M9302 Mathematical Models in Economics

HANDOUT 7 - 18.05.2011

1. First-price, sealed-bid auction /static game of incomplete information/

Consider a classical market mechanism called first-price, sealed-bid auction. There is a single good offered for sale. Two bidders have independent valuations of the good. The valuation of a bidder is private information, nobody else than the bidder knows it but all have prior beliefs about it. The bidders simultaneously submit their bids (in a sealed envelope). After the bids are submitted, the auctioneer opens the envelopes. The higher bidder wins the good and pays the (first) price she bid; the other bidder gets and pays nothing. In case of a tie, the winner is determined by a flip of a coin. The bidders are risk-neutral. All of this is common knowledge.

Normal-form Representation:

PLAYERS: bidder i=1,2
TYPE SPACES: bidder *i*'s valuation $\boldsymbol{v_i}$ of the good sold
BELIEFS: $\boldsymbol{v_i}$ is independently and uniformly distributed on [0,1]
ACTION SPACES: bidding function $b_i(v_i) \geq 0$
PAYOFFS: Bidder *i*'s payoff function is given by the following expression:

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i \text{ if } b_i > b_j \\ (v_i - b_i)/2 \text{ if } b_i = b_j \\ 0 \text{ if } b_i < b_j \end{cases}$$

Solution:

We search for Bayesian Nash Equilibrium where player 1's strategy is a best response to player 2's strategy, and vice versa.

First, let's construct the players' strategy spaces.
In a static Bayesian game, a strategy is a function that maps player's type into her action.

Here, a **strategy for bidder *i*** is a function $\boldsymbol{b_i(v_i)}$ specifying the bid that each of *i*'s types (i.e., valuations) would choose. Respectively, the pair of strategies $(b_1(v_1), b_2(v_2))$ is a **Bayesian Nash equilibrium** if for each v_i in [0,1], $b_i(v_i)$ maximizes bidder *i*'s expected payoff given the common prior beliefs:

$$\max_{b_i} (v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\} \qquad (*)$$

Let's check for a linear equilibrium that is a Bayesian Nash equilibrium where both bidders choose a linear strategy as follows:

$b_1(v_1) = a_1 + c_1 v_1$ and $b_2(v_2) = a_2 + c_2 v_2$

N.B. The latter assumption does not imply that there could not be non-linear equilibrium solution. It just says that here we are mainly interested in checking if linear equilibrium exists.

The objective function (*) could be simplified by using the fact that when $b_j(v_j) = a_j + c_j v_j$ and v_i is believed to be uniformly distributed, b_j must also be uniformly distributed (i.e. $\text{Prob}\{b_i = b_j(v_j)\}$=0). Namely it becomes:

$$\max_{b_i}(v_i - b_i)\text{Prob}\{b_i > b_j(v_j) = a_j + c_j v_j\} \qquad (**)$$

Since it is pointless for player *i* to bid below player *j*'s minimum bid and stupid for *i* to bid above *j*'s maximum bid, we have $a_j \le b_i \le a_j + c_j$ (***)
Then,

$$\text{Prob}\{b_i > a_j + c_j v_j\} = \text{Prob}\left\{v_j < \frac{b_i - a_j}{c_j}\right\} = \frac{b_i - a_j}{c_j}$$

which after substitution to (**) and taking first-order condition for optimality yields:

$$b_i(v_i) = \frac{v_i + a_j}{2}$$

That in order to hold together with (***) the following restriction on a_j need to be introduced:

$$a_j \le 0$$

Hence, by comparison we get $a_i = \frac{a_j}{2}$, $c_i = \frac{1}{2}$. However, the solution is symmetric. Therefore, we also have $a_i \le 0$, $a_j = \frac{a_i}{2}$, $c_j = \frac{1}{2}$. These, however, only hold when $a_i = a_j = 0$. That is, the best response strategy for bidder *i* is $b_i(v_i) = \frac{v_i}{2}$.
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Asymmetric discrete-value, first-price, sealed-bid auction:

Consider an auction with 2 bidders. There is one object for sale.
The value which bidder 1 attaches to the object is 1 with probability ½, and 5 with probability ½.
The value which bidder 2 attaches to the object is 0 with probability ¼, 3 with probability ½ and 7 with probability ¼.
Bids can only be integers: 0, 1, 2, 3, … etc. to infinity.

Find the Bayesian Nash equilibrium i.e. the optimal bids in this auction game.

Hint: For each type of bidder i (i = 1, 2), compute the expected payoffs of a feasible bid, starting from the lowest and working your way up to the highest integer.

