AAU - Business Mathematics I

Lecture #1, February 27, 2010

1 Numbers, Sets, Algebraic Expressions

1.1 Constants, Variables, and Sets

A constant is something that does not change, over time or otherwise: a fixed value. Mathematical constant, a number that arises naturally in mathematics, such as

$\pi = 3.1415$ (the ratio of any circle's perimeter to its diameter)

and $e = 2.7182$ (Euler's number - the base of the natural logarithm - $\ln x = \log_e x$; with inverse function being exponential function e^x)

A variable is a symbol that stands for a value that may vary; the term usually occurs in opposition to constant, which is a symbol for a non-varying value, i.e. completely fixed or fixed in the context of use. For instance, in the formula $x + 1 = 5$, x is a variable which represents an "unknown" number.

Set: collection of distinct objects which are called elements (numbers, people, letters of alphabet)

Two ways of defining sets:

• list each member of the set (e.g. {4,2,15,6}, {red, blue, white}, ...)

The order in which the elements of a set are listed is irrelevant, as are any repetitions in the list. For example,

$$\{6, 11\} = \{11, 6\} = \{11, 11, 6, 11\}$$

are equivalent, because the specification means merely that each of the elements listed is a member of the set.

• rule (e.g. A = set of even numbers, $B = \{n^2, n \in N, 0 \leqq n \leqq 5\}$, ...)

Membership: some elements belong to a set and some do not

• $4 \in A$, $15 \in \{4, 2, 15, 6\}$, $16 \in B$ (read "$\in$" as "belongs to")

• $5 \notin A$, $5 \notin B$, green $\notin$ {red, blue, white} (read "$\notin$" as "does not belong to")

Subsets:

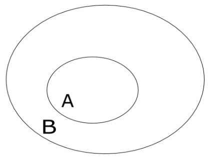
• $A \subseteq B$ if every member of A is in B as well (A is subset of B)

• if $A \subseteq B$ but $A \neq B$, then A is a proper subset of B, $A \subset B$

• $\{1, 2\} \subseteq \{1, 2, 3, 4\}$ and also $\{1, 2\} \subset \{1, 2, 3, 4\}$

• $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$ but it is not true that $\{1, 2, 3, 4\} \subset \{1, 2, 3, 4\}$

• set of men is a proper subset of the set of all people

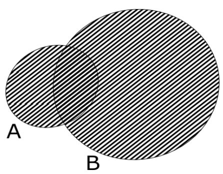
Venn diagram: 

Note: $A \subseteq A$, $\varnothing \subseteq A$ for every set A

1.2 Basic Operations on Sets

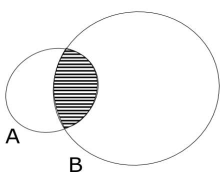
There are ways to construct new sets from existing ones. Two sets can be "added" together, "subtracted", etc.

• **Union:** $A \cup B$ elements that belong to A or B.



Example: $\{1, 2\} \cup \{\text{blue,red}\} = \{1, 2, \text{blue,red}\}$

• **Intersection:** $A \cap B$ elements that belong to A and B at the same time.



Example: $\{1, 2\} \cap \{\text{blue,red}\} = \varnothing$

$\{1, 2\} \cap \{1, 2, 4, 7\} = \{2\}$

Special Sets:

P - primes, N - natural numbers, Z - integers, $Q = \{\frac{a}{b}, a, b \in Z, b \neq 0\}$ - rational, R - real, I - irrational

π and e are examples of irrational numbers, i.e. their value cannot be expressed exactly as a fraction m/n, where m and n are integers. Consequently, its decimal representation never ends or repeats.

$P \subset N \subset Z \subset Q \subset R$

Note: We will use the concept of sets and membership in sections concerning functions and their domains.

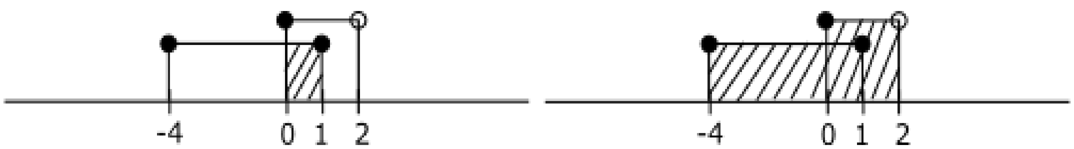
Real numbers (R): represented on real line with origin 0

Intervals: subsets of a real line

closed - e.g. [2,5] - 2 and 5 belong to the interval open - e.g. (3,9) - 3 and 9 do not belong to the interval

Intersection: $[-4, 1] \cap [0, 2) = [0, 1]$

Union: $[-4, 1] \cup [0, 2) = [-4, 2)$



Intersection Union

Example:

- $A = [-5, 3], B = [1, 10] \;\rightarrow\; A \cap B = [1, 3],\; A \cup B = [-5, 10]$
- $A = (-\infty, 2), B = (0, 4] \;\rightarrow\; A \cap B = (0, 2),\; A \cup B = (-\infty, 4]$
- $A = [-2, 8], B = (3, 10), C = (9, 15) \;\rightarrow\; A \cap B \cap C = \emptyset,\; A \cup B \cup C = [-2, 15)$

Exercise 2: Find intercept and union of the following intervals:

- $A = (0, 7], B = [1, 6]$
- $A = [-3, 4], B = (3, 10), C = (-1, 7)$

1.3 Algebraic Operations

Operations with Negative Numbers

A negative number is a real number that is less than zero, such as -3.

