Solutions to Exercises in *Introduction to Economic Growth*

(Second Edition)

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1 Introduction

No problems.

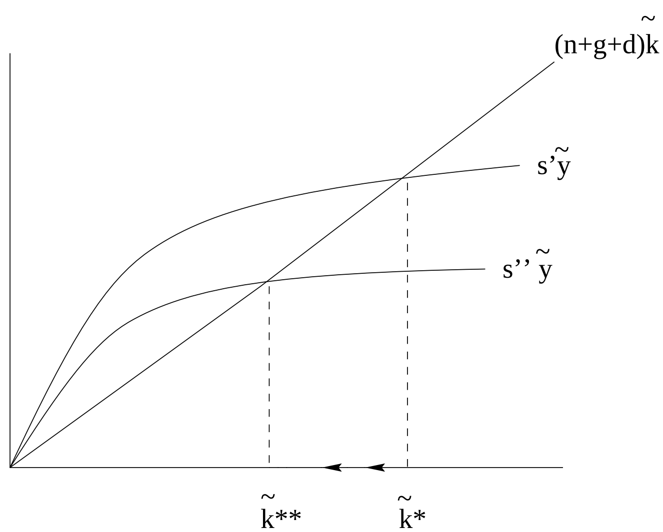
2 The Solow Model

Exercise 1. *A decrease in the investment rate*.

A decrease in the investment rate causes the $s\tilde{y}$ curve to shift down: at any given level of $\tilde{k}$, the investment-technology ratio is lower at the new rate of saving/investment.

Assuming the economy began in steady state, the capital-technology ratio is now higher than is consistent with the reduced saving rate, so it declines gradually, as shown in Figure 1.

Figure 1: A Decrease in the Investment Rate



The log of output per worker y evolves as in Figure 2, and the dynamics of the growth rate are shown in Figure 3. Recall that $\log \tilde{y} = \alpha \log \tilde{k}$ and $\dot{\tilde{k}}/\tilde{k} = s''\tilde{k}^{\alpha-1} - (n+g+d)$.

The policy permanently reduces the level of output per worker, but the growth rate per worker is only temporarily reduced and will return to g in the long run.

Figure 2: $y(t)$

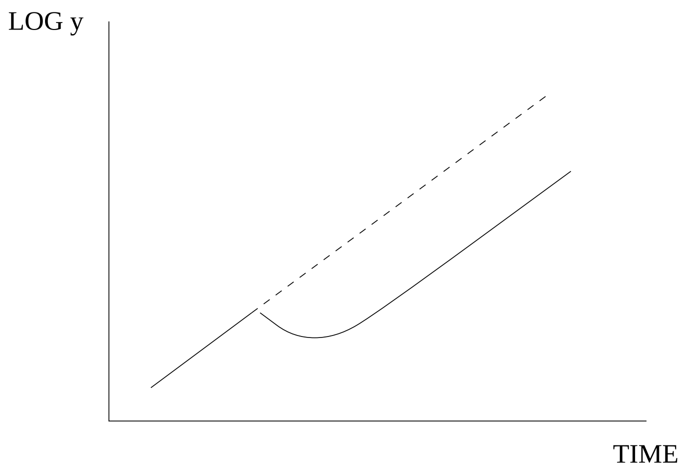
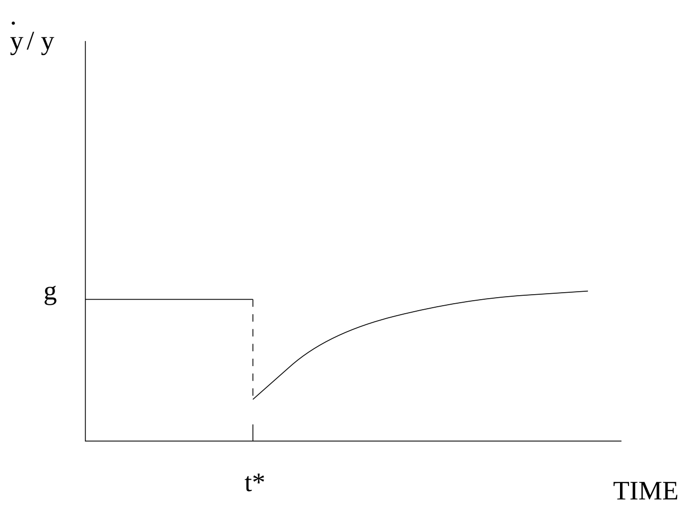


Figure 3: Growth Rate of Output per Worker

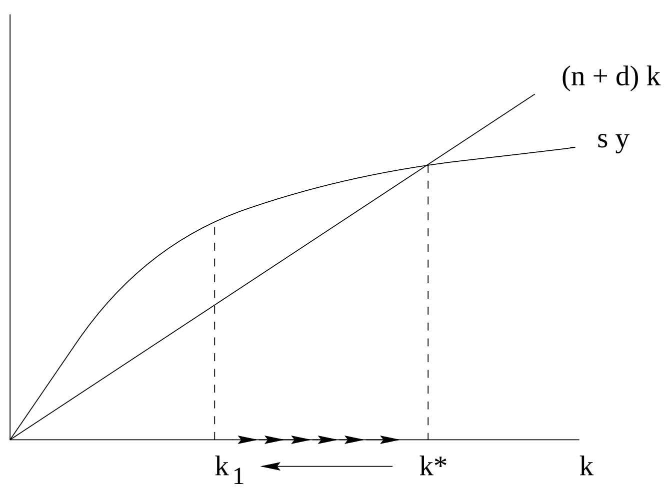


Exercise 2. *An increase in the labor force*.

The key to this question is to recognize that the initial effect of a sudden increase in the labor force is to reduce the capital-labor ratio since $k \equiv K/L$ and K is fixed at a moment in time. Assuming the economy was in steady state prior to the increase in labor force, k falls from k^* to some new level k_1. Notice that this is a movement *along* the sy and $(n+d)k$ curves rather than a *shift* of either schedule: both curves are plotted as functions of k, so that a change in k is a movement along the curves. (For this reason, it is somewhat tricky to put this question first!)

At k_1, $sy > (n + d)k_1$, so that $\dot{k} > 0$, and the economy evolves according to the usual Solow dynamics, as shown in Figure 4.

Figure 4: An Increase in the Labor Force

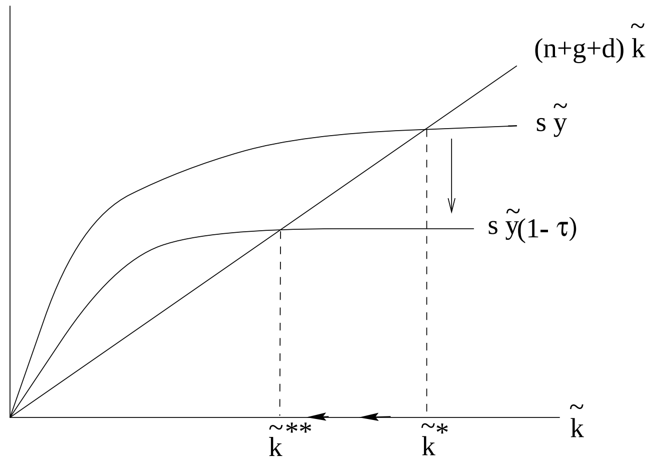


In the short run, per capita output and capital drop in response to a inlarge flow of workers. Then these two variables start to grow (at a decreasing rate), until in the long run per capita capital returns to the original level, k^*. In the long run, nothing has changed!

Exercise 3. *An income tax*.

Assume that the government throws away the resources it receives in taxes. Then an income tax reduces the total amount available for investing and shifts the investment curve down as shown in Figure 5.

Figure 5: An Income Tax



The tax policy permanently reduces the level of output per worker, but the growth rate per worker is only temporarily lowered. Notice that this experiment has basically the same results as that in Exercise 2.

For further thought: what happens if instead of throwing away the resources it collects the government uses all of its tax revenue to undertake investment?

Exercise 4. *Manna falls faster*.

Figure 6 shows the Solow diagram for this question. It turns out, however, that it's easier to answer this question using the transition dynamics version of the diagram, as shown in Figure 7. When g rises to g', $\dot{\tilde{k}}/\tilde{k}$ turns negative, as shown in Figure 7 and $\dot{A}/A = g'$, the new steady-state growth rate.

To see what this implies about the growth rate of y, recall that

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + \frac{\dot{A}}{A} = \alpha\frac{\dot{\tilde{k}}}{\tilde{k}} + g'.$$

So to determine what happens to the growth rate of y at the moment of the change in g, we have to determine what happens to $\dot{\tilde{k}}/\tilde{k}$ at that moment. As can be seen in Figure 7, or by algebra, this growth rate falls to $g - g' < 0$ — it is the negative of the difference between the two horizontal lines.

