4 Linear and Rational Inequalities

4.1 Linear Inequalities in One Variable

$3(x-5) \geq 5(x+7)$, $-4 \leq 3-2x < 7$, ...

Properties of inequality:

1. if $a < b$ then $a + c < b + c$ — addition
2. if $a < b$ then $a - c < b - c$ — subtraction
3. if $a < b$ then $ca < cb$ for $c > 0$
   $ca > cb$ for $c < 0$ — multiplication
4. if $a < b$ then $a/c < b/c$ for $c > 0$
   $a/c > b/c$ for $c < 0$ — division
5. if $a < b$ and $b < c$ then $a < c$ — transitivity

Problem: Solve $2(2x+3) - 10 < 6(x-2)$

Solution: Solving linear inequalities is almost the same as solving linear equalities. Only if you multiply or divide by a negative number, change the sign!!!

$$\begin{aligned} 2(2x+3) - 10 &< 6(x-2) \\ 4x + 6 - 10 &< 6x - 12 \\ -2x &< -8 \qquad /(-2) \qquad \text{Change the sign of the inequality!} \\ x &> 4 \end{aligned}$$

The inequality holds for all $x \in (4, \infty)$.

Problem: Solve $-6 < 2x + 3 \leq 5x - 3$

Solution: We divide this problem into two parts and solve simultaneously these two inequalities: $-6 < 2x + 3$ and $2x + 3 \leq 5x - 3$

$$\begin{array}{rl} -6 < 2x + 3 & 2x + 3 \leq 5x - 3 \\ -9 < 2x & -3x \leq -6 \\ -9/2 < x & x \geq 2 \end{array}$$

The inequality holds for all $x \in (-9/2, \infty)$ and at the same time $x \in [2, \infty)$. So the solution is $x \in [2, \infty)$.

Exercise 1: Solve Solve $1 < 3x - 5 \leq 2x + 5$

Problem: Apple Inc. produces 100% apple juice. Its production function is $J \leq 12A - 4$, where J is quantity of juice in liters and A is quantity of apples in kilograms. For what quantity does Apple Inc. produce at most 20 liters of juice?

Solution:

$$\begin{aligned} J &\leq 20 \\ 12A - 4 &\leq 20 \\ 12A &\leq 24 \\ A &\leq 2 \end{aligned}$$

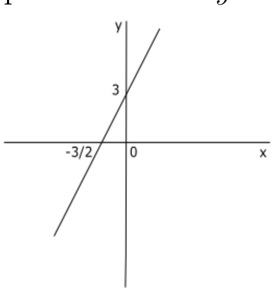
In order to produce at most 20 liters of juice, Apple Inc. can use at most 2 kilograms of apples.

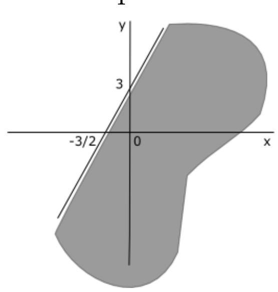
4.2 Linear Inequalities in Two Variables

Graphing linear inequalities on the number line: For instance, graph $x > 2$. First, draw the number line, find the "equals" part (in this case, $x = 2$), mark this point with the appropriate notation (an open dot, indicating that the point $x = 2$ wasn't included in the solution), and then you'd shade everything to the right, because "greater than" meant "everything off to the right". The steps for graphing two-variable linear inequalities are very much the same.

Problem: Graph the solution to $y \leq 2x + 3$.

Solution: Just as for number-line inequalities, first find the "equals" part. For two-variable linear inequalities, the "equals" part is the graph of the straight line; in this case, that means the "equals" part is the line $y = 2x + 3$ which is depicted on the left hand picture below:





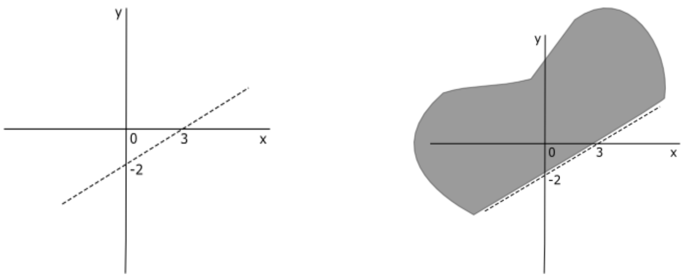
We have the graph of the "or equal to" part (it's just the line); now we need "y less than" part. In other words, we need to shade one side of the line or the other. If we need y LESS THAN the line, we want to shade below the line as it is depicted on the right hand picture above.

