M9302 Mathematical Models in Economics

HANDOUT 1 - 26.02.2010

SLOT 1 – 10:00-10:50 a.m.

1. Discussion on the Syllabus

-> 7 meetings: every second Friday from now on, 2 hours before noon from 10:00 to 11:50, and 2 hours afternoon, from 2 to 3:50 p.m.
I suggest organizing the lectures in 4 slots, each of 50 minutes?

 -> Regular presence on the lectures is highly recommended – students who miss a lecture are expected to fill the gap on their own
 -> Consultation hours in the second half of the lunch break, 1-2 p.m.?
 -> consultations on unattended topic would not be given!
 -> Sending to me the questions in advance by e-mail will be appreciated!
 -> the course is organized as introduction to GT, so if somebody of you feels it too elementary and need to explore further beyond the basic topics, please, let me know, I could provide you with further readings – however, for passing the exam, covering the material discussed at lectures should be enough.

-> 3 homework assignments (worth 30%), 14 days for each
 -> HW1 due to 26.03.
 -> HW2 due to 23.04.
 -> HW3 due to 21.05.

-> Final exam, a mixture of problems similar to the ones in HWs

2. What is Game Theory?

(Introduction to the course based on Dixit and Nalebuff's (1990) best seller *Thinking Strategically*, p. 1-3)

3. Where is Game Theory coming from?

(based on Binmore's (1992) *Fun and Games*, pp. 11-13)

4. Let's start!

(From here by the end of the lecture, description of the static games, based on Gibbons (1992) A Primer in Game Theory, see the Gibbons' (1997) JEP paper at: http://www.cas.buffalo.edu/classes/psc/fczagare/PSC 504/Gibbons.pdf)

5. The aim of the first lecture is to show:

 a. How to describe a game?
 b. How to solve the resulting game-theoretic problem?

6. How to describe a game? – Students' Dilemma Game

-> The simultaneous-move games of complete information are usually represented in the so-called normal form which consists of the following elements:

- PLAYERS

e.g. YOU versus OTHERS (Students' Dilemma Game[1])

- STRATEGIES available to each player – $s_i \in S_i$, S_i stands for *i*'s strategy space, s_i is an arbitrary member of this set

e.g. $S_i = \{Easy, Hard\}$ (Students' Dilemma Game)

- PAYOFFS – the payoff $u_i = u_i(s_1,...,s_n)$ received by each player for each combination of strategies that could be chosen by the players

e.g. $u_i = u_i(s_i, s_{-i}) = LEISURE_i(s_i) - GRADE_i(s_i, s_{-i})$ (Students' Dilemma Game)

Hard study scheme devotes twice less time for leisure than the easy one. Hence, the payoffs of student *i* could be computed as follows:

Player i's choice	Others' choice	LEISURE	GRADE	Player *i*' payoff
Easy	All Easy	2	3	**-1**
	At least one Hard	2	5	**-3**
Hard	At least one Easy	1	1	**0**
	All Hard	1	3	**-2**

Apparently, it does not matter how many of the other (-*i*) students would choose to study hard. At the end, this would result in an average performance higher than the one of an "easy-taker". Similarly, the presence of at least one easy-taker among the rest of the students would ensure all the "hard crammers" with A. So, without loss of generality, the number of players could be reduced to two by introducing a representative player on the place of the other (-*i*) students. This would allow us to demonstrate the set of all possible payoffs of these two players in a single bi-matrix as shown below:

		OTHERS	
		Easy	Hard
YOU	Easy	-1,-1	-3,0
	Hard	<u>0</u>,-3	<u>-2</u>,-2

-> We denote the game of n-players by G = $\{S_1,...,S_n;u_1,...,u_n\}$

[1] Modification of the classical simultaneous move game called Prisoners' Dilemma Game (PDG).

M9302 Mathematical Models in Economics

HANDOUT 1 - 26.02.2010

SLOT 2 – 11:00-11:50 a.m.

7. How to solve a GT problem?

Solution concepts:

- Strategic Dominance
- Nash Equilibrium (NE) in static games of complete information
- Subgame Perfect Nash Equilibrium (SPNE) in dynamic games of complete information
- Bayesian Nash Equilibrium (BNE) in static games of incomplete information
- Perfect Bayesian Equilibrium (PBNE) in dynamic games of incomplete information

Now, we will consider the first two solution concepts which refer to the static games of complete information.

8. Iterated Elimination of Strictly Dominated Strategies

<u>Definition (STRICTLY DOMINATED STRATEGY)</u>:

In the normal-form game G = $\{S_1,\ldots,S_n;u_1,\ldots,u_n\}$, let $s_i^{'}$ and $s_i^{''}$ be feasible strategies for player *i* (i.e., $s_i^{'}$ and $s_i^{''}$ are members of S_i). Strategy is strictly dominated by strategy if for each feasible combination of the other players' strategies, *i*'s payoff from playing $s_i^{'}$ is strictly less than *i*'s payoff from playing $s_i^{''}$:

$$u_i(s_1,\ldots,s_{i-1},s_i^{'},s_{i+1},\ldots,s_n) < u_i(s_1,\ldots,s_{i-1},s_i^{''},s_{i+1},\ldots,s_n)$$

for each $(s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n)$ that can be constructed from the other players' strategy spaces $S_1,\ldots,S_{i-1},S_{i+1},\ldots,S_n$.