- **Addition and subtraction**: For purposes of addition and subtraction, one can think of negative numbers as debts.

  Adding a negative number is the same as subtracting the corresponding positive number:

  5 + (-3) = 5 - 3 = 2

  Subtracting a negative is equivalent to adding the corresponding positive:

  5 - (-2) = 5 + 2 = 7

- **Multiplication**: The square of a smaller number can be larger than the square of a larger number. For example, how could the square of -3 be larger than the square of -2, since -3 is smaller than -2.

  Multiplication of a negative number by a positive number yields a negative result. Multiplication of two negative numbers yields a positive result: (-4) × (-3) = 12.

- **Division**: Division is similar to multiplication. A positive number divided by a negative number is negative. If dividend and divisor have the same sign, the result is positive, even if both are negative.

Operations with Fractions

A fraction is a number that can represent part of a whole and is usually written as a pair of numbers, the top number called the **numerator** and the bottom number called the **denominator**.

- **Addition**: The first rule of addition is that only like quantities can be added; for example, various quantities of quarters. Unlike quantities, such as adding thirds to quarters, must first be converted to like quantities as described below: Imagine a pocket containing two quarters, and another pocket containing three quarters; in total, there are five quarters. Since four quarters is equivalent to one (dollar), this can be represented as follows:

  $$\frac{2}{4} + \frac{3}{4} = \frac{5}{4} = 1\frac{1}{4}$$

  If $\frac{1}{2}$ of a cake is to be added to $\frac{1}{4}$ of a cake, the pieces need to be converted into comparable quantities, such as cake-eighths or cake-quarters.

- **Adding unlike quantities**: To add fractions containing unlike quantities (e.g. quarters and thirds), it is necessary to convert all amounts to like quantities. It is easy to work out the type of fraction to convert to; simply multiply together the two denominators (bottom number) of each fraction.

  For adding quarters to thirds, both types of fraction are converted to $\frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$ (twelfths).

  Consider adding the following two quantities:

  $$\frac{3}{4} + \frac{2}{3} = \frac{9}{12} + \frac{8}{12} = \frac{17}{12} = 1\frac{5}{12}$$

  This method always works, but sometimes there is a smaller denominator that can be used (a least common denominator). For example, to add $\frac{3}{4}$ and $\frac{5}{12}$ the denominator 48 can be used (the product of 4 and 12), but the smaller denominator 12 may also be used, being the least common multiple of 4 and 12.

- **Subtraction**: The process for subtracting fractions is, in essence, the same as that of adding them: find a common denominator, and change each fraction to an equivalent fraction with the chosen common denominator. The resulting fraction will have that denominator, and its numerator will be the result of subtracting the numerators of the original fractions.

- **Multiplication**: When multiplying or dividing, it may be possible to choose to cancel down crosswise multiples (often simply called, 'canceling tops and bottom lines') that share a common factor. For example:

  $$2/7 \times 7/8 = 1/1 \times 1/4 = 1/4$$

  A two is a common factor in both the numerator of the left fraction and the denominator of the right so is divided out of both. A seven is a common factor of the left denominator and right numerator.

1.4 Algebraic Expressions

Algebraic expressions are formed using constants, variables and operators
e.g. $\sqrt{x^3+5}$, $x+y-7$, $(2x-y)^2$, ...

Polynomials are special algebraic expressions which include only addition, subtraction, multiplication and raising to a natural number powers

e.g. $4x^3-2x+7$ (polynomial of 3^{rd} degree), $x^3-3x^2y+xy^2+2y^7$ (7^{th} degree), $2x^3y^2-5x-2y^2$ (5^{th} degree), ...

Polynomial function has the following general form:

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

in which each term contains a coefficient as well as a nonnegative-integer power of the variable x. (We can write $x^1 = x$ and $x^0 = 1$, thus the first two terms may be taken to be a_0x^0 and a_1x^1 respectively.)

Depending on the value of the integer n (which specifies the highest power of x), we have several subclasses of polynomial function:

Case of $n = 0$:	$y = a_0$	*constant* function
Case of $n = 1$:	$y = a_0 + a_1x$	*linear* function
Case of $n = 2$:	$y = a_0 + a_1x + a_2x^2$	*quadratic* function
Case of $n = 3$:	$y = a_0 + a_1x + a_2x^2 + a_3x^3$	*cubic* function

Basic operations on polynomials

Addition: $(3x^3+2x+1)+(7x^2-x+3) = 3x^3+7x^2-x+4$

Subtraction: $(3x^3+2x+1)-(7x^2-x+3) = 3x^3+7x^2+3x-2$

Multiplication: $(2x-3)(3x^2-2x+3) = 2x(3x^2-2x+3)-3(3x^2-2x+3) = 6x^3-4x^2+6x-9x^2+6x-9 = 6x^3-13x^2+12x-9$

Special products: $(a+b)^2 = a^2+2ab+b^2$ NOT a^2+b^2!!!