Problem: Graph the solution to $2x - 3y < 6$.

Solution: First, solve for y:

$$2x - 3y < 6$$
$$-3y < -2x + 6$$
$$y > \frac{2}{3}x - 2$$

Now we need to find the "equals" part, which is the line $y = \frac{2}{3}x - 2$. Note, that here we have strict inequality therefore the line itself does not belong to the set of solutions and hence is graphed as a dashed line. It looks like the left hand picture below.



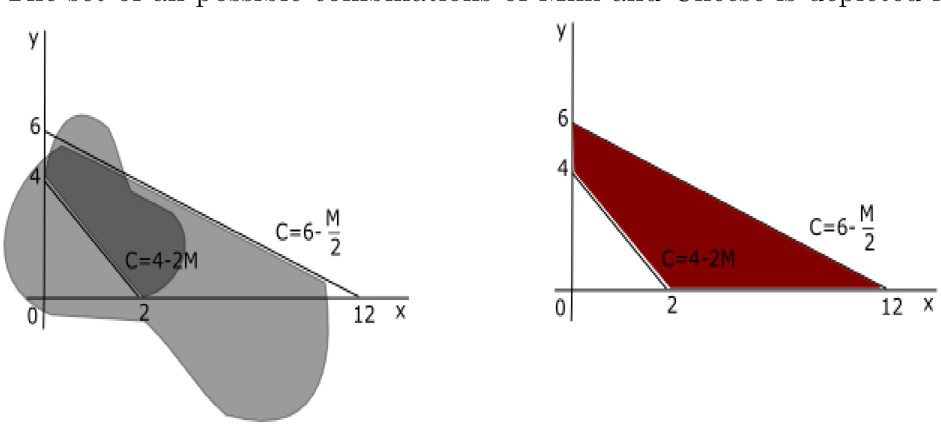
By using a dashed line, we know where the border is, but we also know that the border isn't included in the solution. Since this is a "y greater than" inequality, we need to shade above the line, so the solution looks like the right hand picture above.

Problem: A milk company faces the following problem. It's production function is $M + 2C \leq 12$ where M is the amount of milk produced and C is the amount of cheese. Price of the cheese is 5 and the price of milk is 10. Company wants to reach a level of revenue of at least 20. Draw the set of all possible combinations of milk and cheese.

Solution: This problem is about solving two inequalities, graphing them and finding their intercept.

$$M + 2C \leq 12 \quad \Rightarrow \quad C \leq 6 - \frac{M}{2} \quad \text{Production function}$$
$$P_M M + P_C C \geq 20 \quad \Rightarrow \quad 10M + 5C \geq 20 \quad \Rightarrow \quad C \geq 4 - 2M \quad \text{Revenue requirement}$$

The set of all possible combinations of Milk and Cheese is depicted in red on the graph below.



4.3 Rational Inequalities:

$$\frac{x+1}{x-3} > 1, \quad \frac{x+1}{x^2-3x+5} < 0, \quad \frac{x^2-x-1}{2x^2+4x-3} > 5, \ \ldots$$

Problem: Solve $\frac{2x}{x+2} > 1$

Solution: Note that we can **not** just multiply the inequality by $(x+2)$ and solve the resulting linear inequality $2x > x+2$, because we do not know whether $x+2$ is positive or negative and hence we do not know whether we should change the sign of inequality or not. So we distinguish two cases:

• $x+2 > 0 \quad \Rightarrow \quad x > -2 \quad \ldots \quad 2x > x+2 \quad \Rightarrow \quad x > 2$