Substituting into the equation above, we see that $\dot{y}/y$ immediately after the

Figure 6: An Increase in g

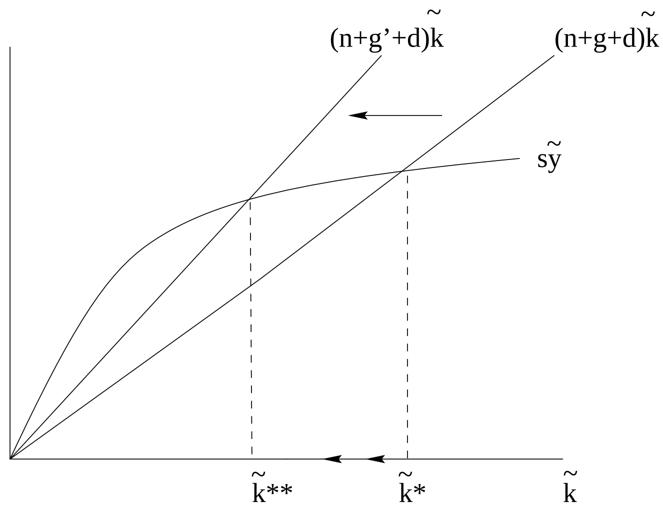
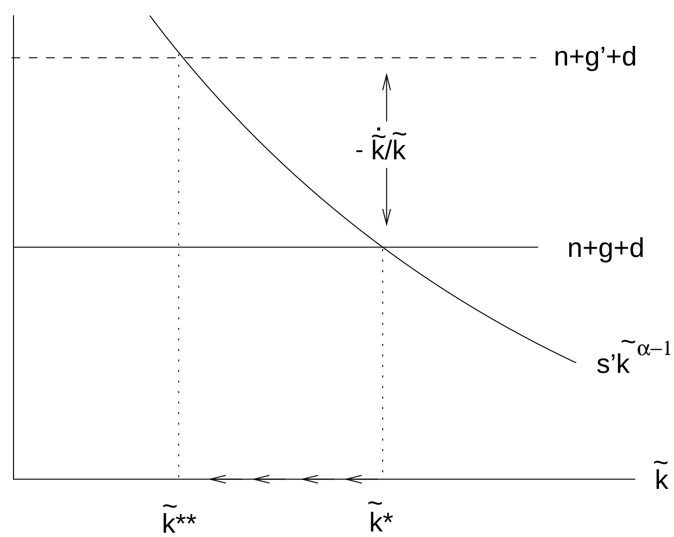


Figure 7: An Increase in g: Transition Dynamics



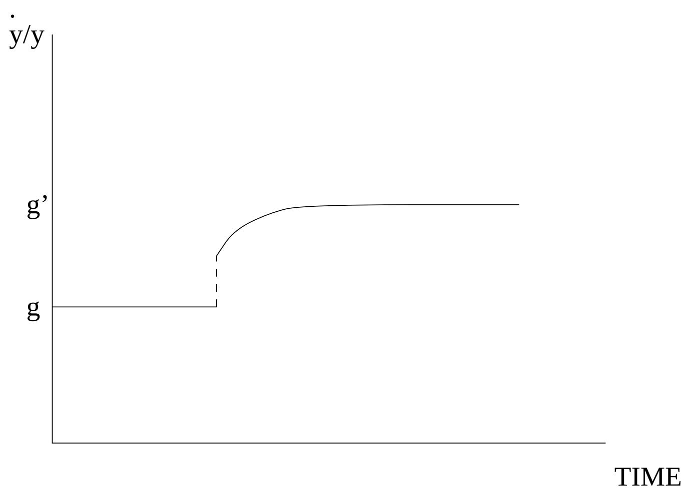
increase in g (suppose this occurs at time $t = 0$) is given by

$$\frac{\dot{y}}{y} |_{t=0} = \alpha(g - g') + g' = (1 - \alpha)g' + \alpha g > 0.$$

Notice that this value, which is a weighted average of g' and g, is strictly less than g'.

After time $t = 0$, $\dot{y}/y$ rises up to g' (which can be seen by looking at the dynamics implied by Figure 6). Therefore, we know that the dynamics of the growth rate of output per worker look like those shown in Figure 8.

Figure 8: Growth Rate of Output per Worker



Exercise 5. *Can we save too much?*

From the standard Solow model, we know that steady-state output per capita is given by $y^* = (\frac{s}{n+d})^{\frac{\alpha}{1-\alpha}}$. Steady-state consumption per worker is $(1-s)y^*$, or

$$c^* = (1-s)\left(\frac{s}{n+d}\right)^{\frac{\alpha}{1-\alpha}}.$$

From this expression, we see that an increase in the saving rate has two effects. First, it increases steady-state output per worker and therefore tends to increase consumption. Second, it reduces the amount of output that gets consumed.

To maximize c^*, we take the derivative of this expression with respect to s and set it equal to zero:

$$\frac{\partial c^*}{\partial s} = -\left(\frac{s}{n+d}\right)^{\frac{\alpha}{1-\alpha}} + (1-s)\frac{\alpha}{1-\alpha}\frac{s^{\frac{\alpha}{1-\alpha}-1}}{(n+d)^{\frac{\alpha}{1-\alpha}}} = 0.$$

Rearrange the equation, we have

$$1 = \frac{1-s^*}{s^*}\frac{\alpha}{1-\alpha},$$

and therefore

$$s^* = \alpha.$$

The saving rate which maximizes the steady-state consumption equals α.

Now turn to the marginal product of capital, MPK. Given the production function $y = k^\alpha$, the marginal product of capital is $\alpha k^{\alpha-1}$. Evaluated at the steady state value k^*,

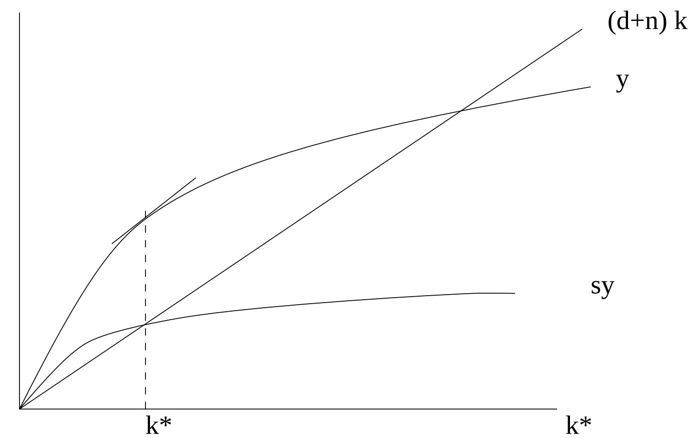
$$MPK = \alpha\,(k^*)^{(\alpha-1)} = \alpha\left(\frac{n+d}{s}\right).$$

When the saving rate is set to maximize consumption per person, $s^* = \alpha$, so that the marginal product of capital is

$$MPK^* = n + d.$$

That is, the steady-state marginal product of capital equals $n + d$ when consumption per person is maximized. Alternatively, this expression suggests that the *net* marginal product of capital — i.e. the marginal product of capital net of depreciation — is equal to the population growth rate. This relationship is graphed in Figure 9.

Figure 9: Can We Save Too Much?



If $s > \alpha$, then steady-state consumption could be increased by reducing the saving rate. This result is related to the diminishing returns associated with capital accumulation. The higher is the saving rate, the lower is the marginal product of capital. The marginal product of capital is the return to investing — if you invest one unit of output, how much do you get back? The intuition is clearest if we set $n = 0$ for the moment. Then, the condition says that the marginal product of capital should equal the rate of depreciation, or the net return to capital should be zero. If the marginal product of capital falls below the rate of depreciation, then you are getting back less than you put in, and therefore you are investing too much.

Exercise 6. *Solow (1956) versus Solow (1957).*

a) This is an easy one. Growth in output per worker in the inital steady state is 2 percent and in the new steady state is 3 percent.

b) Recall equation (2.15)

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \frac{\dot{B}}{B}$$

	$\frac{\dot{y}}{y}$	$\alpha\frac{\dot{k}}{k}$	$\frac{\dot{B}}{B}$
Initial S.S.	.02	1/3*(.02)	2/3*(.02)=.0133
New S.S.	.03	1/3*(.03)	2/3*(.03)=.0200
Change	.01	1/3*(.01)	2/3*(.01) = .0067

In other words, Solow (1957) would say that 1/3 of the faster growth in output per worker is due to capital and 2/3 is due to technology.

c) The growth accounting above suggests attributing some of the faster growth to capital and some to technology. Of course this is true in an accounting sense. However, we know from Solow (1956) that faster growth in technology is itself the cause of the faster growth in capital per worker. It is in this sense that the accounting picture can sometimes be misleading.

3 Empirical Applications of Neoclassical Growth Models

Exercise 1. *Where are these economies headed?*

From equation (3.9), we get

$$\hat{y}^* = \left(\frac{\hat{s}_K}{\hat{x}}\right)^{\frac{\alpha}{1-\alpha}} \hat{h}\,\hat{A} = \left(\frac{\hat{s}_K}{\widehat{(n+0.075)}}\right)^{\frac{\alpha}{1-\alpha}} e^{\psi(u-u_{U.S.})}\hat{A},$$

where the ($\hat{}$) is used to denote a variable relative to its U.S. value and $x = n+g+d$. The calculations below assume $\alpha = 1/3$ and $\psi = .10$, as in the chapter.

Applying this equation using the data provided in the exercise leads to the following results for the two cases: Case (a) maintains the 1990 TFP ratios, while case (b) has TFP levels equalized across countries. The Ratio column reports the ratio of these steady-state levels to the values in 1997.

	$\hat{y}_{97}$	(a) $\hat{y}^*$	Ratio	(b) $\hat{y}^*$	Ratio
U.S.A.	1.000	1.000	1.000	1.000	1.000
Canada	0.864	1.030	1.193	1.001	1.159
Argentina	0.453	0.581	1.283	0.300	0.663
Thailand	0.233	0.554	2.378	0.259	1.112
Cameroon	0.048	0.273	5.696	0.064	1.334

The country furthest from its steady state will grow fastest. (Notice that by furthest we mean in percentage terms). So in case (a), the countries are ranked by their rates of growth, with Cameroon predicted to grow the fastest and the United States predicted to grow the slowest. In case (b), Cameroon is still predicted to grow the fastest while Argentina is predicted to grow the slowest.

Exercise 2. *Policy reforms and growth.*

The first thing to compute in this problem is the approximate slope of the relationship in Figure 3.8. Eyeballing it, it appears that cutting output per worker in half relative to its steady-state value raises growth over a 37-year period by about 2 percentage points. (Korea is about 6 percent growth, countries at the 1/2 level are about 4 percent, and countries in their steady state are about 2 percent).

a) Doubling A will cut the current value of y/A in half, pushing the economy that begins in steady state to 1/2 its steady state value. According to the calculation above, this should raise growth by something like 2 percentage points over the next 37 years.

b) Doubling the investment rate s_K will raise the steady state level of output per worker by a factor of $2^{\alpha/(1-\alpha)}$ according to equation (3.8). If $\alpha = 1/3$, then this is equal to $\sqrt{2} \approx 1.4$. Therefore the ratio of current output per worker to steady-state output per worker falls to $1/1.4 \approx .70$, i.e. to seventy percent of its steady-state level. Dividing the gap between 1/2 and 1.0 into tenths, we are 3/5ths of the way towards 1/2, so growth should rise by $3/5 * (.02) = 1.2$ percentage points during the next 37 years.

c) Increasing u by 5 years of schooling will raise the steady state level of output per worker by a factor of $exp\psi * 5$ according to equation (3.8). If $\psi = .10$, then this is equal to 1.65, and the ratio of current output per worker to steady-state output per worker falls to $1/1.65 \approx .60$, i.e. to sixty percent of its steady-state level. Dividing the gap between 1/2 and 1.0 into tenths, we are 4/5ths of the way towards 1/2, so growth should rise by $4/5 * (.02) = 1.6$ percentage points during the next 37 years.

