 - 2^5}{1 - 2} = 31$$

Problem: Find the sum of the numbers: 3, 7, 11, 15, …, 99.

Solution: In this example, $a_1 = 3$, $a_n = 99$, and $d = 4$. To find n we solve the following equation:

$$3 + (n - 1)4 = 99$$

to get $n = 25$. Then the sum of the numbers is:

$$\Sigma_{i=1}^{n} = \frac{(a_1 + a_n) \cdot n}{2} = \frac{(3 + 99) \cdot 25}{2} = 1275$$

Example: Infinite geometric series. Find the sum of the numbers: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

In this case $r = \frac{1}{2} < 1$. If $r < 1$ then the sum of infinite geometric series exists and it can be found as:

$$\Sigma_{i=1}^{\infty} = \frac{a_1}{1 - r} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

(If $r > 1$ then the sum is equal to infinity.)

Example: Find a decimal form of the number $0.77\overline{7}$.

Notice that:

$$0.77\overline{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$$

So r in this case is $\frac{1}{10}$ and hence:

$$\Sigma_{i=1}^{\infty} = \frac{a_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$$

This means that $\frac{7}{9} = 0.77\overline{7}$.

Similarly, $0.09090\overline{909} = \frac{09}{99} = \frac{1}{11}$; $0.1438\overline{1438} = \frac{1438}{9999}$

Problem: Find the sum of the first 30 terms of $5 + 9 + 13 + 17 + \ldots$

Solution: We know that $n = 30$ and $a_1 = 5$, and we need the 30th term. Use the definition of an arithmetic sequence.

$a_{30} = 5 + 29 \times 4 = 121$. Therefore, $S_{30} = 30(5 + 121)/2 = 1890$.

10 Financial Mathematics, Simple and Compound Interest

The central theme of these notes is embodied in the question, What is the value today of a sum of money which will be paid at a certain time in the future? Since the value of a sum of money depends on the point in time at which the funds are available, a method of comparing the value of sums of money which become available at different points of time is needed. This methodology is provided by the theory of interest.

A typical part of most insurance contracts is that the insured pays the insurer a fixed premium on a periodic (usually annual or semiannual) basis. Money has time value, that is, \$1 in hand today is more valuable than \$1 to be received one year hence. A careful analysis of insurance problems must take this effect into account. The purpose of this section is to examine the basic aspects of the theory of interest. A thorough understanding of the concepts discussed here is essential.

In this last context the interest rate i is called the nominal annual rate of interest. The effective annual rate of interest is the amount of money that one unit invested at the beginning of the year will earn during the year, when the amount earned is paid at the end of the year.

10.1 Simple and Compound Interest

Interest

Interest is a fee paid on borrowed capital. The fee is compensation to the lender for foregoing other useful investments that could have been made with the loaned money. Instead of the lender using the assets directly, they are advanced to the borrower. The borrower then enjoys the benefit of the use of the assets ahead of the effort required to obtain them, while the lender enjoys the benefit of the fee paid by the borrower for the privilege. The amount lent, or the value of the assets lent, is called the principal. This principal value is held by the borrower on credit. Interest is therefore the price of credit, not the price of money as it is commonly - and mistakenly - believed to be. The percentage of the principal that is paid as a fee (the interest), over a certain period of time, is called the interest rate. (wikipedia.org)

Simple interest

Simple Interest is calculated only on the principal, or on that portion of the principal which remains unpaid. The amount of simple interest is calculated according to the following formula:

$$A = P(1 + in)$$

where

A is the amount of money to be paid back

P is the principal

i is the interest rate (expressed as decimal number)

n the number of time periods elapsed since the loan was taken

Simple interest is often used over short time intervals, since the computations are easier than with compound interest.

For example, imagine Jim borrows \$23,000 to buy a car and that the simple interest is charged at a rate of 5.5% per annum. After five years, and assuming none of the loan has been paid off, Jim owes:

$$A = 23000(1 + 0.055 \times 5) = 29325$$

At this point, Jim owes a total of \$29,325 (principal plus interest).

