Economics II: Micro Winter 2009
Exercise session 4 Aslanyan: VŠE

1 Review

Game of strategy: A player is engaged in a game of strategy if that individual's payoff (utility) is determined not by that individual's action, but also by the action of others.

Extensive form: The extensive form of the game describes the rules of the game and payoff contingencies. It specifies the sequence of moves (i.e., who moves first, second, etc.), and the information each player has when it is that player's turn to move.

Perfect information games: Games of perfect information are those for which when it is any player's turn to move or chose an action, that player knows all the previous moves that have been made by other players.

Incomplete information games: Games where some players do not know the previous moves of others are called incomplete information games.

Information set: The information set represents the information available to an agent when that agent is about to choose an action.

Normal form: The normal form of a game states who the players of the game are, what their available actions are and the payoffs to each player as a function of the actions chosen by the players. Unlike the extensive form of a game, the normal form does not indicate the order of moves and also does not indicate what info the player has about the previous moves of the other players.

Nash Equilibrium: A Nash equilibrium to a game is a collection of strategies, one for each player, such that no player has an incentive to change their strategy if the others are playing their equilibrium strategies. In other words, in a Nash equilibrium the strategy of each player is the best for that player given what the strategies of the other players are.

2 Problems

Problem 1 (Cuban Missile Crisis) *There are two players: USA and USSR. The USA must decide whether to Blockade the ports of Cuba or have an Airstrike to eliminate the missiles. The USSR must decide whether to Withdraw the missiles or to Maintain them. The payoffs are as in the matrix below:*

		USSR	
		Withdraw	*Maintain*
USA	*Blockade*	$3, 3$	$2, 4$
	Airstrike	$4, 2$	$0, 0$

1. *How many Nash equilibria are there?*

Solution: If USA plays Blockade, the best response of USSR would be Maintain (as the payoff in that case is 4 as opposed to 3 in case of Withdraw). If USA plays Airstrike, the best response would be Withdraw. Thus $BR_{SU}(Blockade) = Maintain$ and $BR_{SU}(Airstrike) = Withdraw$. With the same logic: $BR_{US}(Withdraw) = Airstrike$ and $BR_{US}(Maintain) = Blockade$. Thus, as $BR_{SU}(Blockade) = Maintain$ and $BR_{US}(Maintain) = Blockade$, (*Blockade, Maintain*) is a Nash equilibrium. With the same logic (*Airstrike, Withdraw*) is another equilibrium. Hence, in this game there are two Nash equilibria: (*Blockade, Maintain*) and (*Airstrike, Withdraw*).

Second version of the solution: (*Blockade, Withdraw*) is not a Nash equilibrium as both the US and the SU have incentive to deviate in case the other player does not, *i.e.,* if US keeps playing Blockade, it would be better for the SU to play Maintain. Alternatively, if the SU keeps playing Withdraw, the US has incentive to play something other than Blockade (in our example Airstrike). Next, (*Blockade, Maintain*) is an equilibrium as no player has an incentive to deviate given the co-player's choice: The SU would not want to play anything else but Maintain if the US plays Blockade, and the US would not want to play anything else if the SU plays Maintain. With similar logic (*Airstrike, Withdraw*) is a Nash equilibrium, while (*Airstrike, Maintain*) is not. Once again we have two equilibria: (*Blockade, Maintain*) and (*Airstrike, Withdraw*).

Problem 2 (Prisoner's Dilemma) *The set-up is described at class*[1]. *Payoffs*

[1] In the lecture-notes version of the Prisoner's Dilemma we have years-in-prison written in the Game Matrix. In the text it is implicitly mentioned that the players are trying to minimise the years spent in the jail (a very realistic assumption), so while solving we are in a search for the smallest number, thus 5 (years in the prison) is preffered to 10. In the problem presented here we have more traditional notation of payoffs and payoff maximising behaviour.

as follows:

		Player B	
		Silence	*Confess*
Player A	*Silence*	$3, 3$	$1, 4$
	Confess	$4, 1$	$2, 2$

1. *Is (Silence, Silence) a Nash Equilibrium?*
2. *Is Confess dominant strategy for player B?*
3. *Is Silence dominant strategy for Player A?*

Solution: This is a standard Prisoner's dilemma.

1. (Silence, Silence) is not a Nash equilibrium as the Player A has an incentive to deviate from playing Silence if Player B plays Silence (same argument is true in case of Player B).
2. Yes, because Player B's payoffs are larger in case he plays Confess (compared to playing Silence) independent of Player A's choice.
3. No, because playing Silence brings less payoff.