Exercise 3. *What are state variables?*

Consider the production function

$$Y = K^{\alpha}(AH)^{1-\alpha}.$$

Dividing both sides by AL yields

$$\frac{y}{A} = \left(\frac{k}{A}\right)^{\alpha} h^{1-\alpha}.$$

Use the $\tilde{()}$ to denote the ratio of a variable to A and rewrite this equation as

$$\tilde{y} = \tilde{k}^{\alpha} h^{1-\alpha}.$$

Now turn to the standard capital accumulation equation:

$$\dot{K} = s_K Y - dK.$$

Using the standard techniques, this equation can be rewritten in terms of the capital-technology ratio as

$$\dot{\tilde{k}} = s_K \tilde{y} - (n + g + d)\tilde{k}.$$

In steady state, $\dot{\tilde{k}} = 0$ so that

$$\tilde{k} = \frac{s_K}{n+g+d}\tilde{y} = \frac{s_K}{n+g+d}\tilde{k}^{\alpha}h^{1-\alpha},$$

and therefore

$$\tilde{k}^* = \left(\frac{s_K}{n+g+d}\right)^{\frac{1}{1-\alpha}} h.$$

Substituting this into the production function $\tilde{y} = \tilde{k}^{\alpha}h^{1-\alpha}$ we get

$$\tilde{y}^* = \left(\frac{s_K}{n+g+d}\right)^{\frac{\alpha}{1-\alpha}} h^{\alpha}h^{1-\alpha} = \left(\frac{s_K}{n+g+d}\right)^{\frac{\alpha}{1-\alpha}} h.$$

Finally, note that $\tilde{y} = y/A$, hence

$$y^*(t) = \left(\frac{s_K}{n+g+d}\right)^{\frac{\alpha}{1-\alpha}} hA(t),$$

which is the same as the equation (3.8).

Exercise 4. *Galton's fallacy*.

In the example of the heights of mother and daughter, it is true that tall mothers tend to have shorter daughters and vice versa. Under the assumption of independent, identical (uniform) distributions of the heights of mothers and daughters, we have the following chart:

mother's height	5'1	5'2	5'3	5'4	5'5	5'6	5'7	5'8	5'9	5'10
probability of shorter daughter	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$

Mothers with height 5'1" have zero chance of having shorter daughters because no one can be shorter than 5'1". Mothers with height 5'2" have $\frac{1}{10}$ chance of having daughters with height 5'1". Other cases can be reasoned in the same way.

In the above example, there is clearly no convergence or narrowing of the distribution of heights: there is always one very tall person and one very short person, etc., in each generation. However, we just showed that in spite of the fact that the heights of mothers and daughters have the same distribution (non-converging), we still can observe the phenomenon that tall mothers tend to have shorter daughters, and vice versa. Let the heights correspond to income levels, and consider observing income levels at two points in time. Galton's fallacy implies that even though we observe that countries with lower initial income grow faster, this does not necessarily mean that the world income distribution is narrowing or converging.

The figures in this chapter are not useless, but Galton's fallacy suggests that care must be taken in interpreting them. In particular, if one is curious about whether or not countries are converging, then simply plotting growth rates against initial income is clearly not enough. The figures in the chapter provide other types of evidence. Figure 3.3, for example, plots per capita GDP for several different industrialized economies from 1870 to 1994. The narrowing of the gaps between advanced countries is evident in this figure. Similarly, the ratios in Figure 3.9 suggest a lack of any narrowing in the distribution of income levels for the world as a whole.

Exercise 5. *Reconsidering the Baumol results*.

As in Figure 3.3, William Baumol (1986) presented evidence of the narrowing of the gaps between several industrialized economies from 1870. But De Long (1988) argues that this effect is largely due to "selection bias". First, only countries that were rich at the end of the sample (i.e., in the 1980s) were included. To see the problem with this selection, suppose that countries' income levels were like women's heights in the previous exercise. That is, they are random numbers in each period, say drawn with equal probability from 1,2,3,...,10. Suppose we look only at countries with income levels greater than or equal to 6 in the second period. Because of this randomness, knowing that a country is rich in the second period implies nothing about its income in the first period — hence the distribution will likely be "wider" in the first period than in the second, and we will see the appearance of convergence even though in this simple experiment we know there is no convergence. The omission of Argentina from Baumol's data is a good example of the problem. Argentina was rich in 1870 (say a relative income level of 8) but less rich in 1987 (say a relative income level of 4). Because of its low income in the last period, it is not part of the sample and this "divergent" observation is missing.

This criticism applies whenever countries are selected on the basis of the last observation. What happens if countries are selected on the basis of being rich for the first observation? The same argument suggests that there should be a bias toward divergence. Therefore, to the extent that the OECD countries were already rich in 1960, the OECD convergence result is even stronger evidence of convergence.

For the evidence related to the world as a whole, there is clearly no selection bias — all countries are included.

Exercise 6. *The Mankiw-Romer-Weil (1992) model*.

From the Mankiw-Romer-Weil (1992) model, we have the production function:

$$Y = K^{\alpha} H^{\beta} (AL)^{1-\alpha-\beta}.$$

Divide both sides by AL to get

$$\frac{y}{A} = \left(\frac{k}{A}\right)^{\alpha} \left(\frac{h}{A}\right)^{\beta}.$$

Using the $(\tilde{\ })$ to denote the ratio of a variable to A, this equation can be rewritten as

$$\tilde{y} = \tilde{k}^{\alpha} \tilde{h}^{\beta}.$$

Now turn to the capital accumulation equation:

$$\dot{K} = s_K Y - dK.$$

As usual, this equation can be written to describe the evolution of $\tilde{k}$ as

$$\dot{\tilde{k}} = s_K \tilde{y} - (n + g + d)\tilde{k}.$$

Similarly, we can obtain an equation describing the evolution of $\tilde{h}$ as

$$\dot{\tilde{h}} = s_H \tilde{y} - (n + g + d)\tilde{h}.$$

In steady state, $\dot{\tilde{k}} = 0$ and $\dot{\tilde{h}} = 0$. Therefore,

$$\tilde{k} = \frac{s_K}{n + g + d}\tilde{y},$$

and

$$\tilde{h} = \frac{s_H}{n + g + d}\tilde{y}.$$

Substituting this relationship back into the production function,

$$\tilde{y} = \tilde{k}^{\alpha} \tilde{h}^{\beta} = \left(\frac{s_K}{n + g + d}\tilde{y}\right)^{\alpha} \left(\frac{s_H}{n + g + d}\tilde{y}\right)^{\beta}.$$

Solving this equation for $\tilde{y}$ yields the steady-state level

$$\tilde{y}^{*} = \left\{ \left(\frac{s_K}{n + g + d}\right)^{\alpha} \left(\frac{s_H}{n + g + d}\right)^{\beta} \right\}^{\frac{1}{1-\alpha-\beta}}.$$

Finally, we can write the equation in terms of output per worker as

$$y^*(t) = \left\{ \left(\frac{s_K}{n+g+d} \right)^{\alpha} \left(\frac{s_H}{n+g+d} \right)^{\beta} \right\}^{\frac{1}{1-\alpha-\beta}} A(t).$$

Compare this expression with equation (3.8),

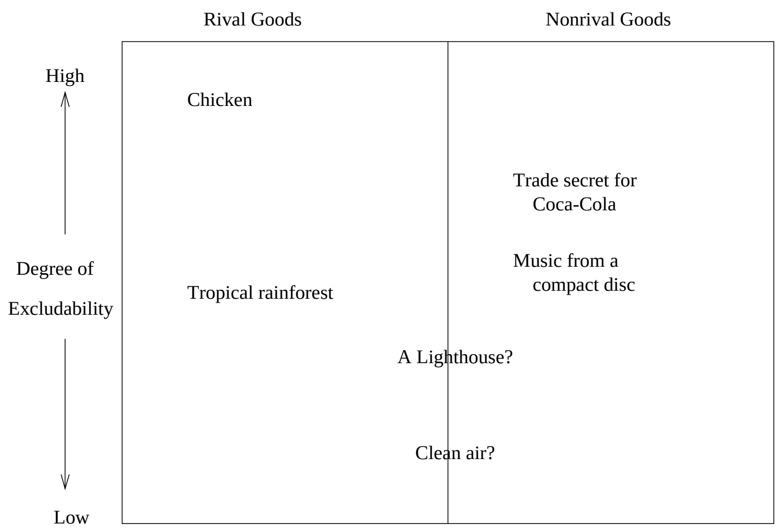
$$y^*(t) = \left\{ \frac{s_K}{n+g+d} \right\}^{\frac{\alpha}{1-\alpha}} hA(t).$$

In the special case $\beta = 0$, the solution of the Mankiw-Weil-Romer model is different from equation (3.8) only by a constant h. Notice the symmetry in the model between human capital and physical capital. In this model, human capital is accumulated by foregoing consumption, just like physical capital. In the model in the chapter, human capital is accumulated in a different fashion — by spending time instead of output.

4 The Economics of Ideas

Exercise 1. *Classifying goods*.

Figure 10:



A chicken and a rainforest are clearly rivalrous — consumption of either by one person reduces the amount available to another. Private goods like a chicken have well-defined property rights which make them excludable to a very high degree. For some rainforests, property rights appear to be less well-defined.