Solution:

Expected payoffs of the two bidders are given by the following general expressions:

$$Eu_1(b_1) = (v_1 - b_1)\text{Prob}\{b_1 > b_2\} + \frac{(v_1 - b_1)}{2}\text{Prob}\{b_1 = b_2\} =$$
$$= (v_1 - b_1)\left\{\left(\frac{1}{4}\text{Prob}\{b_1 > b_2(0)\} + \frac{1}{2}\text{Prob}\{b_1 > b_2(3)\} + \frac{1}{4}\text{Prob}\{b_1 > b_2(7)\}\right) + \right.$$
$$\left. + \left(\frac{1}{8}\text{Prob}\{b_1 = b_2(0)\} + \frac{1}{4}\text{Prob}\{b_1 = b_2(3)\} + \frac{1}{8}\text{Prob}\{b_1 = b_2(7)\}\right)\right\}$$

$$Eu_2(b_2) = (v_2 - b_2)\text{Prob}\{b_2 > b_1\} + \frac{(v_1 - b_1)}{2}\text{Prob}\{b_2 = b_1\} =$$
$$= (v_2 - b_2)\left(\frac{1}{2}\text{Prob}\{b_2 > b_1(1)\} + \frac{1}{2}\text{Prob}\{b_2 > b_1(5)\} + \frac{1}{4}\text{Prob}\{b_2 = b_1(1)\} + \frac{1}{4}\text{Prob}\{b_2 = b_1(5)\}\right)$$

Let v_2 = 0. Hence, the optimal bid for bidder 2 is $\mathbf{b_2(0) = 0}$ because anything else would yield a negative payoff.

Let v_1=1, b_1=1. Then, $Eu_1(1)$=0

Let v_1=1, b_1=0. Then, $Eu_1(0) = (1-0)\left(\frac{1}{8}\cdot 1 + \frac{1}{4}\text{Prob}\{b_2(3) = 0\} + \frac{1}{8}\text{Prob}\{b_2(7) = 0\}\right)$

since $\text{Prob}\{b_1 > b_2(0)\} = \text{Prob}\{b_1 > b_2(3)\} = \text{Prob}\{b_1 > b_2(7)\} = 0$ and $\text{Prob}\{b_2(0) = b_1 = 0\} = 1$

Hence, the optimal bid for bidder 2 is $\mathbf{b_1(1) = 0}$.

Let v_2=3, b_2=3. Then, $Eu_2(3)$=0

Let v_2=3, b_2=2. Then, $Eu_2(2) = (3-2)\left(\frac{1}{2}\cdot 1 + \frac{1}{2}\text{Prob}\{2 > b_1(5)\} + \frac{1}{4}\text{Prob}\{2 = b_1(5)\}\right)$

since $\text{Prob}\{2 > b_1(1) = 0\} = 1$ and $\text{Prob}\{2 = b_1(1) = 0\} = 0$

Let v_2=3, b_2=1. Then, $Eu_2(1) = (3-1)\left(\frac{1}{2}\cdot 1 + \frac{1}{2}\text{Prob}\{1 > b_1(5)\} + \frac{1}{4}\text{Prob}\{1 = b_1(5)\}\right) =$

$$= 2\cdot\left(\frac{1}{2}\cdot 1 + \frac{1}{2}\text{Prob}\{0 = b_1(5)\} + \frac{1}{4}\text{Prob}\{1 = b_1(5)\}\right)$$

since $\text{Prob}\{1 > b_1(1) = 0\} = 1$ and $\text{Prob}\{1 = b_1(1) = 0\} = 0$

Let v_2=3, b_2=0. Then, $Eu_2(0)=(3-0)\left(\frac{1}{4}\cdot 1+\frac{1}{4}\text{Prob}\{0=b_1(5)\}\right)$

since $\text{Prob}\{0>b_1(1)=0\}=0$ and $\text{Prob}\{0=b_1(1)=0\}=1$

Hence, the optimal bid for bidder 2 is $\mathbf{b_2(3) = 1}$.