Problem 3 *Players A and B must make bids for the division of a pot of gold equal to £1000. The rules are as follows: Each must bid an amount between zero and £1000. If the sum of their bids is less than or equal to £1000 then they both get their bids (e.g., if A bids £400 and B bids £100 then A gets £400 and B gets £100). If the sum of their bids exceeds £1000 then they both get nothing. Which pairs of bids are Nash equilibria? (For simplicity you can assume that players can make bids only in hundreds of pounds, i.e. £200 or £500 but not £350.)*

Solution: For solving this problem we have to consider several cases (similar to the problems above we need to discuss each and all choice combinations). First, suppose the choices of bids that sum up to £1000 (*e.g.*, 100 and 900, 500 and 500, and the like). All of those are Nash equilibria, as no player can get better off given the other player's choice: One of the players can ask less (and get less as the sum will still be less than £1000), but that would not be in her own interest; she also could ask for more, but then she will not get anything. Second, consider the case when both made a choice that in sum is less than £1000: These cannot be Nash equilibria, as at least one player would want to change her choice for a higher bid. Third, suppose that in sum they have asked for more than £1000 while each player made a bid for less than £1000: This is clearly not a Nash equilibrium as one of them can decrease the bid to a level where the sum is £1000 and will have a non-zero payoff (without caring about the other). For instance, if the bid was (800,800) the first player gets zero, while if he would ask £200 or less he will make more than 0. Fourth, suppose that in sum they have asked for more than £1000 while one player solely made a bid for more than £1000: This is again not a Nash equilibrium, as that player would want to decrease the size of the bid and have a non-zero payoff. And fifth, suppose both players made a bid exceeding £1000: this is a Nash equilibrium: No player has an incentive to deviate as no other bid will bring non-zero payoff. (Note that if both of them change their bids both of them would be better off; however, for a Nash equilibrium only one player may change her bid.)

Problem 4 (Battle of the Sexes) *There are two players, Husband and Wife, who must decide whether to go to watch Ballet or Football. The payoffs are as in the matrix below:*

		Wife	
		Ballet	*Football*
	Ballet	3, 4	1, 1
Husband			
	Football	2, 2	4, 3

1. *is (Football, Ballet) a Nash equilibrium?*
2. *is (Football, Football) a Nash equilibrium?*
3. *does player Wife have a dominant strategy?*
4. *Now consider the extensive form where Husband moves first, and then Wife, having observed Husband's move, then plays second. What's the (sub-game perfect) Nash equilibrium to this extensive form game?*

Solution: Questions 1.-3. are clear: (Ballet, Ballet) and (Football, Football) are equilibria and the other two are not; and obviously there is no dominant strategy in the game. With the fourth part of the problem you have to notice that now the player Husband is going to choose on the first period knowing the behaviour of the player Wife in the second period. Thus, for subgame perfect first we consider the moves of wife. If the Wife finds herself in the node where the Husband has played Ballet, she will also play Ballet. Thus the Husband knows that if he plays Ballet the payoff is (3,4). And similarly, if he plays Football, the wife will choose Football too and the payoffs are (4,3). Knowing this the Husband will play Football, as he will have payoff 4 as opposed to 3 in case of Ballet. Thus the subgame perfect Nash Equilibrium is (Football, Football) with payoffs (4,3). (We can notice here that 'sub-game-perfection' eliminated one of the Nash equilibria. Also we should notice the importance of timing!)

Problem 5 *Suppose a man and a woman each choose whether to go to a Football match or a Ballet performance. The man would rather go to Football and the woman to Ballet. What is more important to them however is that the man wants to show up to the same event as the woman (he adores her) but the woman wants to avoid him (she cannot stand his... whatever... something... cologne). Construct a game matrix to illustrate this game, choosing numbers to fit the preferences described verbally.*

Solution: Try to think yourself! The answer in general is:
Assume the matrix:

		Man	
		Ballet	Football
Woman	Ballet	α, β	γ, δ
	Football	ε, ζ	η, θ

The third sentence tells us:

$$\begin{array}{ll} \beta > \delta & \theta > \delta \\ \beta > \zeta & \theta > \zeta \\ & \\ \alpha < \varepsilon & \eta < \gamma \\ \alpha < \gamma & \eta < \varepsilon \end{array}$$

The second sentence is summarised as follows:

$$\begin{array}{ll} \gamma > \varepsilon & \theta > \beta \\ & \\ \delta > \zeta & \eta < \alpha \end{array}$$