The trade secret for Coca-Cola is a nonrivalrous idea. Although not protected by a patent, the good is protected by trade secrecy (although Pepsi and other soft drinks do imitate the formula). Music from a compact disc is fundamentally a collection of 0's and 1's and so is also nonrivalrous. The degree of excludability is a function of the property rights system. Within the U.S. the enforcement appears to be fairly strong, but this is less true in some other countries, where pirating of compact discs is an issue.

The lighthouse (a tower flashing lights to provide guidance to ships at night) is sometimes thought of as a public good. Notice that it is not truly nonrivalrous — if a million ships wanted to use one lighthouse, there would be some crowding effects. Excludability is, as always, a function of the markets in place. See the next exercise for a discussion of this point by Ronald Coase (the 1991 Nobel Prize winner in economics).

Similarly, clean air is not truly nonrivalrous. If I breathe a molecule of air, then you cannot breathe the same molecule (at least at the same moment in time). In terms of excludability, however, it is very difficult to monitor an individual's consumption of air and charge her for it.

Exercise 2. *Provision of goods.*

Chicken are rivalrous and highly excludable. The market does a good job of providing chicken on a supply and demand basis.

The trade secret for Coca-Cola is nonrivalrous and partially excludable. One might think that the disclosure of the trade secret would seriously weaken the competitive position of Coca-Cola, but in practice, this doesn't seem to be the case. Other soft drinks, while not produced with exactly the same ingredients, are close substitutes, yet Coca-Cola is a large and prosperous company. Similarly, the lack of use of any official mechanism like patents to protect intellectual property rights does not appear to be a serious problem in this industry — we see innovations like diet soft drinks and New Coke.

Music from a compact disc is nonrivalrous and partially excludable (however, the specific compact disc is rivalrous). Markets should provide music because there is an incentive for profits in the music production business. However, once a compact disc is produced, it is easy to replicate. Governments typically intervene to protect intellectual property rights so that individuals can benefit from their music talents and will have the incentives to produce better music. An interesting issue is whether or not China should protect the intellectual property rights of U.S. musicians.

A tropical rainforest is rivalrous but only partially excludable. For example, the pollution that occurs when rainforests are burned is a serious externality on neighbors (not always within the same country). Does the owner of the land have the right to burn it and pollute the neighbor's air, or does the neighbor have the right to clean air on his property? If these rights are well-defined, then the Coase theorem suggests that negotiation might achieve the efficient outcome. In the absence of well-defined property rights on this issue, some government involvement may be necessary. Other issues related to tropical rainforests also arise, such as biodiversity and global warming.

Similar issues apply to clean air more generally.

The lighthouse has been used (by Mill, Pigou, Samuelson, and others) as a classic example of a public good that should be provided by the government. The claim is that it is impossible to charge passing ships for their use of the lighthouse and that the marginal cost of allowing one more ship to pass is zero.

Ronald Coase in "The Lighthouse in Economics" (*Journal of Law and Economics*, October 1974:357-376) provides an excellent discussion of the history of lighthouses as an example of a public good in economics. Coase shows that in fact lighthouses in Britain in the 17th and 18th centuries were often very successfully provided by the private market system in conjunction with patents granted by the government. Individuals would apply to the government for authorization to build and maintain a lighthouse in exchange for the right to charge any ship that docked in nearby ports a specified fee (based on the size of the ship, etc.).

Exercise 3. *Pricing with increasing returns to scale*.

a) $C = wL = w(\frac{5}{100} + F)$
b) $C(Y) = w(\frac{Y}{100} + F)$
c) $\frac{dC}{dY} = \frac{w}{100}$
d) $\frac{C}{Y} = \frac{w}{100} + \frac{F}{Y}w, \;\; \frac{d(C/Y)}{dY} < 0,$
e) $\pi = PY - C(Y) \; = \frac{w}{100}Y - w(\frac{Y}{100} + F) \; = -wF < 0.$

5 The Engine of Growth

Exercise 1. *An increase in the productivity of research.*

Figure 11: An Increase in δ

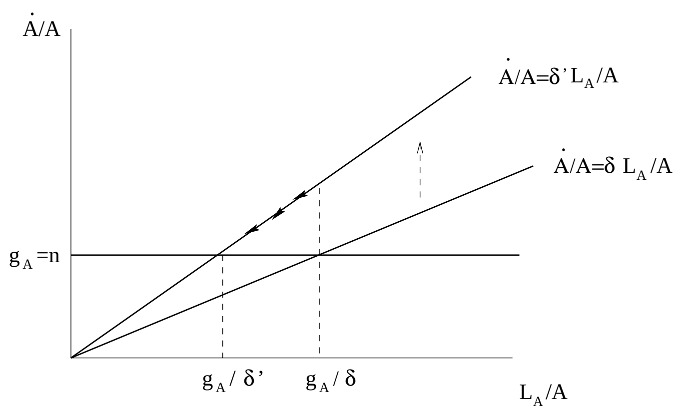
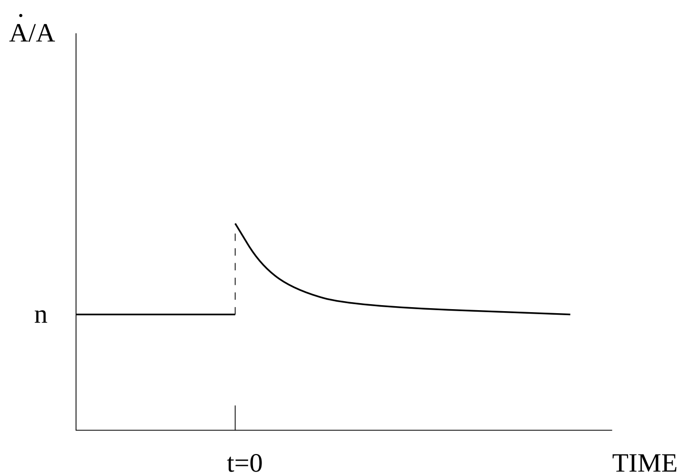
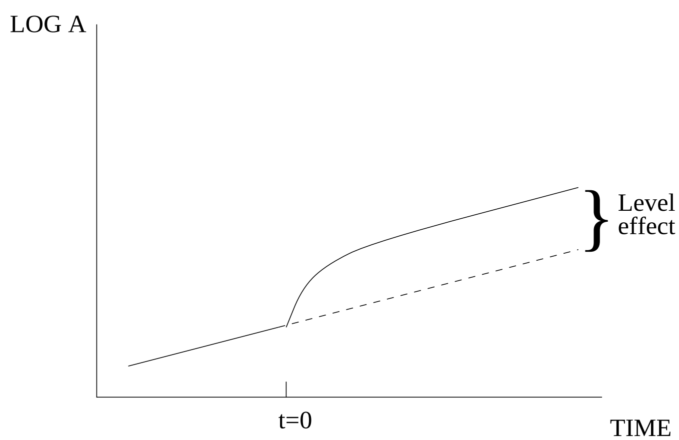


Figure 12: The Growth Rate of Technology



As shown in Figures 11 and 12, an increase in δ causes a temporary increase in the growth rate of technology: at the initial level of L_A, research is more productive and the economy produces more ideas. Over time, the growth rate falls as $\frac{L_A}{A}$

Figure 13: The Level of Technology



decreases when $\frac{\dot{A}}{A} > n$. In the long run, the growth rate of technology returns to n. The long-run level effect of an increase in δ is shown in Figure 13.

Exercise 2. *Too much of a good thing?*

Solution.

From equation (5.11), we have

$$y^*(t) = \left(\frac{s_K}{n + g_A + d}\right)^{\frac{\alpha}{1-\alpha}} (1 - s_R)\frac{\delta\, s_R}{g_A} L(t).$$

To maximize output per worker along a balanced growth path, take the derivative with respect to s_R:

$$\frac{\partial\, y^*(t)}{\partial\, s_R} = B\, \frac{\partial\,(1 - s_R)s_R}{\partial\, s_R},$$

where

$$B = \left(\frac{s_K}{n + g_A + d}\right)^{\frac{\alpha}{1-\alpha}} \frac{\delta L(t)}{g_A}.$$

The maximum occurs when this derivative is equal to zero, and $\frac{\partial\, y^*(t)}{\partial\, s_R} = 0$ implies that

$$1 - 2s_R^* = 0,$$

and therefore

$$s_R^* = \frac{1}{2}.$$

Notice that the first time s_R appears, it enters negatively to reflect the fact that more researchers mean fewer workers producing output. The second time, it enters positively to reflect the fact that more researchers mean more ideas, which increases the productivity of the economy. $s_R^* = \frac{1}{2}$ achieves the balance between these two effects. If $s_R^* > \frac{1}{2}$, the negative effect of more researchers overpowers the positive effect. As a result, we can have too much of a good thing.

Exercise 3. *The future of economic growth.*

a) From equation (5.6), we have

$$0 = \lambda \frac{\dot{L_A}}{L_A} - (1-\phi)\frac{\dot{A}}{A},$$

which implies that

$$\frac{\lambda}{1-\phi} = \frac{\dot{A}}{A} / \frac{\dot{L_A}}{L_A} = \frac{2}{3}.$$

b) From equation (5.7), we have

$$g_A = \frac{\lambda n}{1-\phi}.$$

Assuming $\frac{\lambda}{1-\phi}$ in the world economy is the same as that calculated from the advanced countries, we get

$$g_A = \frac{2}{3} \times .01 = \frac{2}{3} = .0067.$$

The long-run steady-state growth rate of the per capita output in the world economy is also equal to this value.

c) g_y is different from $\frac{\dot{A}}{A}$ given in the question because $\frac{\dot{L_A}}{L_A}$ is substantially higher than population growth. This means that the advanced countries are in transition stage where L_A/L is rising in the data. g_y is the long-run steady-state growth rate, while the current level of $\frac{\dot{A}}{A}$ is reflects transition dynamics. The implication is that, holding everything else constant, growth rates may decline substantially in the future.

d) The fact that many developing countries are starting to engage in R&D suggests that the decline in the growth in L_A may not occur for a long time. The decline in growth suggested by part (c) may therefore be postponed for some time.