In the example of Students' Dilemma Problem, for each of the two strategies composing her opponent's strategy space, one of the players gains higher payoff from playing Hard than Easy. Hence, Easy is strictly dominated by Hard.

The Strategic Dominance solution concept is based on the principle that:

Rational players do not play strictly dominated strategies, because there is no belief that a player could hold such that it would be optimal for her to play a dominated strategy.

Based on this principle, it could be concluded that if rational students will not take it easy but will study hard[2]. The solution process is called *iterated elimination of strictly dominated strategies*.

[2] Dominant ('surviving') strategies are underlined in the payoff bi-matrix of point 6.

9. Students' Dilemma Game - 2

Let's consider a small modification of the Students' Dilemma Game.

We compensate the mediocre students by grading them B (2) keeping the hard-crammers without leisure time:

		OTHERS Easy	OTHERS Hard
YOU	Easy	<u>0</u>,0	-3,-1
YOU	Hard	-1,-3	<u>-2</u>,-2

Here, we do not have a dominant strategy, neither strictly, or weakly. Another solution concept needs to be introduced which deals with such problems.

10. Nash Equilibrium

->We need a stronger solution concept which produces much tighter predictions. If such a concept exists in Game Theory which provides a unique solution to the modified version of the Students Dilemma problem (previous point 9.), it must be that each player would be willing to choose the strategy predicted by the theory.

->In other words, each player's chosen strategy must be the best response to the predicted strategies of the other players.

-> Such a prediction could be called strategically STABLE or *self-enforcing* – therefore its prediction is called EQUILIBRIUM – Nash Equilibrium (NE)

M9302 Mathematical Models in Economics

HANDOUT 1 - 26.02.2010

Definition (NE): In the n-player normal form game G = $\{S_1,\dots,S_n;u_1,\dots,u_n\}$, the strategies $(s_1^*,\dots,s_n^*)$ are a Nash equilibrium if, for each player *i*, s_i^* is (at least tied for) player *i*'s best response to the strategies specified for the n-1 other players, $(s_1^*,\dots,s_{i-1}^*,s_i^*,s_{i+1}^*\dots,s_n^*)$:

$$u_i\left(s_1^*,\dots,s_{i-1}^*,s_i^*,s_{i+1}^*\dots,s_n^*\right)\geq u_i\left(s_1^*,\dots,s_{i-1}^*,s_i,s_{i+1}^*\dots,s_n^*\right)$$

for every feasible strategy s_i in S_i; that is, s_i^* solves

$$\max_{s_i\in S_i} u_i\left(s_1^*,\dots,s_{i-1}^*,s_i^*,s_{i+1}^*\dots,s_n^*\right)$$

11. Which combination of strategies does not satisfy the definition for a Nash equilibrium of G?

Saying that given suggested solution $(s_1^{'},\dots,s_n^{'})$ is not a Nash equilibrium of G is equivalent to saying that there exists some player *i* such that is not a best response to $(s_1^{'},\dots,s_{i-1}^{'},s_{i+1}^{'}\dots,s_n^{'})$. That is, there exists some $s_i^{''}$ in S_i such that

$$u_i\left(s_1^{'},\dots,s_{i-1}^{'},s_i^{'},s_{i+1}^{'}\dots,s_n^{'}\right)< u_i\left(s_1^{'},\dots,s_{i-1}^{'},s_i^{''},s_{i+1}^{'}\dots,s_n^{'}\right).$$

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

12. Brute-force approach to finding a game's Nash equilibria is simply to check whether each possible combination of strategies satisfies the condition for NE.

		OTHERS	
		Easy	Hard
YOU	Easy	0,0	-3,-1
	Hard	-1,-3	-2,-2

The payoffs from best responses are underlined and the cells of the bi-matrix which contain two underlined payoffs corresponds to the strategies which satisfy the definition for Nash equilibrium. In this case, we have 2 NE.

13. Relation between Strategic Dominance and Nash Equilibrium

However, the initial version of the game has a unique NE which corresponds to the solution derived through iterated elimination of strictly dominated strategies. This is a general condition as stated by proposition A below.

PROPOSITION A: In the n-player normal form game G = $\{S_1,...,S_n;u_1,...,u_n\}$, if iterated elimination of strictly dominated strategies eliminates all but the strategies $(s_1^*,...,s_n^*)$, then these strategies are the unique Nash equilibrium of the game.

Moreover, the players' strategies in a Nash equilibrium always survive iterated elimination of strictly dominated strategies as stated in proposition B below:

PROPOSITION B: In the n-player normal form game G = $\{S_1,...,S_n;u_1,...,u_n\}$, if the strategies $(s_1^*,...,s_n^*)$ are a unique Nash equilibrium, then they survive iterated elimination of strictly dominated strategies.

For rigorous proof: see the additional reading materials.
****

SLOT 3 & 4 – 2:00-3:50 p.m.

For Cournot and Bertrand Models of Duopoly: See additional reading materials.