Let v_1=5, b_1=5. Then, $Eu_1(5)=0$

Let v_1=5, b_2=4. Then, $Eu_2(4)=(5-4)\left(\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 1+\frac{1}{4}\text{Prob}\{4>b_2(7)\}+\frac{1}{8}\text{Prob}\{4=b_2(7)\}\right)$

Let v_1=5, b_1=3. Then, $Eu_2(3)=(5-3)\left(\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 1+\frac{1}{4}\text{Prob}\{3>b_2(7)\}+\frac{1}{4}\text{Prob}\{3=b_2(7)\}\right)$

Let v_1=5, b_1=2. Then, $Eu_2(2)=(5-2)\left(\frac{1}{2}\cdot 1+\frac{1}{4}\cdot 1+\frac{1}{4}\text{Prob}\{2>b_2(7)\}+\frac{1}{4}\text{Prob}\{2=b_2(7)\}\right)$

Let v_1=5, b_1=1. Then, $Eu_2(1)=(5-1)\left(\frac{1}{4}\cdot 1+\frac{1}{4}\cdot 1+\frac{1}{4}\text{Prob}\{1>b_2(7)\}+\frac{1}{8}\text{Prob}\{1=b_2(7)\}\right)$

Let v_1=5, b_1=0. Then, $Eu_2(0)=(5-0)\left(\frac{1}{8}\cdot 1+\frac{1}{8}\text{Prob}\{0=b_2(7)\}\right)$

Hence, the optimal bid for bidder 2 is $\mathbf{b_1(5) = 2}$.

Let v_2=7, b_2=7. Then, $Eu_2(7)=0$

Let v_2=7, b_2=6. Then, $Eu_2(6)=(7-6)=1$

Let v_2=7, b_2=5. Then, $Eu_2(5)=(7-5)=2$

Let v_2=7, b_2=4. Then, $Eu_2(4)=(7-4)=3$

Let v_2=7, b_2=3. Then, $Eu_2(3)=(7-3)=4$

Let v_2=7, b_2=2. Then, $Eu_2(2)=(7-2)\left(\frac{1}{2}+\frac{1}{4}\right)=\frac{15}{4}\approx 3.7$

Let v_2=7, b_2=1. Then, $Eu_2(1)=(7-1)\left(\frac{1}{2}\right)=3$

Let v_2=7, b_2=0. Then, $Eu_2(0)=(7-0)\left(\frac{1}{4}\right)=\frac{7}{4}\approx 1.7$

Hence, the optimal bid for bidder 2 is $\mathbf{b_2(7) = 3}$.

Finally, the Bayesian Nash equilibrium is given by the following pair of discrete functions:

$$b_1(v_1)=\begin{cases}0, \text{ for } v_1=1\\ 2, \text{ for } v_1=5\end{cases} \text{ and } b_2(v_2)=\begin{cases}0, \text{ for } v_2=0\\ 1, \text{ for } v_2=3\\ 3, \text{ for } v_2=7\end{cases}$$

2. Signaling game /dynamic game of incomplete information/

A signaling game is a dynamic game of incomplete information involving two players: a Sender (S) and a Receiver (R). The timing of the game is as follows:

1) Nature draws a type t_i for the Sender from a set of feasible types $T = \{t_1,\ldots,t_I\}$ according to a probability distribution $p(t_i)$, where $p(t_i)>0$ for every i and $p(t_1)+\ldots+p(t_I)=1$.

2) The Sender observes t_i and then chooses a message m_j from a set of feasible messages $M = \{m_1,\ldots,m_I\}$.

3) The Receiver observes m_j (but not t_i) and then chooses an action a_k from a set of feasible actions $A = \{a_1,\ldots,a_K\}$.

4) Payoffs are given by $U_S(t_i,m_j,a_k)$ and $U_R(t_i,m_j,a_k)$.

The strategies in the signaling game are different for the Sender and the Receiver and are given by the functions:

A pure strategy for the Sender is a function $m(t_i)$ specifying which message will be chosen for each type that nature might draw. A pure strategy for the Receiver is a function $a(m_j)$ specifying which action will be chosen for each message that the Sender might send.

Pure-strategy perfect Bayesian equilibrium in a signaling game – a pair of strategies $m^*(t_i)$ and $a^*(m_j)$ and a belief $\mu(t_i|m_j)$ satisfying Signaling Requirements (1), (2R), (2S) and (3), as follows:

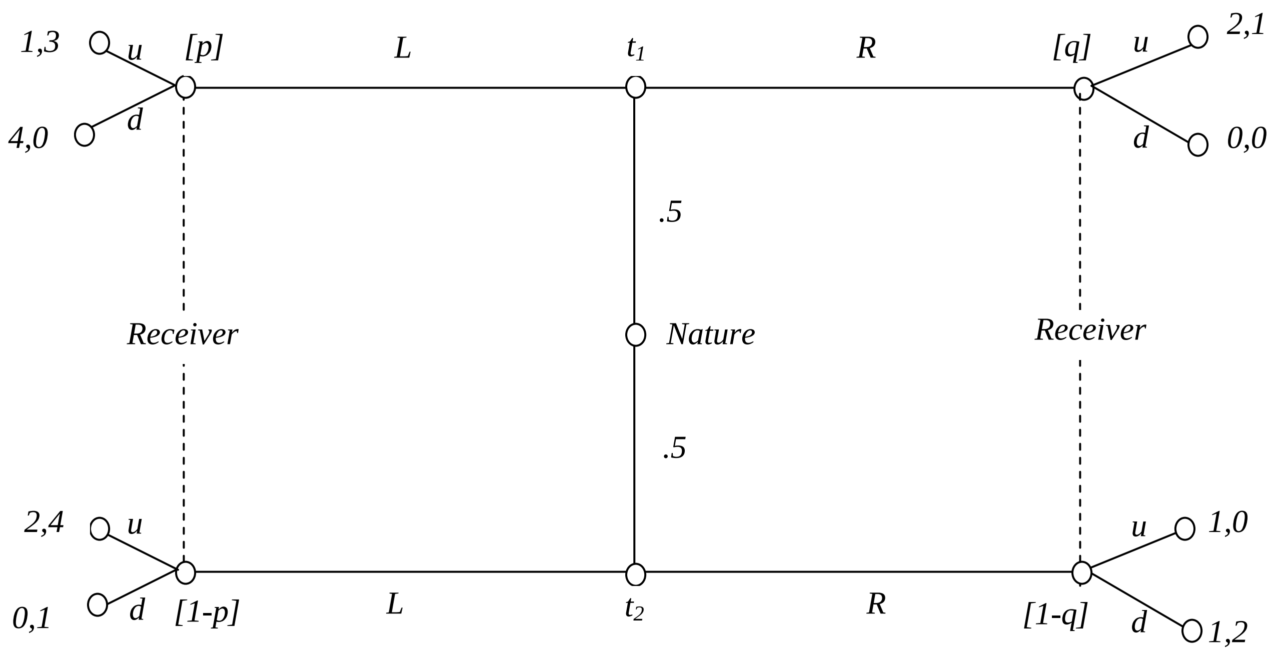
<u>Signaling Requirement 1:</u> After observing any message m_j from M, the Receiver must have a belief about which types could have sent m_j. This belief is denoted by the probability distribution $\mu(t_i|m_j)$, where $\mu(t_i|m_j)\geq 0$ for each t_i in T, and $\sum_{t_i\in T}\mu(t_i|m_j)=1$.

<u>Signaling Requirement 2R:</u> For each m_j in M, the Receiver's action $a^*(m_j)$ must maximize the Receiver's expected utility, given the belief $\mu(t_i|m_j)$ about which types could have sent m_j. That is, $a^*(m_j)$ solves $\max_{a_k\in A}\sum_{t_i\in T}\mu(t_i|m_j)U_R(t_i,m_j,a_k)$.

<u>Signaling Requirement 2S:</u> For each t_i in T, the Sender's message $m^*(t_i)$ must maximize the Sender's utility, given the Receiver's strategy $a^*(m_j)$. That is, $m^*(t_i)$ solves $\max_{m_j\in M}U_S(t_i,m_j,a^*(m_j))$.

Signaling Requirement 3: For each m_j in M, if there exists t_i in T such that $m^*(t_i)=m_j$, then the Receiver's belief at the information set corresponding to m_j must follow from Bayes' rule and the Sender's strategy i.e. $\mu(t_i|m_j)=\dfrac{p(t_i)}{\sum_{t_i \in T_i} p(t_i)}$.

Extensive-form representation of a signaling game:



Distinct from a standard dynamic game, the extensive-form representation of the signaling game starts not from a node on top but from node in the middle of the game tree. First move is made by Nature. It chooses the type of the Sender given a general prior belief about how likely is she to be any given type possible. The figure above presents an extensive form of the simplest possible case of a signaling game with only two possible types (t_1 and t_2) which are believed to be equally probable to be chosen. In the second stage, the sender observes its type but receiver does not observe it. So, the sender decides what a message to send to the receiver. In the example on the figure above, the Sender has two possible actions (L and R) to choose from. Respectively, after observing the message from the Sender, the Receiver updates its beliefs about how likely is she to be any of the two types given the message received and chooses the action which maximizes her expected payoff based on the updated (posterior) beliefs of the Receiver. The Receiver chooses between two possible actions (u and d), as well.

Solution:

Pooling on L:
Suppose that there is a (pooling) equilibrium in which the Sender's strategy is (L,L) i.e. chooses to send message L no matter what is her type.