Solution of the system of 12 inequalities results in:

$$\begin{cases} \theta > \beta > \delta > \zeta \\ \gamma > \varepsilon > \alpha > \eta \end{cases}$$

So any $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$ that satisfy the system above will be a solution. Example, $\theta = 3$, $\beta = 2$, $\delta = 1$, $\zeta = 0$, $\gamma = 3$, $\varepsilon = 2$, $\alpha = 1$, $\eta = 0$ or $\theta = -1$, $\beta = -2$, $\delta = -3$, $\zeta = -4$, $\gamma = 3000$, $\varepsilon = 2000$, $\alpha = 10$, $\eta = 2$.

Problem 6 (2x5 game) *Consider the normal form game presented below. Find Nash equilibria for the 2x5 game? (Yes, each player has 5 actions!)*

		Mrs. Column				
		$\beta\,(1)$	$\beta\,(2)$	$\beta\,(3)$	$\beta\,(4)$	$\beta\,(5)$
	$\alpha\,(1)$	$0,4$	$18,1$	$8,9$	$7,5$	$11,7$
	$\alpha\,(2)$	$11,5$	$8,8$	$0,8$	$13,12$	$2,0$
Mr. Row	$\alpha\,(3)$	$11,3$	$10,5$	$9,4$	$9,2$	$10,1$
	$\alpha\,(4)$	$11,5$	$19,3$	$1,9$	$9,8$	$2,10$
	$\alpha\,(5)$	$12,4$	$15,0$	$5,1$	$5,1$	$4,3$

Solution:

		Mrs. Column				
		$\beta\,(1)$	$\beta\,(2)$	$\beta\,(3)$	$\beta\,(4)$	$\beta\,(5)$
	$\alpha\,(1)$	$0,4$	$18,1$	$8,9^*$	$7,5$	$11^*,7$
	$\alpha\,(2)$	$11,5$	$8,8$	$0,8$	*13*,12**	$2,0$
Mr. Row	$\alpha\,(3)$	$11,3$	$10,5^*$	$9^*,4$	$9,2$	$10,1$
	$\alpha\,(4)$	$11,5$	$19^*,3$	$1,9$	$9,8$	$2,10^*$
	$\alpha\,(5)$	*12*,4**	$15,0$	$5,1$	$5,1$	$4,3$

Problem 7 *Provide an example of a simple two-person game, with each player having two actions, to illustrate that the following statement is NOT always true: 'If your opponent does NOT play her Nash equilibrium strategy you should, however, still always play yours since it is always in your best interest to do so.'*

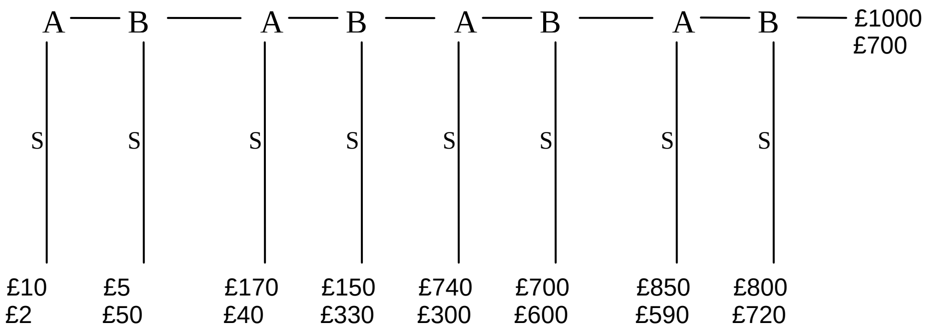
Solution: There are infinitely many answers. One general answer (still not the only) is, considering the matrix

		Player 2	
		Left	Right
	Up	α,β	γ,δ
Player 1			
	Down	ε,ζ	η,θ

and assuming that $\alpha > \varepsilon$ and $\beta > \delta$, i.e. (Up, Left) as an Nash equilibrium. Further assume that $\gamma > \eta$ and $\theta > \zeta$. So in this case if Player 1 is not going to play her Nash equilibrium strategy *Up*, then it is better for Player 2 also not to play his. A numerical example:

		Player 2	
		Left	Right
Player 1	Up	$5,4$	$2,0$
	Down	$1,3$	$0,4$

Problem 8 (Centipede Game) *Consider the following extensive form game: There are two players, A and B. Players take turns in making decisions begining with Player A. At each decision node each player must choose either the action <C> meaning 'continue' in which case the other player has a chance to make a decision; or to choose <S> meaning 'stop,' in which case the game ends and the payoffs indicated in the extensive form (the game tree) are received. As indicated bellow, the game will automatically end if <C> is played by each player 4 times and the payoff will then be (1000,700) for A and B, respectively. What is the sub-game perfect Nash equilibrium to this extensive form game?*



Solution: As during each choice the next player is going to prefer to stop the game, at the very beginning Player A will stop the game and the payoff will be (10,2).