• $x+2 < 0 \quad \Rightarrow \quad x < -2 \quad \ldots \quad 2x < x+2 \quad \Rightarrow \quad x < 2$

Alternative solution:

$$\frac{2x}{x+2} > 1$$
$$\frac{2x}{x+2} - 1 > 0$$
$$\frac{x-2}{x+2} > 0$$

Here, we need the ratio of two numbers be positive. This is the case if both numerator and denominator are positive or if both numerator and denominator are negative:

• $x-2 > 0$ and $x+2 > 0 \quad \Leftrightarrow \quad x > 2$ and $x > -2 \quad \Rightarrow x > 2$

OR

• $x-2 < 0$ and $x+2 < 0 \quad \Leftrightarrow \quad x < 2$ and $x < -2 \quad \Rightarrow x < -2$

Hence the solution to the problem is $x \in (-\infty, -2) \cup (2, \infty)$.

Exercise 2: Solve $\frac{x^2-3x-10}{1-x} \geq 2$.

4.4 Answers

Exercise 1: $x \in (2, 10]$

Exercise 2: $x \in (-\infty, -3] \cup (1, 4]$

5 Quadratic Equations, Inequalities

5.1 Quadratic Equations

Quadratic equation has the following form: $ax^2 + bx + c = 0$

Equations with the second power of a variable; e.g.

$x^2 - 6x + 9 = 0$

$y^2 + 3y - 1 = 2y^2 - 4y - 3$

5.1.1 Solving by Square Root

Problem: Solve $3x^2 - 27 = 0$.

Solution:

$$\begin{aligned}
&3x^2 - 27 = 0\\
&3x^2 = 27\\
&x^2 = 9\\
&x = \pm\sqrt{9}\\
&x_{1,2} = \pm 3
\end{aligned}$$

Note: if $a^2 = b$, then $a \neq \sqrt{b}$!!!, but $a = \pm\sqrt{b}$

5.1.2 Solving by Quadratic Formula

Problem: Solve $ax^2 + bx + c = 0$.

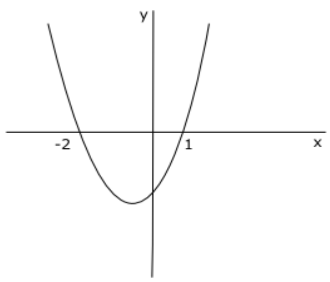
Solution:

$$\begin{aligned}
&ax^2 + bx + c = 0\\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \quad \text{where} \quad D = b^2 - 4ac\\
&x^2 - x - 6 = 0\\
&D = b^2 - 4ac = 1 - 4 \times 1 \times (-6) = 25\\
&x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 5}{2} = -2, 3
\end{aligned}$$

Problem: Solve the following equation: $x^2 + x - 2 = 0$.

Solution:

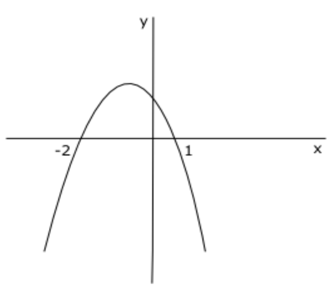
$$x^2 + x - 2 = 0$$
$$D = b^2 - 4ac = 1 - 4 \times 1 \times (-2) = 9$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-1 \pm 3}{2} = 1, -2$$



Problem: Solve the following equation: $-x^2 - x + 2 = 0$.

Solution:

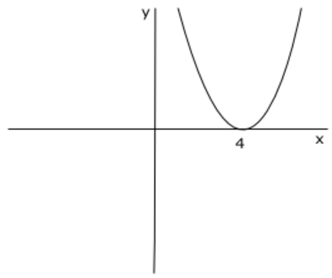
$$-x^2 - x + 2 = 0$$
$$D = b^2 - 4ac = (-1)^2 - 4 \times (-1) \times 2 = 9$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{1 \pm 3}{-2} = 1, -2$$



Problem: Solve the following equation: $x^2 - 8x + 16 = 0$.