$(a-b)^2 = a^2-2ab+b^2$ NOT a^2-b^2!!!

$a^2-b^2 = (a+b)(a-b)$

Factoring: Factor of an algebraic expression is one of two or more algebraic expressions whose product is the given algebraic expression; e.g. $x^2-4 = (x+2)(x-2)$. $(x+2)$ and $(x-2)$ are factors.

In general, quadratic function $y = a_0+a_1x+a_2x^2$ can be written as $(x-r_1)(x-r_2)$; cubic function $y = a_0+a_1x+a_2x^2+a_3x^3$ can be written as $(x-r_1)(x-r_2)(x-r_3)$ and so on. Numbers r_1, r_2, r_3 are called *roots*. Note that if all roots are integers then all roots must be divisors of a_0 (Why?).

Factoring by grouping

Here, the general idea, as the name of the method suggests, is to group in some sense similar terms together. The method is illustrated on the following examples:

Example: $x^3 + x^2 + x + 1 = x^2(x + 1) + (x + 1) = (x + 1)(x^2 + 1)$

Example: $x^3 - x^2 - 4x + 4 = x^2(x - 1) - 4(x - 1) = (x - 1)(x^2 - 4) = (x - 1)(x - 2)(x + 2)$

1.5 Rational Expressions

Rational expressions are fractional expressions whose numerator and denominator are polynomials.

Problem: Simplify $\frac{x^2-6x+9}{x^2-9}$.

Solution:

$$\frac{x^2 - 6x + 9}{x^2 - 9} = \frac{(x - 3)^2}{(x + 3)(x - 3)} = \frac{x - 3}{x + 3} \quad \text{for all} \quad x \neq \pm 3$$

Note: We use condition $x \neq \pm 3$ because for ± 3 there would be zero in denominator and the expression would not be well defined. Details will be explained in the section 7.2 below.

Problem: Reduce $\frac{6x^4(x^2+1)^2-3x^2(x^2+1)^3}{x^6}$ to the lowest terms.

Solution:

$$\frac{6x^4(x^2 + 1)^2 - 3x^2(x^2 + 1)^3}{x^6} = \frac{(x^2 + 1)^2[6x^4 - 3x^2(x^2 + 1)]}{x^6} =$$
$$= \frac{(x^2 + 1)^2 3x^2[2x^2 - x^2 - 1]}{x^6} = \frac{3(x^2 + 1)^2(x^2 - 1)}{x^4} \quad \text{for all} \quad x \neq 0$$

Exercise 3: Reduce $\frac{x^2+4x+4}{x+2}$ to the lowest terms.

Exercise 4: Reduce $\frac{x^2-2x+1}{1-x}$ to the lowest terms.

1.6 Least Common Denominator

Least common denominator: is found as follows: Factor each denominator completely; identify each different prime factor from all the denominators; form a product using each different factor to the highest power that occurs in any one denominator. This product is the LCD.

Example:
$$\frac{x^2}{x^2 + 2x + 1} + \frac{x - 1}{3x + 3} - \frac{1}{6} = \frac{x^2}{(x + 1)^2} + \frac{x - 1}{3(x + 1)} - \frac{1}{6} =$$
$$= \frac{6x^2 + 2(x + 1)(x - 1) - (x + 1)^2}{6(x + 1)^2} = \frac{7x^2 + 2x - 3}{6(x + 1)^2} \quad \text{for all} \quad x \neq -1$$

Exercise 5: Simplify by finding the LCD

$$\frac{x}{(x-2)(x+3)} + \frac{x^2}{x^2-2x} - \frac{2(x+1)}{x^2+6x+9}$$

More problems:

$$\begin{array}{lll}
x^a x^b = x^{a+b} & x^2x^4 = x^6 & 2^2 2^3 = 4.8 = 32 = 2^5 \\
(x^a)^b = x^{ab} & (x^2)^3 = x^6 & (2^2)^3 = 4^3 = 64 = 2^6 \\
x^{-a} = \dfrac{1}{x^a} & x^{-2} = \dfrac{1}{x^2} & 2^{-2} = \dfrac{1}{2^2} = \dfrac{1}{4} \\
x^{1/2} = \sqrt{x} & 9^{1/2} = \sqrt{9} = 3 &
\end{array}$$

Problem: Simplify:

$$\begin{array}{ll}
(a) & (x^6y^4)^{1/2}x^{-3}y^{-1} \\
(b) & (ab)^{-1} + \dfrac{a^3b^4}{a^2b^3}
\end{array}$$

Solution:

$$\begin{array}{ll}
(a) & (x^6y^4)^{1/2}x^{-3}y^{-1} = \dfrac{(x^6)^{1/2}(y^4)^{1/2}}{x^3y^1} = \dfrac{x^3y^2}{x^3y} = y, \quad x, y \neq 0 \\
(b) & (ab)^{-1} + \dfrac{a^2b^3}{a^3b^4} = \dfrac{1}{ab} + \dfrac{1}{ab} = \dfrac{2}{ab}, \quad a, b \neq 0
\end{array}$$

1.7 Answers

Exercise 1: The identity (a) holds and the identity (b) does not hold (to see this draw Venn diagrams).

