8 Matrices, Determinants, Cramer's Rule

8.1 Matrices

Matrices: In mathematics, a matrix (plural matrices) is a rectangular table of elements (or entries), which may be numbers or, more generally, any abstract quantities that can be added and multiplied. Matrices are mostly used to describe linear equations and solve systems of equations in a more efficient way. Matrices can be added, multiplied, and decomposed in various ways, making them a key concept in linear algebra and matrix theory.

The horizontal lines in a matrix are called rows and the vertical lines are called columns. A matrix with m rows and n columns is called an $m-by-n$ matrix (written $m \times n$) and m and n are called its dimensions. The dimensions of a matrix are always given with the number of rows first, then the number of columns.

Almost always capital letters denote matrices with the corresponding lower-case letters with two indices representing the entries. For example, the entry of a matrix A that lies in the i-th row and the j-th column is written as $a_{i,j}$ and called the i,j entry or (i,j)-th entry of A.

Example:

$$A = \begin{pmatrix} 8 & 9 & 6 \\ 1 & 2 & 7 \\ 9 & 2 & 4 \\ 6 & 0 & 5 \end{pmatrix}$$

is a 4×3 matrix. The element $a_{2,3}$ is 7.

$a_{1,1} = 8, a_{1,2} = 9, a_{1,3} = 6$
$a_{2,1} = 1, a_{2,2} = 2, a_{2,3} = 7$
$a_{3,1} = 9, a_{3,2} = 2, a_{3,3} = 4$
$a_{4,1} = 6, a_{4,2} = 0, a_{4,3} = 5$

More examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 5 & 7 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Square matrix	Column matrix	Row matrix	Zero matrix	Identity matrix
3×3	4×1	1×4	2×3	3×3

Relationship between system of equations and matrix:

$$2x - 3y = 5$$
$$x + 2y = -3$$

In matrix notation:

$$\left(\begin{array}{cc|c} 2 & -3 & 5 \\ 1 & 2 & -3 \end{array}\right)$$

8.2 Operations on/with Matrices

Elementary Row Operations producing Row-Equivalent Matrices:

1. Two rows are interchanged
2. A row is multiplied by a non-zero constant
3. A constant multiple of one row is added to another row.

Example: Solve the following system by using matrix:

$$3x + 4y = 1$$
$$x - 2y = 7$$

Solution: We start by writing corresponding matrix form:

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right)$$

Our objective is to use row operations as described above to transform matrix into the following form (which is called *reduced form*):

$$\left(\begin{array}{cc|c} 1 & 0 & \text{m} \\ 0 & 1 & \text{n} \end{array}\right)$$

where m and n are some real numbers. The solution to our system is then obvious because if we rewrite the matrix form into the system form we get:

$$1x + 0y = m$$
$$0x + 1y = n$$

or equivalently

$$x = m$$
$$y = n$$

which is the solution that we were looking for. So the only problem to be solved is to transform

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right) \text{ into } \left(\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right)$$

Step 1: To get 1 in the upper left corner, we interchange rows 1 and 2:

$$\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right)$$

Step 2: To get 0 in the lower left corner, we multiply row 1 by (-3) and add to row 2:

$$\left(\begin{array}{cc|c} 1 & -2 & 7 \\ 3 & 4 & 1 \end{array}\right) \begin{array}{c} (-3) \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right)$$

Step 3: To get 1 in the second row, second column, we multiply row 2 by 1/10:

$$\left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 10 & -20 \end{array}\right) \begin{array}{c} \\ 1/10 \end{array} \sim \left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right)$$

Step 4: To get 0 in the first row, second column, we multiply row 2 by 2 and add the result to row 1:

$$\left(\begin{array}{cc|c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right) \begin{array}{c} \nwarrow \\ 2 \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right)$$

The last matrix is the matrix for:

$x = 3$

$y = -2$

Exercise 1: Solve the following system by using matrix method:

$$\begin{aligned} 2x + 3y &= 11 \\ x - 2y &= 2 \end{aligned}$$

Example: Solve the following system by using matrix method:

$$\begin{aligned} x + y + z &= 6 \\ 2x + y - z &= 1 \\ 3x + y + z &= 8 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 3 & 1 & 1 & 8 \end{array}\right) \begin{array}{cc} (-2) & (-3) \\ \swarrow & \\ & \swarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & -2 & -2 & -10 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 2 & 2 & 10 \end{array}\right) \begin{array}{c} \\ (-2) \\ \swarrow \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -4 & -12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{c} \\ \nwarrow \\ (-3) \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{c} \nwarrow \\ (-1) \\ (-1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \Longrightarrow \begin{array}{c} x = 1 \\ y = 2 \\ z = 3 \end{array}$$

8.3 Gauss Elimination Method

Gauss elimination method is a method of solving systems of equations. The process of Gaussian elimination has two parts. The first part reduces a given system to either triangular form (zeros below the main diagonal of the matrix). The second step uses back substitution to find the solution of the system above.