Then, the Receiver's posterior belief (p, 1-p) will coincide with the prior one after observing L. Hence the expected payoffs of the Receiver from playing u and d respectively are as follows:

$$Eu_R(u) = 0.5 \cdot 3 + 0.5 \cdot 4 = 3.5$$
$$Eu_R(d) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$$

So, the best response of the Receiver to message L is to take action *u*.

Now, it remains to check if the pooling strategy (L,L) is optimal for the both types of the Sender or there is an incentive for at least one of them to deviate to R.

For the purpose, we need to identify first the values of the Receiver's belief off the equilibrium path (i.e. after observing message R), for which the Receiver would take action *u* or *d*, respectively. The value of q for which both strategies yield the same expected payoff to the Receiver is given by the solution of the following equation:

$$Eu_R(u) = q \cdot 1 + (1-q) \cdot 0 = q \cdot 0 + (1-q) \cdot 2 = Eu_R(d)$$

i.e. $q = \frac{2}{3}$

Apparently, if $q > \frac{2}{3}$, the Receiver would find it optimal to take action *u* given message R, which would make type t_1 of the Sender better off of deviating to R, since this will bring her 2 rather than 1 given message L was sent. However for any $q \leq \frac{2}{3}$, the Receiver would take action d as optimal, so the Sender would not have an incentive to deviate. Hence, there exists a **pooling perfect Bayesian Nash equilibrium [(L,L),(u,d),p=.5,** $q \leq \frac{2}{3}$ **]**.

Pooling on R:
Suppose that there is a (pooling) equilibrium in which the Sender's strategy is (R,R) i.e. chooses to send message R no matter what is her type.

Then, the Receiver's posterior belief (q, 1-q) will coincide with the prior one after observing R. Hence the expected payoffs of the Receiver from playing u and d respectively are as follows:

$$Eu_R(u) = 0.5 \cdot 1 + 0.5 \cdot 0 = 0.5$$
$$Eu_R(d) = 0.5 \cdot 0 + 0.5 \cdot 2 = 1$$

So, the best response of the Receiver to message R is to take action *d*.

Now, it remains to check if the pooling strategy (R,R) is optimal for the both types of the Sender or there is an incentive for at least one of them to deviate to L.

For the purpose, we need to identify first the values of the Receiver's belief off the equilibrium path (i.e. after observing message L), for which the Receiver would take action *u* or *d*, respectively. It is obvious that *u* is a strictly dominant strategy of the Receiver for any p:

$$Eu_R(u) = p \cdot 3 + (1-p) \cdot 4 > p \cdot 0 + (1-p) \cdot 1 = Eu_R(d)$$

In this case, however, type t_1 of the Sender would be better-off of sending message L, since this will bring her 1 rather than 0 given that message R was sent. Hence, there is no equilibrium in which the Sender plays (R,R).

<u>Separation with t_1 playing L:</u>
Suppose that there is a (separating) equilibrium in which the Sender's strategy is (L,R) i.e. t_1 chooses to send message L, while t_2 chooses to send message R. After observing the message L or R, the updated belief of the Receiver will be p=1 or q=0, respectively. Then, the best response of the Receiver is *u* to message L and *d* to message R.

Now, it remains to check if the pooling strategy (L,R) is optimal for the both types of the Sender or there is an incentive for at least one of them to deviate given the Receiver's optimal strategy (*u*,*d*). Apparently, type t_2 has an incentive to deviate by sending message L rather than R, to which the Receiver would respond with *u*, and the Sender of type t_2 would earn a payoff of 2, which exceeds t_2's payoff of 1 from playing R. Hence, there is no equilibrium in which the Sender plays (L,R).

<u>Separation with t_1 playing R:</u>
Suppose that there is a (separating) equilibrium in which the Sender's strategy is (R,L) i.e. t_1 chooses to send message R, while t_2 chooses to send message L. After observing the message L or R, the updated belief of the Receiver will be p=0 or q=1, respectively. Then, the best response of the Receiver is *u* both to message L and R which gives both types of the Sender a payoff of 2.

Now, it remains to check if the pooling strategy (R,L) is optimal for the both types of the Sender or there is an incentive for at least one of them to deviate given the Receiver's optimal strategy (*u*,*d*). Apparently, type t_1 has no incentive to deviate by sending message L rather than R, to which the Receiver would respond with *u*, and the Sender of type t_2 would earn a payoff of only 1, which is less than t_2's payoff of 2 from playing R. The same is true for t_1 having no incentive to deviate by sending message R instead of L. Hence, **there exists a separating perfect Bayesian Nash equilibrium [(R,L),(u,u),p=0,q=1]**.