Exercise 2:

- $A = (0,7], B = [1,6] \ \rightarrow \ A \cap B = [1,6], \ A \cup B = (0,7]$
- $A = [-3,4], B = (3,10), C = (-1,7) \ \rightarrow \ A \cap B \cap C = (3,4], \ A \cup B \cup C = [-3,10)$

Exercise 3: $x+2$.

Exercise 4: $1-x$.

Exercise 5:

$$\frac{x}{(x-2)(x+3)}+\frac{x^2}{x^2-2x}-\frac{2(x+1)}{x^2+6x+9}=\frac{x}{(x-2)(x+3)}+\frac{x}{x-2}-\frac{2(x+1)}{(x+3)^2}=$$
$$=\frac{x(x+3)+x(x+3)^2-2(x+1)(x-2)}{(x-2)(x+3)^2}=\frac{x^2+3x+x^3+6x^2+9x-2x^2+2x+4}{(x-2)(x+3)^2}=$$
$$=\frac{x^3+5x^2+14x+4}{(x-2)(x+3)^2}\quad \text{for all} \quad x\neq -3,0,2$$

2 Linear Equations

2.1 Numerical Solution

Equation: mathematical statement that relates two algebraic expressions involving at least one variable.

- $5x+3=2-x$; $x^3+3x^2-1=7+x-x^2$; $\frac{3}{x^2-x+1}=x+2$

Properties of equality:

1. if $a=b$ then $a+c=b+c$ — addition
2. if $a=b$ then $a-c=b-c$ — subtraction
3. if $a=b$ then $ca=cb$, $c\neq 0$ — multiplication
4. if $a=b$ then $\frac{a}{c}=\frac{b}{c}$, $c\neq 0$ — division
5. if $a=b$ then they can be used interchangeably — substitution

Linear equation has the following form: $ax+b=0$

To solve linear equations in one variable we use the properties of equality. Remember, that whatever you do with one side of the equation has to be done with the other side as well.

$7x-4=3$	add 4 to both sides of equation
$7x-4+4=3+4$	
$7x=7$	dived both sides of equation by 7
$\frac{7x}{7}=\frac{7}{7}$	
$x=1$	

Exercise 1: Solve the following equation and check if your result is correct: $6x+2=2x+14$.

Problem: Find 5 consecutive natural numbers such that their sum is 50.

Solution: Let's denote the first number x. Then the four remaining numbers are $x+1, x+2, x+3$ and $x+4$. Their sum is supposed to be equal to 50. So we have the following equation:

$$x + (x+1) + (x+2) + (x+3) + (x+4) = 50$$
$$5x + 10 = 50$$
$$5x = 40$$
$$x = 8$$

Hence the numbers are 8, 9, 10, 11 and 12.

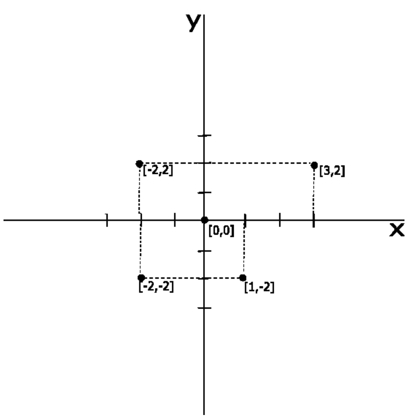
Exercise 2: Find 4 consecutive odd integers such that the sum of the last two is equal to 2 times the sum of the first two numbers.

2.2 Graphical Representation

Cartesian coordinate system, point, line

Cartesian coordinate system is formed by two real lines, one horizontal and one vertical, which cross through their origins. These two lines are called the horizontal axis and vertical axis.

Point: Every point is represented by two numbers - coordinates. The first number represents the value on axis x and the second number represents the value on axis y.



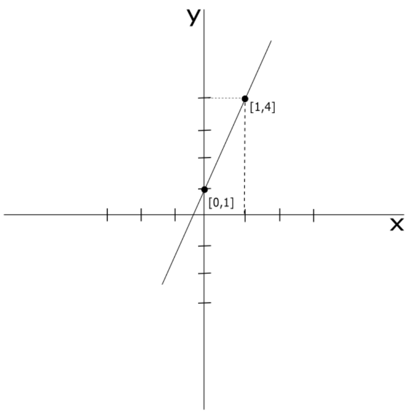
Linear function - Straight line:

Generally, linear function has the following form: $y = ax + b$. This can be graphically represented by a straight line. Any straight line can be represented by two points. If we find two points lying on the line, we can draw the whole line. Coefficient a is called *slope*. The bigger (smaller) a the steeper (flatter) the line.

Example: $y = 3x + 1$.