Compound interest

In the short run, compound Interest is very similar to Simple Interest, however, as time goes on difference becomes considerably larger. The conceptual difference is that the principal changes with every time period, as any interest incurred over the period is added to the principal. Put another way, the lender is charging interest on the interest.

$$A = P(1 + i)^n$$

In this case Jim would owe principal of \$30,060.

Generally: If a principal P is invested at an annual rate r (expressed in decimal form) compounded m times a year, then the amount A in the account at the end of n years is given by:

$$A = P\left(1 + \frac{i}{m}\right)^{nm}$$

10.2 Savings, Loans, Project Evaluations

Time value of money

The time value of money represents the fact that, loosely speaking, it is better to have money today than tomorrow. Investor prefers to receive a payment today rather than an equal amount in the future, all else being equal. This is because the money received today can be deposited in a bank account and an interest is received.

Present value of a future sum

$$PV = \frac{FV}{(1+i)^n}$$

where:

PV is the value at time 0

FV is the value at time n

i is the rate at which the amount will be compounded each period

n is the number of periods

Present value of an annuity

The term annuity is used in finance theory to refer to any terminating stream of fixed payments over a specified period of time. Payments are made at the end of each period.

$$PV(A) = A\frac{1}{(1+i)} + A\frac{1}{(1+i)^2} + \ldots + A\frac{1}{(1+i)^n} = A\frac{1}{(1+i)}\frac{1-\frac{1}{(1+i)^n}}{1-\frac{1}{1+i}} = A\frac{1-\frac{1}{(1+i)^n}}{i}$$

where:

$PV(A)$ is the value of the annuity at time 0

A is the value of the individual payments in each compounding period

i is the interest rate that would be compounded for each period of time

n is the number of payment periods

Present value of a perpetuity

A perpetuity is an annuity in which the periodic payments begin on a fixed date and continue indefinitely. It is sometimes referred to as a "perpetual annuity" (UK government bonds).

The value of the perpetuity is finite because receipts that are anticipated far in the future have extremely low present value (present value of the future cash flows). Unlike a typical bond, because the principal is never repaid, there is no present value for the principal. The price of a perpetuity is simply the coupon amount over the appropriate discount rate or yield, that is

$$PV(P) = \frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \ldots = \frac{\frac{A}{1+i}}{1 - \frac{1}{1+i}} = \frac{A}{i}$$

Future value of a present sum

$$FV = PV(1+i)^n$$

Future value of an annuity

$$FV(A) = A(1+i)^{(n-1)} + A(1+i)^{(n-2)} + \ldots + A = A\frac{1-(1+i)^n}{1-(1+i)} = A\frac{(1+i)^n - 1}{i}$$

Example: One hundred euros to be paid 1 year from now, where the expected rate of return is 5% per year, is worth in today's money:

$$PV = \frac{FV}{(1+i)^n} = \frac{100}{1.05} = 95.23.$$

So the present value of 100 euro one year from now at 5% is 95.23.

Example: Consider a 10 year mortgage where the principal amount P is \$200,000 and the annual interest rate is 6%. What will be a monthly payment?

The number of monthly payments is

$$n = 10 \text{ years} \times 12 \text{ months} = 120 \text{ months}$$

The monthly interest rate is

$$i = \frac{6\% \text{ per year}}{12 \text{ monhs per year}} = 0.5\% \text{ per month}$$

$$PV(A) = A\frac{1 - \frac{1}{(1+i)^n}}{i} \quad \Rightarrow A = PV(A)\frac{i}{1 - \frac{1}{(1+i)^n}} = PV(A)\frac{i(1+i)^n}{(1+i)^n - 1}$$

$$A = 200000\frac{0.005(1+0.005)^{120}}{(1+0.005)^{120} - 1} = \$2220.41 \text{ per month.}$$

Example: Consider a deposit of \$ 100 placed at 10% annually. How many years are needed for the value of the deposit to double?

$$
\begin{aligned}
&FV = PV(1+i)^n \\
&200 = 100(1+0.1)^n \\
&1.1^n = \frac{200}{100} = 2 \\
&\ln 1.1^n = \ln 2 \\
&n \ln 1.1 = \ln 2 \\
&n = \frac{\ln 2}{\ln 1.1} = 7.27 \text{ years}
\end{aligned}
$$