Example: Solve the following system by using Gauss elimination method:

$$\begin{aligned} &x + y + z = 0 \\ &x - 2y + 2z = 4 \\ &x + 2y - z = 2 \end{aligned}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 4 \\ 1 & 2 & -1 & 2 \end{array}\right) \begin{array}{cc} (-1) & (-1) \\ \swarrow & | \\ & \leftarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 1 & -2 & 2 \end{array}\right) \begin{array}{c} \\ \searrow \\ \times 3 \swarrow \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 0 & -5 & 10 \end{array}\right)$$

This matrix is equivalent to the following system of equations:

$$\begin{aligned} &x + y + z = 0 \\ &-3y + z = 4 \\ &-5z = 10 \quad \Rightarrow \quad z = -2 \\ &-3y + z = 4 \quad \Rightarrow \quad -3y - 2 = 4 \quad \Rightarrow \quad y = -2 \\ &x + y + z = 0 \quad \Rightarrow \quad x - 2 - 2 = 0 \quad \Rightarrow \quad x = 4 \end{aligned}$$

Exercise 2: Solve the following system by using Gauss elimination method:

$$\begin{aligned} &2x + y - z = -3 \\ &-x + 2y - 3z = 4 \\ &x + 3y + 2z = 1 \end{aligned}$$

8.4 Determinant

Determinants: Determinant is a real number associated with each square matrix. If A is a square matrix, then the determinant of A is denoted by **det A** of by writing the array of elements in A using vertical lines in place of square brackets. For example, if

$$A = \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix}$$

then the determinant is denoted

$$det \begin{pmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{pmatrix} = \begin{vmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{vmatrix}$$

Value of a second-order determinant:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Examples:

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 3 \times 2 = -2$$

$$\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} = (-1) \times (-4) - (-3) \times 2 = 10$$

Value of a third-order determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} - a_{21}a_{12}a_{33} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}$$

Note: you do not need to remember this formula, there are two options how to calculate a third-order determinant:

Copy the first two lines of the matrix below it:

$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

Now the determinant is just a sum of products of elements on main diagonals with positive sign and elements on secondary diagonals with negative signs:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

8.5 Cramer's Rule

Given the system:

$$\begin{array}{c} a_{11}x + a_{12}y = k_1 \\ a_{21}x + a_{22}y = k_2 \end{array} \quad \text{with} \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$$

then

$$x = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}}{D}$$

Example: Solve the following system using: 1. matrix method 2. Cramer's rule.

$-2x + y = 6$
$x - y = -5$

Solution: 1. matrix method:

$$\left(\begin{array}{cc|c} -2 & 1 & 6 \\ 1 & -1 & -5 \end{array}\right) \begin{array}{c} \nwarrow \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ -2 & 1 & 6 \end{array}\right) \begin{array}{c} /2 \\ \swarrow \end{array} \sim \left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & -1 & -4 \end{array}\right) /(-1) \sim$$

$$\left(\begin{array}{cc|c} 1 & -1 & -5 \\ 0 & 1 & 4 \end{array}\right) \begin{array}{c} \nwarrow \\ \nearrow \end{array} \sim \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array}\right) \quad \Rightarrow \quad \begin{array}{c} x = -1 \\ y = 4 \end{array}$$

2. Cramer's rule:

$$x = \frac{\begin{vmatrix} 6 & 1 \\ -5 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{-1}{1} = -1$$

$$y = \frac{\begin{vmatrix} -2 & 6 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{4}{1} = 4$$

Exercise 3: Solve the following system using: 1. matrix method 2. Cramer's rule.

$3x - 2y = 0$
$x + 2y = 8$

8.6 Answers

Exercise 1: $x = 4$, $y = 1$

Exercise 2:

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ -1 & 2 & -3 & 4 \\ 1 & 3 & 2 & 1 \end{array}\right) \begin{array}{cc} (1/2) & (-1/2) \\ \swarrow & | \\ & \leftarrow \end{array} \sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 0 & 5/2 & -7/2 & 5/2 \\ 0 & 5/2 & 5/2 & 5/2 \end{array}\right) \begin{array}{c} \\ \times(-1)| \\ \swarrow \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 0 & 5/2 & -7/2 & 5/2 \\ 0 & 0 & 6 & 0 \end{array}\right)$$

This matrix is equivalent to the following system of equations:

$$2x + y - z = -3$$
$$5/2y - 7/2z = 5/2$$
$$6z = 0 \quad \Rightarrow \quad z = 0$$
$$5/2y - 7/2z = 5/2 \quad \Rightarrow \quad 5/2y = 5/2 \quad \Rightarrow \quad y = 1$$
$$2x + y - z = -3 \quad \Rightarrow \quad 2x + 1 = -3 \quad \Rightarrow \quad x = -2$$

Exercise 3: $x = 2$, $y = 3$