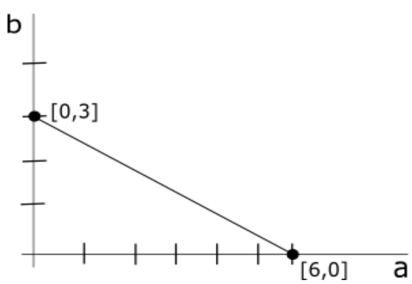
To find two points lying on this line we use 0 and 1 for x and find corresponding values of y from the equation:

x	0	1
y	$3 \times 0 + 1 = 1$	$3 \times 1 + 1 = 4$



In economics, we often deal with the budget constraint. We can draw the budget line or alternatively budget set in the following way:

Example: Assume that there are only two goods: apples and bananas. The price of apples is \$2 and the price of bananas is \$4. You have \$12. If you spend all the money on apples, you can afford to buy 6 of them. If you spend all the money on bananas, you can buy 3. So the budget line goes through points [6,0] and [0,3]. The budget line can be represented by the following equation $2a + 4b = 12$ and graphically:



Budget line represents all combinations of apples and bananas that we can buy spending *exactly* \$12.

The budget set represents all combinations of apples and bananas that we can afford, i.e. that we can buy spending *at most* \$12. This can be represented by inequality $2a + 4b \leq 12$ or graphically it is the triangle below the budget line.

In general, linear equation can be written in the following slope-intercept form: $y = mx + c$, where m is the slope of the line and c is the y-intercept, which is the y-coordinate of the point where the line crosses the y axis. This can be seen by letting $x = 0$, which immediately gives that $y = c$.

Special cases:

1. **y=const:** This is a special case of the general form where $A = 0$ and $B = 1$, or of the slope-intercept form where the slope $m = 0$. The graph is a horizontal line with y-intercept equal to *const*. There is no x-intercept, unless $const = 0$, in which case the graph of the line is the x-axis, and so every real number is an x-intercept.

2. **x=const:** This is a special case of the standard form where $A = 1$ and $B = 0$. The graph is a vertical line with x-intercept equal to *const*. The slope is undefined. There is no y-intercept, unless $const = 0$, in which case the graph of the line is the y-axis, and so every real number is a y-intercept.

2.3 Changes in Linear Equation

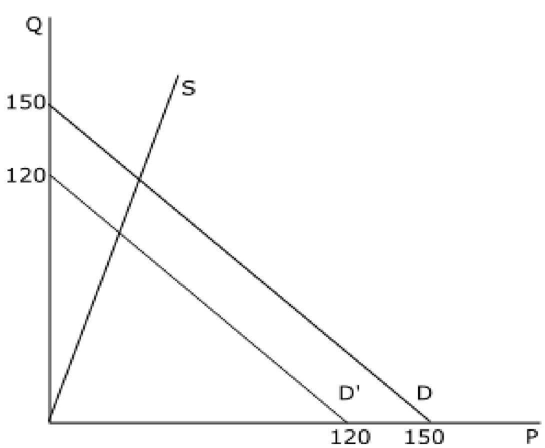
In this section we will look at what happens when parameters of the linear equation change. In particular, what happens with graphical representation of linear equation defined by slope-intercept form if parameters m and/or c change. Or, equivalently, what happens with graphical representation of linear equation defined by general form if parameters A, B and/or C change.

Example: Consider the following supply and demand function for beef.

Supply: $P = 4Q$

Demand: $P = 150 - Q$

Now suppose that people learn about the mad cow disease and decrease the demand to $P = 120-Q$. The situation is illustrated on the picture below. Notice that the two lines representing demand function are parallel. This is because the slope of the demand function remains the same only the intercept changed. In general, changes in intercept cause shifts in the line, changes in the slope "rotate" the line.



2.4 Answers

Exercise 1: $x = 3$.

Exercise 2: Let's denote the first number x. Then we need to solve equation $2[x + (x + 2)] = (x + 4) + (x + 6)$. Solution is $x = 3$ and hence the numbers are 3, 5, 7 and 9.

3 Systems of Linear Equations

3.1 Solving by Substitution

Eliminate one of the variables by replacement when solving a system of equations. Think of it as "grabbing" what one variable equals from one equation and "plugging" it into the other equation.

Problem: Solve the following system of equations:

$$\begin{aligned} &3x + 2y = 12 \\ &4x - y = 5 \end{aligned}$$

Solution:

$$\begin{aligned} &3x + 2y = 12 \\ &4x - y = 5 \qquad \Rightarrow y = 4x - 5 \end{aligned}$$

Now, plug $4x - 5$ for y in the first equation:

$$\begin{aligned} &3x + 2(4x - 5) = 12 \\ &3x + 8x - 10 = 12 \\ &11x = 22 \\ &x = 2 \end{aligned}$$

Now we get back to $y = 4x - 5$ and therefore $y = 4 \times 2 - 5 = 3$.

Problem: Solve the system of linear equations given below using substitution.

Suppose there is a piggybank that contains 57 coins, which are only quarters and dimes. The total number of coins in the bank is 57, and the total value of these coins is \$9.45. This information can be represented by the following system of equations:

$$\begin{aligned} &D + Q = 57 \\ &00.10D + 0.25Q = 9.45 \end{aligned}$$

Determine how many of the coins are quarters and how many are dimes.