Exercise 4. *The share of the surplus appropriated by inventors.*

Denote Π as the profit captured by the monopolist. The monopolist chooses the price P to maximize Π, where

$$\Pi = (P - MC)Q(P) = (P - c)(a - bP).$$

Setting $\partial\Pi/\partial P = 0$ yields the first-order condition:

$$a - bP - b(P - c) = 0.$$

Solving for P, we find the monopolist chooses a price of

$$P^* = \frac{a + bc}{2b}$$

and earns profits

$$\begin{aligned}
\Pi^* &= \left(\frac{a + bc}{2b} - c\right)\left(a - \frac{a + bc}{2}\right) \\
&= \left(\frac{a - bc}{2b}\right)\left(\frac{a - bc}{2}\right) \\
&= \frac{1}{4b}(a - bc)^2.
\end{aligned}$$

The potential consumer surplus, CS, is the area of the shaded triangle in Figure 5.4:

$$CS = \frac{1}{2}(\frac{a}{b} - MC)(a - bMC) = \frac{1}{2b}(a - bc)^2.$$

Therefore, the ratio of profit to consumer surplus is 1/2.

6 A Simple Model of Growth and Development

Exercise 1. *The importance of A versus h in producing human capital*.

Equation (6.8) can be rewritten to isolate the effect of schooling as

$$y^*(t) = Z(t)e^{\frac{\psi}{\gamma}u}, \tag{1}$$

where Z is the collection of all of the other terms (e.g. the one involving the physical investment rate and the $A^*(t)$ term).

This can be compared to the model in Chapter 3, where equation (3.8) and the fact that $h = e^{\psi u}$ in that chapter imply

$$y^*(t) = Z_3(t)e^{\psi u},$$

where Z_3 represents some other terms from Chapter 3, similar to those that make up Z above.

In Chapter 3, we used a fact documented by Mincer (1974) and many other labor economists that an additional year of schooling — u — tends to generate *proportional* effects on the wage, and therefore on output per worker. In particular, the labor market evidence from a wide range of countries suggests that a one year difference in schooling translates into about a ten percent difference in wages. Since the wage is proportional to output per worker (recall that the marginal product of labor in this model is $(1-\alpha)Y/L$), this leads us to choose a value of .10 for ψ in Chapter 3:

$$\frac{\partial \log y}{\partial u} = \psi = .10.$$

Recall that the derivative of a log is like a percentage change, so this expression can be written as

$$\frac{\%\Delta y}{\Delta u} = \psi = .10,$$

i.e. a one unit change in u leads to a 10 percent change in y.

If we apply this same logic to equation (1) above, we see an interesting result:

$$\frac{\partial \log y}{\partial u} = \psi/\gamma.$$

That is, in the model in Chapter 6, a one unit change in u raises output per person — and hence the wage — by ψ/γ units. If we apply the same reasoning as above, this would suggest picking parameter values such that $\psi/\gamma = .10$. This allows our model to match the labor market evidence that one additional year of schooling raises the typical wage by 10 percent. What this means is that even though we've added a new parameter to the model, differences in schooling in the *calibrated* model (i.e. the model once we've picked values for the parameters) will have exactly the same effect on output per worker as in Chapter 3. For this conclusion, we don't need to say anything in particular about γ. All that matters is the ratio ψ/γ.

The parameter γ has important effects on the transition dynamics of the model. For example, in equation (6.5), γ determines how rapidly skill accumulation occurs in a country as a function of how far the country is from the world's knowledge frontier (h/A). One might try to use evidence on transition dynamics to determine this value.

Exercise 2. *Understanding differences in income*.

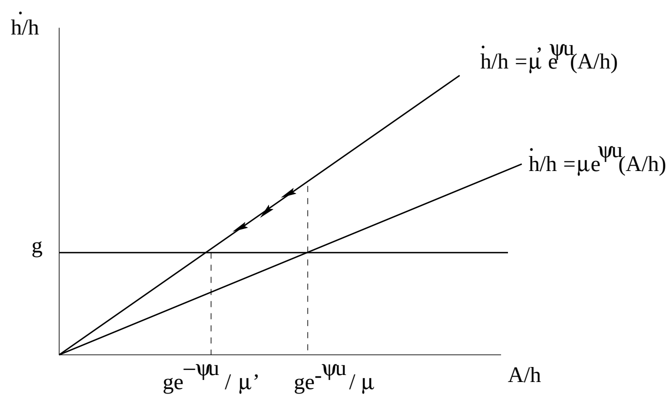
This model explains differences in the level of income across countries by appealing to differences in s_K and u. However, this explanation begs new questions: Why is it that some countries invest more than others and why do individuals in some countries spend more time learning to use new technologies? This model cannot address these questions. In addition, we know from Chapter 3 that differences in s_K and u will not explain entirely the differences in incomes across countries. Recall that differences in productivity were also needed, and the theory in Chapter 6 does not help us to understand why countries have different productivity levels.

Exercise 3. *Understanding differences in growth rates*.

A key implication of the model is that all countries share the same long-run growth rates, given by the rate at which the world technological frontier expands. Even with no differences across countries in the long-run growth rate, we can explain the large variation in rates of growth with transition dynamics. To the extent that countries are changing their positions within the long-run income distribution, they can grow at different rates. It is natural in this model to suspect that countries that are below their steady-state balanced growth paths should grow faster than g, and countries that are above their steady-state growth paths should grow more slowly. Changes in policy can cause an economy to be away from its steady state. [One must be careful in defining distance from steady state in this model since the transition dynamics now involve two dimensions (corresponding to physical capital and human capital.)

Exercise 4. *The role of* μ.

Figure 14: Steady-state A/h ratio



The equation below is true in this model:

$$\left(\frac{h}{A}\right)^* = \left(\frac{\mu}{g}e^{\psi u}\right)^{\frac{1}{\gamma}}.$$

To be sure that the ratio $\frac{h}{A}$ is less than one, we must have

$$\left(\frac{\mu}{g}e^{\psi u}\right)^{\frac{1}{\gamma}} < 1,$$

which implies

$$\mu < g\,e^{-\psi u}.$$

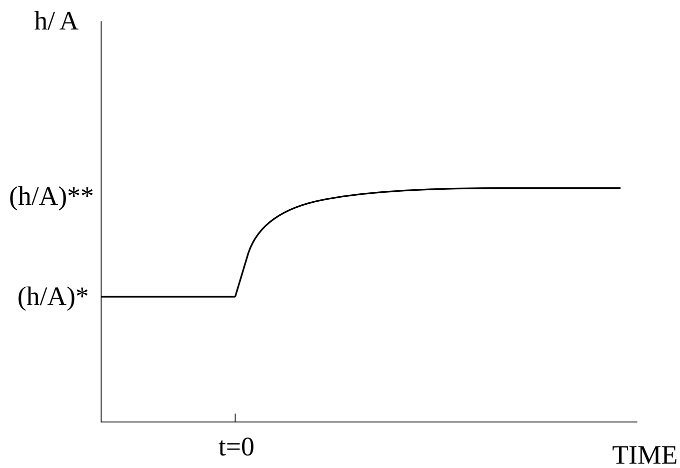
Therefore, we require this condition to hold for the largest possible value of u.

In the model, the parameter μ can be interpreted as the productivity of the economy at using education to learn to use new ideas. A very good educational system, for example, might be reflected by a higher value of μ. Or, as the next question discusses, it might reflect the openness of the economy to technology transfer.

Exercise 5. *An increase in μ.*

a) Figure 14 shows the experiment. The line $\frac{\dot{h}}{h} = \mu\, e^{\psi u}\left(\frac{A}{h}\right)$ says that the growth rate of accumulated skills is proportional to the ratio of A to h, which is the closeness between current skill and the knowledge frontier. The line $\frac{\dot{h}}{h} = g$

Figure 15: h/A Over Time



shows the growth rate of accumulated skills in steady state. The intersection point of these two lines denotes the steady-state value of the $\frac{A}{h}$ ratio.

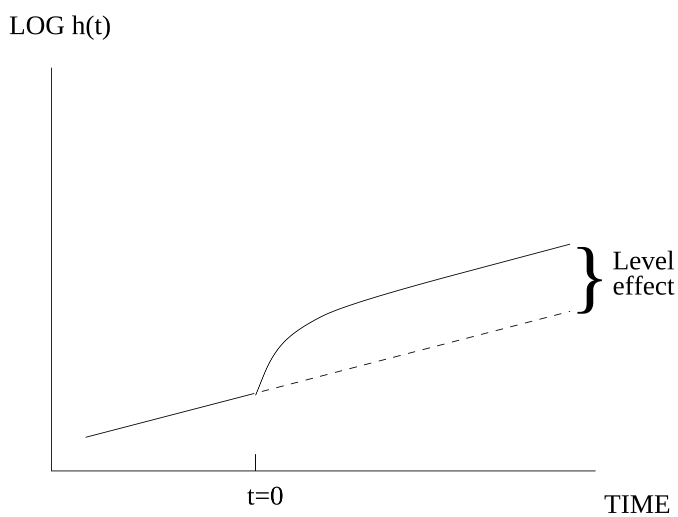
b) Starting from the steady state, an increase in μ causes the growth rate of h to be higher than g. In the intermediate run, h grows faster than A, hence $\frac{A}{h}$ decreases over time until the growth rate of h returns back to g. An increase in μ has a level effect on the ratio of $\frac{A}{h}$, but no long-run growth effect.

c) The behavior of h/A over time is graphed in Figure 15.

d) The behavior of $h(t)$ over time is graphed in Figure 16.

e) An increase in μ can be understood as an increase in the openness of the economy to technology transfers. As illustrated above, an increase in the openness of the economy can shrink the distance between steady-state skill and the frontier technology, thus raising a country's steady-state income.

Figure 16: $h(t)$ Over Time



7 Social Infrastructure and Long-run Economic Performance

Exercise 1. *Cost-benefit analysis.*

Solution
a) $F = \$1000$,
$\Pi = \frac{100}{(1+r)}(1 + \frac{1}{(1+r)} + \cdots) = \frac{100}{r} = 2000$,
$\Pi > F$, so the investment is worth undertaking.
b) $F = 5000$.
$\Pi < F$, so the investment is not worth undertaking.
c) The cutoff value is \$2000.