Problem 9 (3x2 game) *Consider the following three-person game in which player 1 chooses the row, player 2 chooses the column, and player 3 chooses the matrix that will be played.*

$\mathfrak{L}$	L	R
l	*6,3,2*	*4,8,6*
r	*2,3,9*	*4,2,0*

$\mathfrak{R}$	L	R
l	*8,1,1*	*0,0,5*
r	*9,4,9*	*0,0,0*

The first number i each cell is the payoff to player 1, the second number is the payoff to player 2, and the third number is the payoff to player 3. Find the Nash equilibrium for this game.

Solution: This problem is very easy to solve once you know how to work with the game matrices. We are in search for the best responses. So once again, the best response of Player 3 (choosing matrices) in case Player 1 plays l, and Player 2 plays L is the highest of 2 and 1:

$\mathfrak{L}$	L	R
l	6,3,**2**	4,8,6
r	2,3,9	4,2,0

$\mathfrak{R}$	L	R
l	8,1,**1**	0,0,5
r	9,4,9	0,0,0

Thus, the best response to (l,L) is $\mathfrak{L}$. To (r,L) is both $\mathfrak{L}, \mathfrak{R}$, to (l,R) is $\mathfrak{L}$, to (r,R) is both $\mathfrak{L}, \mathfrak{R}$. Best response for Player 2 (columns) to ($\mathfrak{L}$,l) is R (found by comparing 3 to 8):

$\mathfrak{L}$	L	R
l	6,**3**,2	4,**8**,6
r	2,3,9	4,2,0

R	L	R
l	8,1,1	0,0,5
r	9,4,9	0,0,0

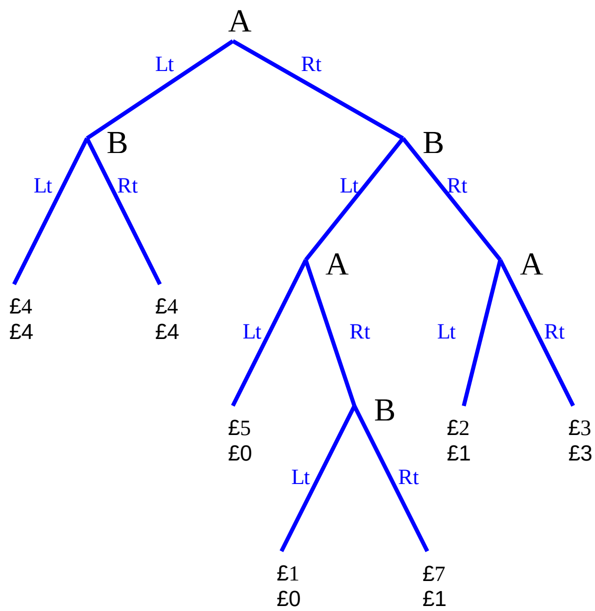
Finally, the Nash equilibria are: (l,R,$\mathfrak{L}$) and (r,L,$\mathfrak{R}$) :

$\mathfrak{L}$	L	R
l	6,3,2	*4,8,6*
r	2,3,9	4,2,0

$\mathfrak{R}$	L	R
l	8,1,1	0,0,5
r	*9,4,9*	0,0,0

Alternative method (comparing the cells): if we are in the first cell (l,L,$\mathfrak{L}$), Player 1 would not want to change her strategy to r as in this case she will get 2 instead of 6; Player 2 would want to change his strategy to R as he will get 8 instead of 3, thus this cell cannot be a Nash equilibrium (at least one player has an incentive to deviate). Next consider the cell (l,R,$\mathfrak{L}$) 4 is more than 4 (so player 1 is fine), 8>3 so player 2 is fine, and 6>5 so player 3 is fine: thus a Nash equilibrium. Following the same steps for the other 6 cells you find the same solution the Nash equilibria are: (l,R,$\mathfrak{L}$) and (r,L,$\mathfrak{R}$).

Problem 10 (Backward induction) *Use backward induction to find subgame perfect equilibrium given following game in extensive form:*



Solution 11 *See the picture*