Solution:

$$\begin{aligned} &D + Q = 57 \qquad\qquad \Rightarrow D = 57 - Q \\ &00.10D + 0.25Q = 9.45 \end{aligned}$$

Plug $57 - Q$ for D in the second equation

$$\begin{aligned} 00.10(57-Q)+0.25Q &= 9.45 \\ 5.7-0.1Q+0.25Q &= 9.45 \\ 0.15Q &= 3.75 \\ Q &= 25 \qquad\qquad D = 57-Q = 57-25 = 32 \end{aligned}$$

Exercise 1: Solve the following systems by substitution method:

(a) $x + y = 3, 2x + y = 1$

(b) $x + y = 4, 2x - y = -4$

3.2 Solving by Addition (Elimination) Method

The addition method says we can just add everything on the left hand side and add everything on the right hand side and keep the equal sign in between.

Problem: Solve the following system of equations:

$$3x + y = 14$$
$$4x - y = 14$$

Solution: Add the two equations; i.e sum left hand sides, sum right hand sides and keep equal sign in between. This way, we eliminate variable y and get only one equation in one variable x:

$$3x + 4x + y - y = 14 + 14$$
$$7x = 28$$
$$x = 4$$

Now we plug 4 for x and use any of two equations to determine y:

$$3x + y = 14$$
$$3 \times 4 + y = 14$$
$$y = 2$$

Check:

$$3x + y = 14 \ldots 3 \times 4 + 2 =^{?} 14 \ldots 14 =^{\checkmark} 14$$
$$4x - y = 14 \ldots 4 \times 4 - 2 =^{?} 14 \ldots 14 =^{\checkmark} 14$$

Exercise 2: Solve the following system of equations using elimination method:

$$2x + 2y = 12$$
$$3x - y = 14$$

Exercise 3: Solve the following system of equations using elimination method:

$$x + y = 1$$
$$2x + 2y = 2$$

Problem: Find the equilibrium price of apple and equilibrium quantity consumed if demand and supply equations are as follows:

$$p = -q + 20 \quad \text{Demand equation (consumer)}$$
$$p = 4q - 55 \quad \text{Supply equation (supplier)}$$

Solution:

$$p = -q + 20$$
$$p = 4q - 55 \quad \Rightarrow \quad -q + 20 = 4q - 55 \quad \Rightarrow \quad 5q = 75 \quad \Rightarrow \quad q = 15$$
$$p = -q + 20 = -15 + 20 = 5$$

3.3 Graphical Representation

We know already that an equation represents a straight line. Intuitively, the system of equations represents the system of lines. Solving system of equations means looking for the intercept of lines. See the following example:

Problem: Solve the following system numerically and graphically:

$x + y = 5$
$2x - y = 1$

Numerical solution to this system is $x = 2$ and $y = 3$.

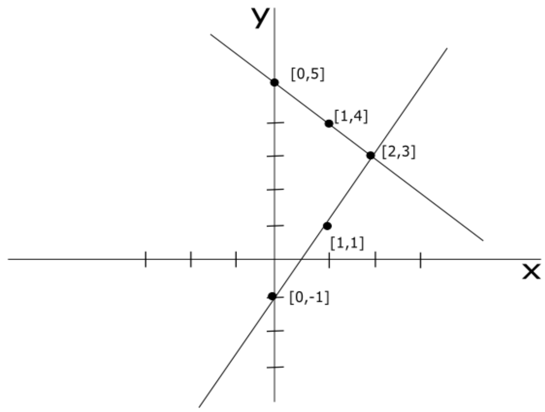
To find graphical solution we first need to draw both lines:

$x + y = 5$ or alternatively $y = 5 - x$

x	0	1
y	5	4

$2x - y = 1$ or alternatively $y = 2x - 1$

x	0	1
y	-1	1



The two lines intercept in point [2,3].

Generally, the system of two equations and two variables can have no solution, exactly one solution (see the example above) or infinitely many solutions.

Problem: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -4$

Solution:

$$
\begin{aligned}
&3x - y = 2 && \Rightarrow \quad y = 3x - 2\\
&-9x + 3y = -4 &&\\
\hline
&-9x + 3(3x - 2) = -4 &&\\
&-9x + 9x - 6 = -4 &&\\
&-6 = -4 &&
\end{aligned}
$$

The last equality does not hold for any values of x and y. This means that this system does not have any solution.

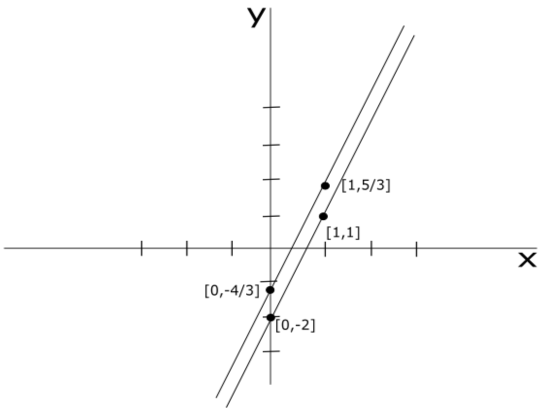
Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -4$ or alternatively $y = \frac{1}{3}(9x - 4)$

x	0	1
y	$-4/3$	$5/3$



From the picture we see that the two lines are parallel, i.e. they do not intercept in any point. That is the reason why the system does not have any solution.