Exercise 2. *Can differences in the utilization of factors of production explain differences in TFP?*

a) For this problem, think of all variables as denoting ratios of the value in the most productive country to the value in the least productive country. With this notation, $I = 10$. If m denotes the fraction of K that is utilized, then

$$Y = (mK)^\alpha (hL)^{1-\alpha} = m^\alpha K^\alpha (hL)^{(1-\alpha)}.$$

Therefore,

$$I = m^\alpha,$$

and

$$m = I^{\frac{1}{\alpha}} = 10^3.$$

Under this assumption we need the utilization of capital to vary by a factor of 1000 to explain the variation in TFP. This means that if one country utilizes all of its physical capital, another country only utilizes 1/1000th, which seems quite unlikely.

b) Suppose the utilization of both physical capital and skills vary in the same way across countries. Then

$$Y = (mK)^\alpha (mhL)^{1-\alpha} = mK^\alpha (hL)^{1-\alpha}.$$

Then,

$$I = m^1,$$

and

$$m = I = 10.$$

Under this assumption, we need a factor of 10 variation in the utilization of both factors. Note that it is important that this variation be perfectly correlated across countries.

c) These calculations suggest that it is unlikely that differences in utilization of physical capital across countries can account fully for differences in productivity. Differences in the utilization of factor inputs have stronger explanatory power if variations in the utilization of skills are introduced, but it still seems unlikely that this is the complete explanation. Another possible explanation is differences in the technologies used in different countries (which could also be due to differences in infrastructure).

Exercise 3. *Social infrastructure and the investment rate.*

a) Suppose a country's investment rate is chosen by international investors who equate rates of return across countries. With this production function, the marginal product of capital is equal to $\alpha Y/K$, so that equating the marginal product of capital across countries is the same as equating the capital-output ratio. Recall that the capital-output ratio in steady state is given by $s/(n+g+d)$, so that if marginal products of capital are equated, then investment rates differ across countries only to the extent that $n+g+d$ differs. In particular, differences in I do not matter for rates of investment in this scenario. They do affect the capital-labor ratio, however, as can be seen by noting that the marginal product of capital can also be written as $\alpha I(K/L)^{\alpha-1}$.

b) One suspects that differences in institutions and government policies — infrastructure — show up in part as differences in tax rates or rates of expropriation (e.g. when a private industry is nationalized with little or no compensation to owners). In this case, it is the *after-tax* rate of return that would likely be equalized across countries by international investors. Let τ denote the effective tax rate on capital income, so that investors receive $(1-\tau)$ times the marginal product of capital. In this case, $(1-\tau)\alpha Y/K$ is equalized across countries, and by the argument given above, differences in τ will lead to differences in investment rates across countries.

Exercise 4. *The meaning of the quotation*.

Firms and individuals will do the best they can given the constraints that face them. One of the fundamental theorems of economics, observed early on by Adam

Smith, is that such self-interested behavior will, under certain circumstances, maximize the welfare of society as a whole. However, such circumstances may not always hold. For example, profit-maximizing firms may do their best to monopolize a market, which may restrict competition and reduce welfare. Or, individuals may try to circumvent poorly defined property rights in order to enhance their own welfare at the expense of others. But if such property rights are easily circumvented, everyone may be worse off.

The quotation also notes that innovation and profit may not always be the same thing. Clearly, the incentives to innovate may be reduced by individuals seeking to circumvent property rights.

8 Alternative Theories of Endogenous Growth

Exercise 1. *Population growth in the AK model.*

a) $Y = AKL^{1-\alpha}$. Substitute this into the standard capital accumulation equation:

$$\dot{K} = sY - dK \Rightarrow \frac{\dot{K}}{K} = sAL^{1-\alpha} - d.$$

Taking logs and derivatives of the production function:

$$\frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L}.$$

In this model, A is some constant, so,

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} = sAL^{1-\alpha} - d + (1-\alpha)n.$$

The growth rate of output is an increasing function of L.

b) If L is growing at some constant rate n then the growth rate of output itself is growing at (approximately) an exponential rate. Empirically, we do not see this occuring.

c) Consider a standard production function,

$$Y = BK^{\alpha}L^{1-\alpha}.$$

Suppose that the accumulation of capital generates new knowledge about production in the economy as a whole, while the utilization of labor produces negative externality. Specifically assume that

$$B = AK^{1-\alpha}L^{\alpha-1} = Ak^{1-\alpha},$$

where A is some constant.

Substituting this equation into the production function, we get

$$Y = AK.$$

As a result, even if $n > 0$,

$$\frac{\dot{Y}}{Y} = sA - d.$$

and output growth is constant rather than growing.

d) Labor affects production in a), but not in c). The reason is that we have specified the externality in terms of the capital-labor ratio instead of in terms of the aggregate capital stock: increases in labor will tend to raise the "private" part of output, but they reduce B in an external fashion in a way that leaves total output unchanged. According to this production function, if the economy doubles its stock of labor, aggregate output is unchanged.

What we see from this exercise is that in this model, in order for changes in labor to affect output (as seems plausible), output growth must itself grow over time with population. This seems counterfactual. On the other hand, to eliminate this result, we must eliminate the effect of labor on output, which also seems implausible. This is one reason why economists studying growth have moved on to more sophisticated models.

Exercise 2. *A permanent increase in s_K in the Lucas model.*

A permanent increase in s_K has a level effect, but not a growth effect. To see the reason, remember that in the Lucas model, h enters the production function of the economy just like labor-augmenting technological change in the original Solow model. So the standard results of the Solow model apply here. The steady-state growth rate is determined by time spent accumulating skill. s_K has no long-run growth effect.

Exercise 3. *Market structure in the Lucas model*

We saw in this chapter that imperfect competition and/or externalities are appealed to whenever a growth model exhibits increasing returns to scale. So this question is really about whether or not the Lucas model exhibits increasing returns.

The production function for output in the Lucas model is

$$Y = K^{\alpha}(hL)^{1-\alpha}.$$

There are constant returns to K and L. Therefore, it would appear that there are increasing returns to K, L, and h taken together. However, this turns out to be a misleading way of viewing the model. The reason is that the h is embodied in L, so that these are not really two separate inputs. For example, if we were to write the production function as $Y = K^{\alpha}H^{1-\alpha}$, with $H = hL$, then there are constant returns to K and H.

Lucas (1988) thinks of firms that hire capital K and workers L with skill level h who get paid a wage that is a function of their skill level. These firms are perfectly competitive, and, at least in the setup described here, there is no need for any externalities.

Exercise 4. *Growth over the very long run*.

Idea-based endogenous growth models suggest that growth is the endogenous outcome of an economy in which profit-seeking individuals who are allowed to earn rents on the fruits of their labors search for newer and better ideas. This story has strong explanatory power for the historical evidence.

The Industrial Revolution — the beginning of sustained economic growth — occurred when the institutions protecting intellectual property rights were sufficiently well-developed that entrepreneurs could capture as a private return some of the enormous social returns their innovations would create. While government incentives such as prizes or public funding could substitute for these market incentives in certain cases, history suggests that it is only when the market incentives were sufficient that widespread innovation and growth took hold.

Other models also contribute in other ways to this understanding. Learning by doing models appropriately suggest that experience in production is an important factor in increasing productivity. The acceleration of growth rates over much of world history is potentially consistent with a large number of models, including the "AK" model discussed in the first exercise in this chapter, the Romer (1990) endogenous growth model, and the semi-endogenous growth version of that model in which per capita growth is proportional to population growth (recall from Figure 4.4 that population growth rates have also accelerated over time).

Exercise 5. *The idea production function.*

The production function is

$$\dot{A} = \delta L_A A^{\phi}.$$

An economic justification for this structure is

1. Both research labor and accumulated knowledge contribute to the production of new ideas.

2. L_A enters linearly to capture constant returns to rivalrous inputs: to double the number of ideas produced at any point in time, we simply set up another research lab with the same number of workers. This neglects any duplication that might occur.

3. $\phi > 0$ reflects a positive knowledge spillover in research. The stock of ideas is a nonrivalrous input into the production of new ideas.

It is this last feature that suggests that increasing returns might apply to the production of new ideas, particularly if there is no congestion effect in research (related to point 2).

9 Natural Resources and Economic Growth

Exercise 1. *Transition dynamics in the land model*.

Define $z \equiv K/Y = \left(BK^{\alpha-1}T^{\beta}L^{1-\alpha-\beta}\right)^{-1}$. Taking logs and derivatives, we have

$$\frac{\dot{z}}{z} = -\left(g_B + (\alpha-1)\frac{\dot{K}}{K} + (1-\alpha-\beta)n\right). \tag{2}$$

From the capital accumulation equation and the definition of z, we also have

$$\frac{\dot{K}}{K} = \frac{s}{z} - d. \tag{3}$$

Substituting this second equation into the first and rearranging terms gives

$$\frac{\dot{z}}{z} = (1-\alpha)\frac{s}{z} - \phi, \tag{4}$$

where $\phi \equiv (1-\alpha-\beta)n + g_B + (1-\alpha)d$ is a constant. Multiplying both sides by z, we have a linear differential equation that is easy to analyze:

$$\dot{z} = (1-\alpha)s - \phi z. \tag{5}$$

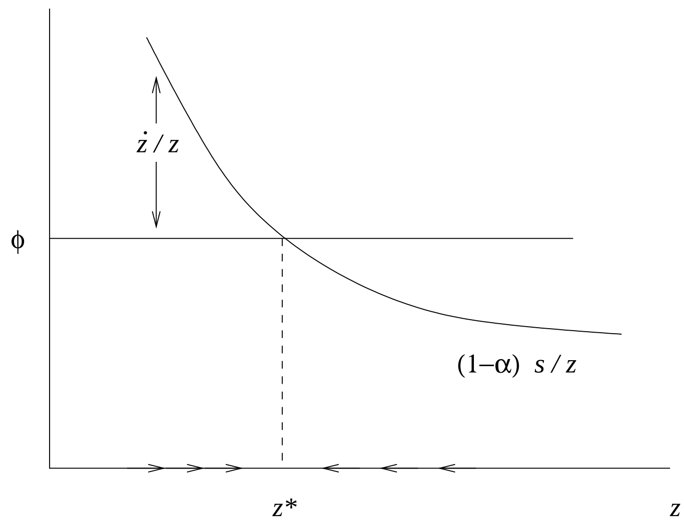
The steady state occurs when $\dot{z} = 0$, so that the capital-output ratio in steady state is given by $z^* = \frac{(1-\alpha)s}{\phi}$. To analyze the transition dynamics, we can look at either equation (5) or equation (4). It turns out to be more convenient to look at the differential equation involving the growth rate of z, so we will focus on equation (4). The stability of the system and the fact that the growth rate of the capital-output ratio declines (or rises) smoothly over time is apparent from Figure 17. Notice that this figure can be analyzed just like the growth version of the Solow diagram in Chapter 2.