Solution:

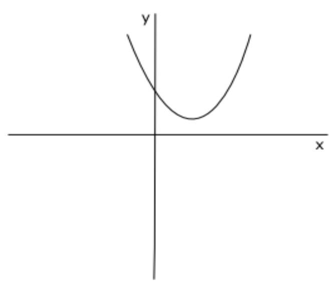
$$x^2 - 8x + 16 = 0$$
$$D = b^2 - 4ac = (-8)^2 - 4 \times 1 \times 16 = 0$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{8 \pm 0}{2} = 4$$



Problem: Solve the following equation: $x^2 - 4x + 10 = 0$.

Solution:

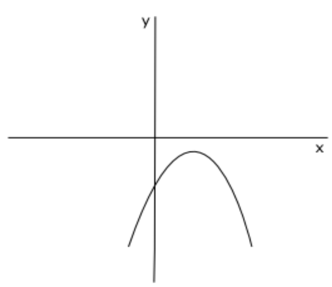
$$x^2 - 4x + 10 = 0$$
$$D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 16 = 16 - 64 = -48$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{8 \pm \sqrt{-48}}{2} \quad \text{the equation does not have any solutions}$$



Problem: Solve the following equation: $-2x^2 + 8x - 20 = 0$.

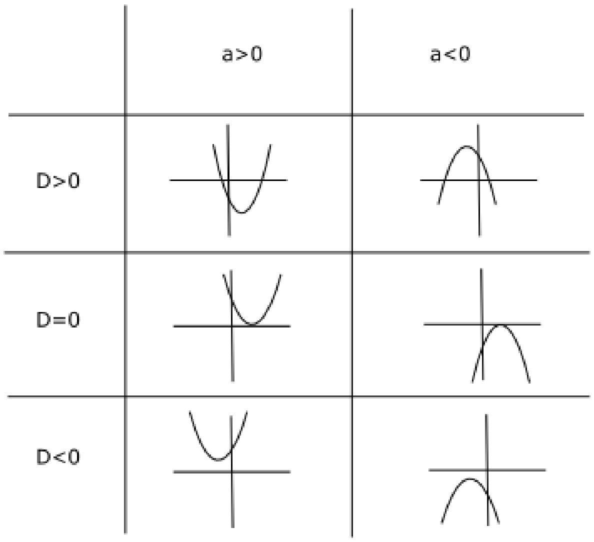
Solution:

$$-2x^2 + 8x - 20 = 0$$
$$D = b^2 - 4ac = 8^2 - 4 \times (-2) \times (-20) = 64 - 160 = -96$$
$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{-8 \pm \sqrt{-96}}{-4} \quad \text{the equation does not have any solutions}$$



Exercise 1: Solve: $x^2 - 10x + 25 = 0$, $x^2 + 2x - 8 = 0$, $x^2 - 2x + 10 = 0$.

Summary:



5.2 Quadratic Inequalities

Quadratic Inequalities have the following form: $ax^2 + bx + c > 0$

We always start solving quadratic inequality with solving a corresponding quadratic equality. This allows for sketching a graph of out quadratic function (parabola). From the graph we can determine the solution easily.

Problem: Solve $x^2 + 5x + 6 > 0$

Solution:

$$\begin{aligned} &x^2 + 5x + 6 = 0 \\ &(x+2)(x+3) = 0 \\ &x = -2, -3 \end{aligned}$$

Therefore, $x^2 + 5x + 6 > 0$ holds for all $x \in (-\infty, -2) \cup (-3, \infty)$.

Problem: Solve $x^2 - 5x + 4 < 0$

Solution:

$$\begin{aligned} &x^2 - 5x + 4 = 0 \\ &(x-1)(x-4) = 0 \\ &x = 1, 4 \end{aligned}$$

Therefore, $x^2 - 5x + 4 < 0$ holds for all $x \in (1, 4)$.

Exercise 2: Solve: $x^2 - 10x + 25 > 0$, $x^2 + 2x - 8 \leq 0$, $x^2 - 2x + 10 > 0$.

5.3 Answers

Exercise 1: 5; 2,-4; no solution

Exercise 2: $R - \{5\}$; [-4,2]; R (all numbers, i.e. infinitely many solutions)