Problem: Solve the following system numerically and graphically:

$3x - y = 2$
$-9x + 3y = -6$

Solution:

$$\begin{aligned} &3x - y = 2 \qquad \Rightarrow \quad y = 3x - 2 \\ &-9x + 3y = -6 \end{aligned}$$

$$\begin{aligned} &-9x + 3(3x - 2) = -6 \\ &-9x + 9x - 6 = -6 \\ &-6 = -6 \end{aligned}$$

The last equality holds for all values of x and y (-6 = -6 no matter what are the values of x and y). This means that this system has infinitely many solutions.

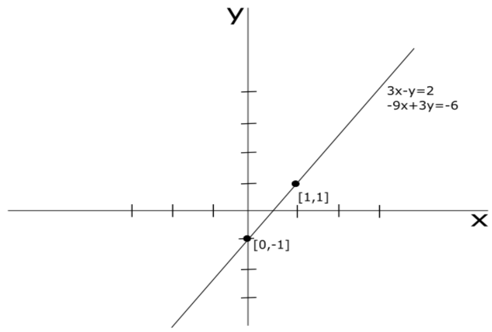
Graphically:

$3x - y = 2$ or alternatively $y = 3x - 2$

x	0	1
y	-2	1

$-9x + 3y = -6$ or alternatively $y = \frac{1}{3}(9x - 6) = 3x - 2$

Note that both lines are represented by the same equation. This means that the two lines coincide and therefore there are infinitely mane points where these two lines intercept and hence the system has infinitely many solutions.



Exercise 4: Solve the following system numerically and graphically:

$2x + 4y = 2, -x - 2y = -1$

Exercise 5: Solve the following system numerically and graphically:

$2x + 4y = 2, -x - 2y = 1$

Exercise 6: Solve the following system numerically and graphically:

$2x + 4y = 2, -x + 2y = 5$

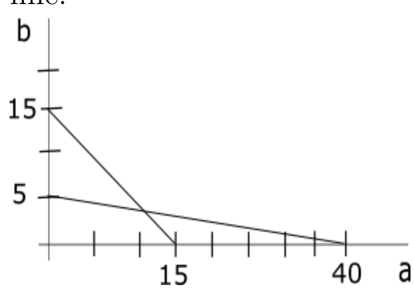
Problem: Assume that there are only two goods apples and bananas. Some company produces apple-banana juice. The budget of the company is \$200. The price of apples is \$5 and the price of bananas is \$40. Further, the company has a limited capacity and can only store 15 pieces of fruit at the time. Sketch the budget set, the production possibilities set and find on the graph all combinations of apples and bananas which are feasible in terms of money and capacity.

Solution:

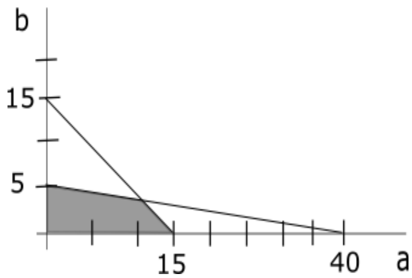
Budget set: The budget set is defined by the following inequality: $5a+40b \leq 200$. If the company buys only apples, it can buy 40 kilograms. If the company spends all the money on bananas only, it can afford 5 kilograms. Therefore, the budget line goes through points [40,0] and [0,5].

Production set: The production set is defined by the inequality: $a + b \leq 15$. If the company buys only apples, it can buy 15 kilograms of apples. Similarly for bananas.

The budget set and production set are depicted on the following figure. Two lines correspond to budget line and production line. Budget (production) set is the area below the budget (production) line.



Combinations of apples and bananas which are feasible in terms of money and capacity are combinations which belong to both sets at the same time. In other words, we find the intercept of two triangles. This intercept is represented be the shaded area on the picture below.



3.4 Changes in Systems of Linear Equations

In this section we look at what happens with the solution to the system of equations if one of them changes.

Problem: Consider the following supply and demand function for beef.

Supply: $P = 4Q$

Demand: $P = 150 - Q$

(a) Find the equilibrium

(b) Suppose now that the government impose taxes \$5 per unit sold. Find new equilibrium

(c) Illustrate the situation in (b) graphically. Does it matter whether the tax is imposed on the producers or the consumers? Explain.