Exercise 2. *Transition dynamics in the energy model*.

This problem is solved just like the first exercise. Define $z \equiv K/Y = \left(BK^{\alpha-1}E^{\gamma}L^{1-\alpha-\gamma}\right)^{-1}$. Taking logs and derivatives, we have

$$\frac{\dot{z}}{z} = -\left(g_B + (\alpha-1)\frac{\dot{K}}{K} + \gamma\frac{\dot{E}}{E} + (1-\alpha-\gamma)n\right). \tag{6}$$

Figure 17: The Dynamics of $z \equiv K/Y$ in the Land Model



From the capital accumulation equation and the definition of z, we also have

$$\frac{\dot{K}}{K} = \frac{s}{z} - d. \tag{7}$$

Substituting this second equation into the first, noting that $\dot{E}/E = -s_E$, and rearranging terms gives

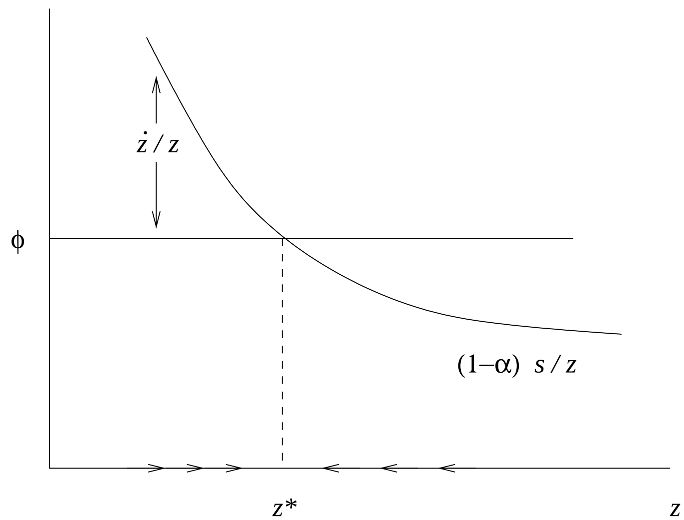
$$\frac{\dot{z}}{z} = (1-\alpha)\frac{s}{z} - \phi, \tag{8}$$

where $\phi \equiv (1-\alpha-\gamma)n + g_B - \gamma s_E + (1-\alpha)d$ is a constant, which will be positive as long as γs_E is not too large (which we assume). Multiplying both sides by z, we have a linear differential equation that is easy to analyze:

$$\dot{z} = (1-\alpha)s - \phi z. \tag{9}$$

The steady state occurs when $\dot{z} = 0$, so that the capital-output ratio in steady state is given by $z^* = \frac{(1-\alpha)s}{\phi}$. To analyze the transition dynamics, we can look at either equation (9) or equation (8). It turns out to be more convenient to look at the differential equation involving the growth rate of z, so we will focus on equation (8). The stability of the system and the fact that the growth rate of the capital-output ratio declines (or rises) smoothly over time is apparent from Figure 18. Notice that this figure can be analyzed just like the growth version of the Solow diagram in Chapter 2.

Figure 18: The Dynamics of $z \equiv K/Y$ in the Energy Model



Exercise 3. *A model with land and energy*.

Using the same procedure as in the previous two questions, it is easy to show that the capital-output ratio will be constant along a balanced growth path. Given this fact, we can derive the steady state growth rate for output per worker as follows:

$$Y = BK^{\alpha}T^{\beta}(s_E R_0 e^{-s_E t})^{\gamma} L^{1-\alpha-\beta-\gamma}$$

$$\begin{aligned}
\Rightarrow Y^{1-\alpha} &= B(\frac{K}{Y})^{\alpha}T^{\beta}(s_E R_0 e^{-s_E t})^{\gamma} L^{1-\alpha-\beta-\gamma} \\
\Rightarrow (\frac{Y}{L})^{1-\alpha} &= B(\frac{K}{Y})^{\alpha}T^{\beta}(s_E R_0 e^{-s_E t})^{\gamma} L^{-\beta-\gamma} \\
\Rightarrow \frac{Y}{L} &= B^{\frac{1}{1-\alpha}}(\frac{K}{Y})^{\frac{\alpha}{1-\alpha}}T^{\frac{\beta}{1-\alpha}}(s_E R_0 e^{-s_E t})^{\frac{\gamma}{1-\alpha}} L^{\frac{\beta+\gamma}{\alpha-1}}
\end{aligned}$$

Taking logs and derivatives of this last equation (and using the facts that the capital-output is constant in a steady state and that T and s_E do not vary with time), we obtain:

$$\begin{aligned}
g_y &= \frac{1}{1-\alpha}g_B - \frac{\gamma}{1-\alpha}s_E - \frac{\beta+\gamma}{1-\alpha}n \\
&= \frac{1}{1-\alpha}[g_B - \gamma s_E - (\beta+\gamma)n]
\end{aligned}$$

Exercise 4. *Optimal extraction rate*.

Define $y \equiv \frac{Y}{L}$. Given that the capital-output ratio is constant, equation 9.17 holds and we know that

$$y(t) = y_0 e^{[g-\bar{\gamma}(s_E+n)]t} = B_0 K_0^{\alpha}(s_E R_0)^{\gamma} L_0^{-\alpha-\gamma} e^{[g-\bar{\gamma}(s_E+n)]t},$$

where B_0, K_0, R_0, and L_0 are the time zero values of B, K, R, and K, respectively. This implies that the present discounted value (PDV) of output per worker is given by

$$\begin{aligned} PDV &= B_0 K_0^{\alpha}(s_E R_0)^{\gamma} L_0^{-\alpha-\gamma} \int_0^{\infty} e^{[g-\bar{\gamma}(s_E+n)-r]t} dt \\ &= B_0 K_0^{\alpha}(s_E R_0)^{\gamma} L_0^{-\alpha-\gamma} [\frac{e^{[g-\bar{\gamma}(s_E+n)-r]t}}{g-\bar{\gamma}(s_E+n)-r}]_0^{\infty} \end{aligned}$$

If $r \geq g - \bar{\gamma}(s_E+n)$, then $\lim_{t\to\infty}[\frac{e^{[g-\bar{\gamma}(s_E+n)-r]t}}{g-\bar{\gamma}(s_E+n)-r}]$ exists and the integral is well-defined. If this is not the case, then we do not have a well-defined maximization problem, so let us assume that $r \geq g - \bar{\gamma}(s_E+n)$ holds. In this case,

$$PDV = \frac{B_0 K_0^{\alpha}(s_E R_0)^{\gamma} L_0^{-\alpha-\gamma}}{r+\bar{\gamma}(s_E+n)-g}.$$

Our task is to find the value of s_E, s_E^*, that maximizes PDV. Noting that PDV is a concave function of s_E over the relevant range, we can find s_E^* by equating the derivative of PDV with respect to s_E equal to zero and solving for s_E :

$$\begin{aligned} \frac{\partial PDV}{\partial s_E} &= \gamma s_E^{\alpha-1} B_0 K_0^{\alpha} R_0^{\gamma} L_0^{-\alpha-\gamma}[r+\bar{\gamma}(s_E+n)-g]^{-1} - \bar{\gamma}[r+\bar{\gamma}(s_E+n)-g]^{-2} \\ & \quad B_0 K_0^{\alpha}(s_E R_0)^{\gamma} L_0^{-\alpha-\gamma} \\ &= 0 \\ \Rightarrow s_E^* &= \frac{(1-\alpha)(r+\bar{\gamma}n-g)}{1-\gamma} \end{aligned}$$

Several interesting interpretations arise. As r increases, future consumption has less present value, so optimizing requires using a greater proportion of non-renewable resources at any given time. As the relative importance of non-renewable resources in production increases (as γ increases), s_E^* increases. This effect is compounded by an increase in the population growth rate, n. As population growth becomes faster, optimizing requires that a greater proportion of non-renewable resources be used at every point in time. Finally, as technological progress accelerates (g_B increases), it becomes optimal to preserve more non-renewable resources for future use, when they will be more productive relative to the present.

Exercise 5. *Solving for τ*.

Equation (9.20) implies that

$$\begin{aligned}
[\frac{e^{(.018-r)t}}{.018-r}]_0^\infty &= (1-\tau)[\frac{e^{(.021-r)t}}{.021-r}]_0^\infty \\
\Rightarrow \tau &= \frac{[\frac{e^{(.021-r)t}}{.021-r}]_0^\infty - [\frac{e^{(.018-r)t}}{.018-r}]_0^\infty}{[\frac{e^{(.021-r)t}}{.021-r}]_0^\infty} \\
&= 1 - \frac{[\frac{e^{(.018-r)t}}{.018-r}]_0^\infty}{[\frac{e^{(.021-r)t}}{.021-r}]_0^\infty} \\
&= 1 - \frac{r-.021}{r-.018} \text{ if } r > .021
\end{aligned}$$

Using the above formula, we can now calculate values of τ for $r = .04$ and $r = .08$:

$$\begin{aligned}
\tau_{.04} &\approx .1364 \\
\tau_{.08} &\approx .0484
\end{aligned}$$

Notice that as r increases, the representative person is less patient and willing to give up less of her income in exchange for faster income growth.

Exercise 6. *Robustness of the growth-drag calculations*.

(Assume that all values, save those for β and γ, are the same as in the text.)