Example: Similarly, the present value formula can be rearranged to determine what rate of return is needed to accumulate a given amount from an investment. For example, \$100 is invested today and \$200 return is expected in five years; what rate of return (interest rate) does this represent?

$$
\begin{aligned}
&FV = PV(1+i)^n \\
&200 = 100(1+i)^5 \\
&(1+i)^5 = \frac{200}{100} = 2 \\
&(1+i) = 2^{1/5} \\
&i = 2^{1/5} - 1 = 0.15 = 15\%
\end{aligned}
$$

Example: A manager of a company has to choose one of two possible projects. Project A requires immediate investment \$500 and yields returns \$200, \$300, and \$400 in the following three years. For project B it is necessary to invest \$400 now and the expected returns in the next three years are \$400, \$100 and \$50. Supposed that an interest rate is 10%. Which project should the manager choose?

Having time value of money in mind, manager should choose project with a higher present value.

$$
PV_A = -500 + \frac{200}{1+i} + \frac{300}{(1+i)^2} + \frac{400}{(1+i)^3} = -500 + \frac{200}{1.1} + \frac{300}{1.1^2} + \frac{400}{1.1^3} =
$$

$$
= -500 + 182 + 248 + 300 = 230
$$

$$
PV_B = -400 + \frac{400}{1+i} + \frac{100}{(1+i)^2} + \frac{50}{(1+i)^3} = -400 + \frac{400}{1.1} + \frac{100}{1.1^2} + \frac{50}{1.1^3} =
$$

$$
= -400 + 364 + 83 + 38 = 85
$$

Project A has a higher present value and hence should be chosen.

Cash flow: Some of the problems consist of analyzing special cases of the following situation. Cash payments of amounts $C_0, C_1, \ldots, C_n$ are to be received at times $0, 1, \ldots, n$. The payment amounts may be either positive or negative. A positive amount denotes a cash inflow; a negative amount denotes a cash outflow.

There are 3 types of questions about this general setting.

(1) If the cash amounts and interest rate are given, what is the value of the cash flow at a given time point?

(2) If the interest rate and all but one of the cash amounts are given, what should the remaining amount be in order to make the value of the cash flow equal to a given value?

(3) What interest rate makes the value of the cash flow equal to a given value?

Problem: Instead of making payments of 300, 400, and 700 at the end of years 1, 2, and 3, the borrower prefers to make a single payment of 1400. At what time should this payment be made if the interest rate is 6% compounded annually?

Solution: Computing all of the present values at time 0 shows that the required time t satisfies the equation of value:

$$\frac{300}{1.06} + \frac{400}{1.06^2} + \frac{700}{1.06^3} = \frac{1400}{1.06^t}$$
$$1.06^t = \frac{1400}{283 + 356 + 588} = \frac{1400}{1227} = 1.141$$
$$\log 1.06^t = \log 1.141$$
$$t = \frac{\log 1.141}{\log 1.06} \approx \frac{0.0573}{0.0253} \approx 2.26$$

Problem: An investor purchases an investment which will pay 2000 at the end of one year and 5000 at the end of four years. The investor pays 1000 now and agrees to pay X at the end of the third year. If the investor uses an interest rate of 7% compounded annually, what is X?

Solution: The equation of value today is:

$$\frac{2000}{1.07} + \frac{5000}{1.07^4} = 1000 + \frac{X}{1.07^3}$$
$$\frac{2000 \times 1.07^3 + 5000}{1.07^4} = \frac{1000 \times 1.07^4 + X \times 1.07}{1.07^4}$$
$$X = \frac{2000 \times 1.07^3 + 5000 - 1000 \times 1.07^4}{1.07} = \frac{2450 + 5000 - 1311}{1.07} = 5737$$

Thus $X = 5737$.

Problem: A three year certificate of deposit carries an interest rate 7% compounded annually. The certificate has an early withdrawal penalty which, at the investors discretion, is either a reduction in the interest rate to 5% or the loss of 3 months interest. Which option should the investor choose if the deposit is withdrawn after 9 months? After 27 months?