Solution:

(a) To solve for market equilibrium we need to find solution to the following system:

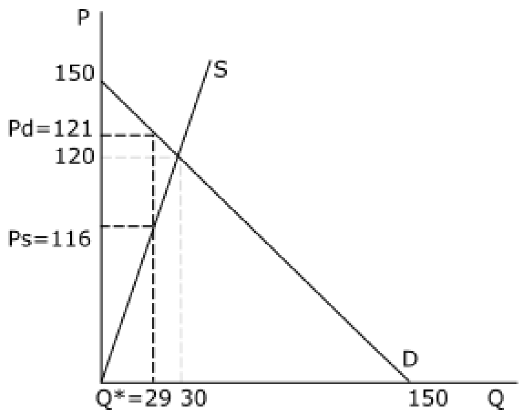
$$P = 4Q$$
$$P = 150 - Q$$
$$4Q = 150 - Q \quad \Rightarrow \quad Q^* = 30, P^* = 120$$

(b) We need to find price paid by buyers P_d, price received by sellers P_s, and quantity Q. Irrespective of who pays the tax (producer or consumer) the difference between P_d and P_s is \$5. So $P_s = P_d - 5$. If the tax is officially paid by producer, the system of two equations is as follows:

$$P_d - 5 = 4Q$$
$$P_d = 150 - Q$$
$$4Q + 5 = 150 - Q \quad \Rightarrow \quad Q^* = 29, \quad P_d^* = 121, \quad P_s^* = 116$$

(c) No, it does not matter. Look at the graph of the market, and put the tax on the graph. The tax puts a wedge between the price paid by buyers and the price received by sellers. If producer pays the tax we leave the demand function the same and shift supply function upwards. On the other hand, if consumer pays the tax we leave supply function unchanged and shift demand function downwards. One way or another the result is the same.

Economic explanation: No matter who formally pays the tax, the costs of the tax are borne by both sides of the transaction, and who pays what share depends on the relative elasticities (slopes of demand and supply curve). If the demand is relative inelastic in comparison to supply (the reaction of consumers on change in price is subtle) most of the tax will be paid by consumers. If on the other hand consumers are very sensitive to changes in price and producers are not, most of the tax will be paid by producers.



3.5 Answers

Exercise 1: (a) $x = -2, y = 5$; (b) $x = 0, y = 4$.

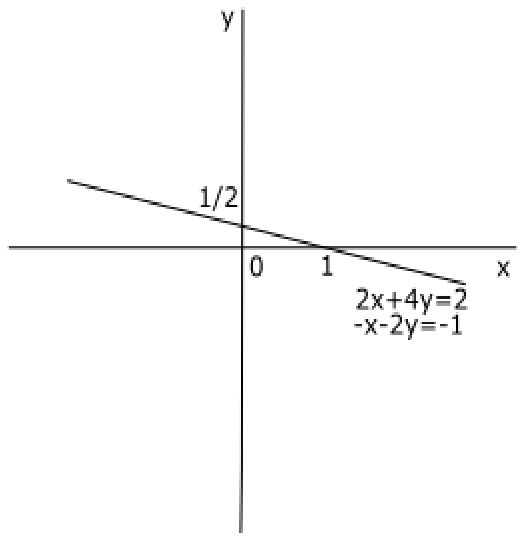
Exercise 2: $x = 5$ and $y = 1$.

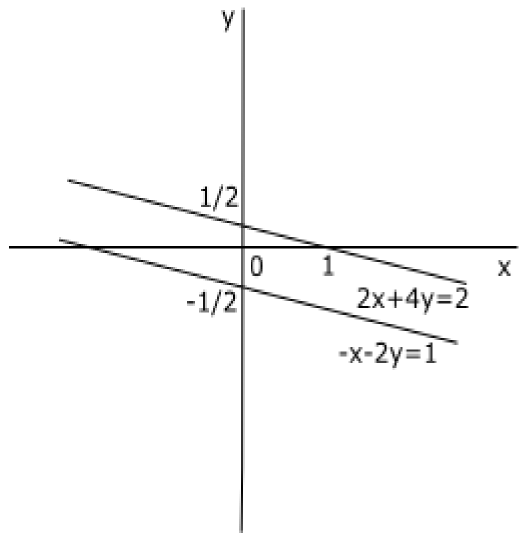
Exercise 3: Infinitely many solutions. All combinations of x and y such that $x + y = 1$.

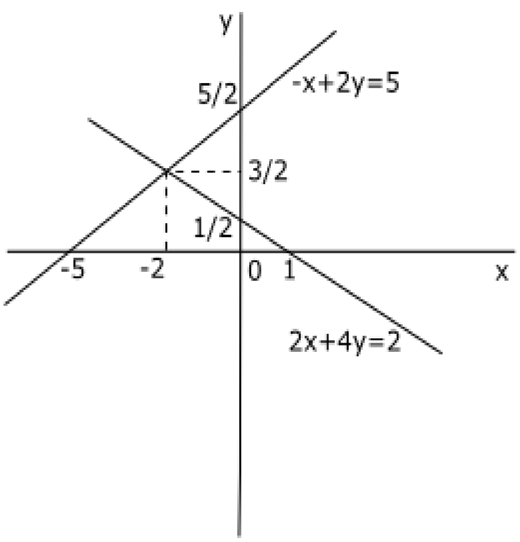
Exercise 4: Infinitely many solutions.

Exercise 5: No solution.

Exercise 6: $x = -2$, $y = 3/2$.


y
1/2
0
1
x
2x+4y=2
-x-2y=-1


y
1/2
0
1
x
-1/2
2x+4y=2
-x-2y=1


y
5/2
-x+2y=5
3/2
1/2
-5
-2
0
1
x
2x+4y=2