Case a) $\beta = .05, \gamma = .05$

Growth Drag $= (\bar{\beta}+\bar{\gamma})n+\bar{\gamma}s_E = (.05/.8+.05/.8).01+(.05/.8).005$ =.0015625.

Note that $r = .06 \geq .0195625$ (see Problem 5). Thus,

$$\tau = 1 - \frac{.06 - .0195625}{.06 - .018} = .037202381 \ .$$

Case b) $\beta = .05, \gamma = .02$

Growth Drag $= (\bar{\beta}+\bar{\gamma})n+\bar{\gamma}s_E = (.05/.8+.02/.8).01+(.02/.8).005$ =.001.

Given $r = .06 \geq .019$,

$$\tau = 1 - \frac{.06 - .019}{.06 - .018} = .0238095238 \, .$$

So it appears that the magnitudes of our results depend crucially on what we assume about the shares of land and energy. However, the qualitative point that the presence of natural resources in production slows growth in a non-trivial fashion still holds even under lowered estimates of β and γ.

Exercise 7. *A changing land share*.

(a) $Y = Y_A + Y_M = T^{\beta}(AL_A)^{1-\beta} + AL_M = T^{\beta}(AsL)^{1-\beta} + A(1-s)L$.

(b) Our problem is:

$$\max_s T^{\beta}(AsL)^{1-\beta} + A(1-s)L,$$

where the maximization is subject to $0 \leq s \leq 1$. Taking the derivative of our objective function with respect to s and setting this equal to zero gives us the first order condition (notice that the second order condition for maximization, namely that the second derivative of our objective is less than zero, is satisfied):

$$\begin{aligned} (1-\beta)ALT^{\beta}(AsL)^{-\beta} - AL = 0 \\ \Rightarrow (1-\beta)T^{\beta}(AsL)^{-\beta} = 1 \\ \Rightarrow s^* = \frac{T}{AL}(1-\beta)^{\frac{1}{\beta}} \, . \end{aligned}$$

(c) As A and L increase, s^* decreases. That is, as A and L increase, the diminishing returns in agriculture associated with the fixed supply of land set in. Therefore, the optimal allocation involves shifting labor out of agriculture and toward manufacturing (which does not use land).

(d) If the price of land is given by its marginal product, then

$$P_T = \beta T^{\beta-1}(AsL)^{1-\beta} \, .$$

Now we have

$$\frac{P_T T}{Y} = \frac{\beta T^\beta (AsL)^{1-\beta}}{T^\beta (AsL)^{1-\beta} + A(1-s)L}.$$

Since $0 \leq \beta \leq 1$, $\frac{P_T T}{Y}$ decreases as A and L increase. The land share of production declines over time as diminishing returns set in and labor is shifted into manufacturing.

Exercise 8. *Energy's share in a CES production function.*

(a) Consider what happens if we increase all factors of production by a factor of N ($N = 2$ if we double all inputs):

$$\begin{aligned} F(NK, NE, NL) &= [(NK)^\rho + (BNE)^\rho]^{\frac{\alpha}{\rho}} (ANL)^{1-\alpha} \\ &= [N^\rho K^\rho + N^\rho (BE)^\rho]^{\frac{\alpha}{\rho}} N^{1-\alpha} (AL)^{1-\alpha} \\ &= N^\alpha [K^\rho + (BE)^\rho]^{\frac{\alpha}{\rho}} N^{1-\alpha} (AL)^{1-\alpha} \\ &= N[K^\rho + (BE)^\rho]^{\frac{\alpha}{\rho}} (AL)^{1-\alpha} \\ &= NF(K, E, L) \end{aligned}$$

Thus, if we double all inputs, output doubles, i.e. this production function exhibits constant returns to scale.

(b) The factor shares for K, L, and E are given by $\frac{P_K K}{Y}$, $\frac{P_L L}{Y}$, and $\frac{P_E E}{Y}$, respectively. Given that factors are paid their marginal products, we have:

$$\begin{aligned} P_K &= \frac{\alpha}{\rho} \rho K^{\rho-1} [K^\rho + (BE)^\rho]^{\frac{\alpha-\rho}{\rho}} (AL)^{1-\alpha} \\ P_E &= \alpha B^\rho E^{\rho-1} [K^\rho + (BE)^\rho]^{\frac{\alpha-\rho}{\rho}} (AL)^{1-\alpha} \\ P_L &= (1-\alpha) A^{1-\alpha} L^{-\alpha} [K^\rho + (BE)^\rho]^{\frac{\alpha}{\rho}} \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{P_K K}{Y} &= \alpha K^\rho [K^\rho + (BE)^\rho]^{-1} \\ \frac{P_E E}{Y} &= \alpha (BE)^\rho [K^\rho + (BE)^\rho]^{-1} \\ \frac{P_L L}{Y} &= 1 - \alpha \end{aligned}$$

Notice that while labor's share is equivalent to the Cobb-Douglas case, the factor shares for capital and energy now depend on how much capital and energy are employed.

(c) With $\rho < 0$, as B increases, $\frac{P_K K}{Y}$ increases, $\frac{P_E E}{Y}$ decreases, and $\frac{P_L L}{Y}$ is unaffected. The declining energy share is offset by a rising share of income paid to capital.

10 Natural Resources and Economic Growth

No problems.

A Mathematical Review

Exercise 1

a) $\frac{\dot{y}}{y} = \frac{\dot{x}}{x} = \frac{0.05e^{0.05t}}{e^{0.05t}} = 0.05,$
b) $\frac{\dot{y}}{y} = \frac{\dot{z}}{z} = \frac{0.01e^{0.01t}}{e^{0.01t}} = 0.01,$
c) $\frac{\dot{y}}{y} = \frac{\dot{x}}{x} + \frac{\dot{z}}{z} = 0.06,$
d) $\frac{\dot{y}}{y} = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} = 0.04,$
e) $\frac{\dot{y}}{y} = \beta\frac{\dot{x}}{x} + (1-\beta)\frac{\dot{z}}{z} = \frac{1}{2} \times 0.05 + \frac{1}{2} \times 0.01 = 0.03,$
f) $\frac{\dot{y}}{y} = \beta\frac{\dot{x}}{x} - \beta\,\frac{\dot{z}}{z} = \frac{1}{3} \times 0.05 - \frac{1}{3} \times 0.01 = .01333.$

Exercise 2
a) $\frac{\dot{y}}{y} = \beta\,\frac{\dot{k}}{k},$
b) $\frac{\dot{y}}{y} = \frac{\dot{k}}{k} - \frac{\dot{m}}{m},$
c) $\frac{\dot{y}}{y} = \beta\,(\frac{\dot{k}}{k} + \frac{\dot{l}}{l} + \frac{\dot{m}}{m}),$
d) $\frac{\dot{y}}{y} = \beta\,(\frac{\dot{k}}{k} + \frac{\dot{l}}{l}) - (1-\beta)\,\frac{\dot{m}}{m}.$

Exercise 3

We are told that $\frac{\dot{x}}{x} = 0.1$. Now integrate both sides of this equation,

$$\int d\log x(t) = \int 0.1dt,$$

which implies that

$$\log x(t) = 0.10\,t + C.$$

Given $x(0) = 2$,

$$x(t) = 2\,e^{0.1t}.$$

By the same reasoning,

$$z(t) = e^{0.02t}.$$

a) $y(t) = x(t)\,z(t),$
b) $y(t) = z(t)\,/x(t)\,,$

t	$x(t)$	$z(t)$	$y = xz$	$y = z/x$
0	2.000	1.000	2.000	0.500
1	2.210	1.020	2.254	0.462
2	2.443	1.041	2.543	0.426
10	5.437	1.221	6.639	0.225

c) $y = x^{\beta} z^{(1-\beta)}$, where $\beta = \frac{1}{3}$

t	x^{β}	$z^{(1-\beta)}$	$x^{\beta} z^{(1-\beta)}$
0	1.260	1.000	1.260
1	1.303	1.013	1.320
2	1.347	1.027	1.383
10	1.758	1.142	2.008

Exercise 4

Note: $\hat{y}$ represents GDP per worker relative to the U.S.A. Plain y denotes the actual level of GDP per worker. Table B.2 informs us that $y^{U.S.}(97) = 40834$ and $y^{U.S.}(60) = 24433$. Therefore, the growth rate for the U.S. is

$$g_y^{US} = 1/37 * (\log(40834) - \log(24433)) = .0139.$$

The growth rate for other countries can be calculated by converting their relative GDPs into absolute levels of GDP, or by calculating directly as

$$g_y^i = g_{\hat{y}}^i + .0139.$$

Country	$\hat{y}(60)$	$\hat{y}(97)$	$y(60)$	$y(97)$	$g(60, 97)$
U.S.	1.00	1.00	24433	40834	0.0139
Canada	0.80	0.86	19484	35267	0.0160
Argentina	0.46	0.45	11339	18485	0.0132
Chad	0.08	0.03	1927	1128	-0.0145
Brazil	0.23	0.30	5549	12153	0.0212
Thailand	0.08	0.23	1884	9505	0.0437

Notice that these numbers differ from those at the back of the book because of rounding errors (the relative income numbers are not very precise — especially for a country like Chad or Thailand where there is only one significant digit).

Exercise 5

Denote the annual growth rate of GDP as g_Y and the labor force growth rate as n. Then

$$g_Y(60,97) = g(60,97) + n.$$

Country	$g(60,97)$	n	$g_Y(60,97)$
U.S.	0.0139	0.0096	0.0235
Canada	0.0160	0.0122	0.0283
Argentina	0.0132	0.0141	0.0273
Chad	-0.0145	0.0276	0.0131
Brazil	0.0212	0.0174	0.0386
Thailand	0.0437	0.0153	0.0590

Note: We are making an error here in that we are using the population growth rate between 1980 and 1997 instead of 1960 and 1997.

Note that the population growth rate and labor force growth rate usually are not the same because of changes in labor force participation rates. These may take the form of retirements, new entry because of a baby boom, and shifts from nonmarket to market work, among many other reasons.

Exercise 6

Figure 19: GDP per Worker and Years of Schooling