Solution: First of all notice that it does not matter at all what is the amount invested in this case. The answer will be the same whether we deposit \$1 or 3250\$ or any other amount of money. If the deposit is not withdrawn for the whole 3 years than it would bring:

$$P \to P(1 + 0.07)^3$$

If the deposit is withdrawn after 9 months in which option do we lose less?

- reduction in interest:

$$P \to P(1 + 0.05)^{9/12} = 1.0372P$$

- loss of 3 months interest:

$$P \to P(1 + 0.07)^{6/12} = 1.0344P$$

In this case reduction in the interest is better because we get higher amount of money with this option.

If the deposit is withdrawn after 27 months in which option do we lose less?

- reduction in interest:

$$P \to P(1 + 0.05)^{27/12} = 1.1160P$$

- loss of 3 months interest:

$$P \to P(1 + 0.07)^{24/12} = 1.1449P$$

In this case we should choose the second option - lose 3 months interest because this way we get higher amount of money.

10.3 Bond Pricing

In finance, a **bond** is a debt security, in which the issuer (borrower) owes the holders (lenders) a debt and is obliged to pay interest (the coupon) and to repay the principal at a later date - maturity.

Par Value (as stated on the face of the bond, F) is the amount that the issuing firm is to pay to the bond holder at the maturity date.

Coupon Yield is simply the coupon payment (C) as a percentage of the face value (F).

Coupon yield = C/F

Current Yield is simply the coupon payment (C) as a percentage of the (current) bond price (P).

Current yield $= C/P$.

Yield to Maturity (YTM) is the discount rate r which returns the market price of the bond. YTM is thus the internal rate of return of an investment in the bond made at the observed price. Since YTM can be used to price a bond, bond prices are often quoted in terms of YTM.

Whatever r is, if you use it to calculate the present values of all payouts and then add up these present values, the sum will equal your initial investment.

In an equation,

$$P = C(1+r)^{-1} + C(1+r)^{-2} + \ldots + C(1+r)^{-n} + F(1+r)^{-n} = \frac{C[1-(1+i)^{-n}]}{i} + F(1+i)^{-n}$$

where

$C =$ annual coupon payment (in dollars, not a percent)

$n =$ number of years to maturity

$F =$ par value

$P =$ purchase price

Problem: Suppose your bond is selling for \$950, and has a coupon rate of 7%; it matures in 4 years, and the par value is \$1000. What is the YTM?

Solution: The coupon payment is \$70 (that's 7% of \$1000), so the equation to satisfy is

$$70(1+r)^{-1} + 70(1+r)^{-2} + 70(1+r)^{-3} + 70(1+r)^{-4} + 1000(1+r)^{-4} = 950$$

We are not really going to solve this, but the result is that r equals 8.53% (If you want, you can plug this number back into equation to make sure it is correct).

Problem: A \$5,000 bond pays the holder an interest rate of 10% payable semi-annually. The bond will be redeemed at par in 10 years. An investor wants to purchase the bond on the bond market to yield a return of 12% payable semi-annually. What would be the purchase price of the bond?

Solution: Since the bond pays 10% on \$5,000 semiannually, the regular interest payment will be: $C = \frac{0.1}{2} \times 5000 = 250$

From the information given, the remaining number of interest periods is 20. The redemption value of the bond in ten years is the par value or the face value of the bond, \$5000.

Now to compute the purchase price, we must calculate the present values of the payments and the redemption value. Since the yield rate is the rate the investor wants to receive, it is the rate we must use to find the present values in determining the purchase price. Substituting the values into our formula, we have:

$$P = \frac{C[1-(1+i)^{-n}]}{i} + F(1+i)^{-n} = \frac{250[1-(1+0.06)^{-20}]}{0.06} + 5000(1+0.06)^{-20} =$$

$$= \$2,867.50 + \$1,559.02 = \$4,426.52$$

Problem: What is the price of the following quarterly bond?

Face value: $1,000

Maturity: 10 years

Coupon rate: 10%

Discount rate: 8%

Solution:

$$\frac{25}{0.08/4}\left[1 - \frac{1}{(1+0.08/4)^{10\cdot 4}}\right] + \frac{1000}{(1+0.08/4)^{10\cdot 4}} = \$1136.78$$