MACRO III
(Summer 2005)

Ex. 5.3, Barro, Sala-i-Martin, p. 211

Externalities in human capital (Lucas 1988)

$$Y = AK^{\alpha} H^{\eta} \bar{H}^{\epsilon}$$
$$I_H = BH$$

$0 < \alpha, \eta < 1$, $0 \le \epsilon < 1$
also need $\eta = 1-\alpha$ for the existence of CE

$$\max \int_0^{\infty} \frac{c(t)^{1-\theta}}{1-\theta} e^{-\rho t} dt, \text{ s.t.}$$

$$Y = AK^{\alpha}(uH)^{\eta} \bar{H}^{\epsilon} = C + \dot{K} + \delta K$$
$$I_H = B(1-u)H = \dot{H} + \delta H$$

$$\mathcal{H} = \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} + \lambda \left[AK^{\alpha}(uH)^{\eta} \bar{H}^{\epsilon} - c - \delta K \right] + \mu \left[B(1-u)H - \delta H \right]$$

FOC: $\frac{\partial \mathcal{H}}{\partial c} = 0$: $\lambda = c^{-\theta} e^{-\rho t}$; $\dot{\lambda} = -\theta c^{-\theta} \frac{\dot{c}}{c} e^{-\rho t} - \rho c^{-\theta} e^{-\rho t}$

$$\dot{\lambda} = -\lambda \left[\theta \gamma_c + \rho \right] \qquad (1)$$

$$\frac{\partial \mathcal{H}}{\partial u} = 0: \quad \lambda A \eta K^{\alpha} u^{\eta-1} H^{\eta} \bar{H}^{\epsilon} - \mu BH = 0 \qquad (2)$$

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial K} = -\lambda\left[A\alpha K^{\alpha-1}(uH)^{\eta}\bar{H}^{\epsilon} - \delta\right] \qquad (3)$$

$$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial H} = -\underbrace{\lambda\left[AK^{\alpha}\eta u^{\eta}H^{\eta-1}\bar{H}^{\epsilon}\right]}_{\mu B u} - \mu\left[B(1-u)-\delta\right] \qquad (4)$$

- combining (3) and (1) and from the constraints we get

$$\gamma_C = \frac{1}{\theta}\left[\alpha A K^{\alpha-1}(uH)^{\eta}\bar{H}^{\epsilon} - \delta - \rho\right] \qquad (5)$$

$$\gamma_K = AK^{\alpha-1}(uH)^{\eta}\bar{H}^{\epsilon} - \frac{C}{K} - \delta \qquad (6)$$

$$\gamma_H = B(1-u) - \delta \qquad (7)$$

- as in steady state the average product of capital is constant, differentiating the first term in (6) gives us the relationship between the growth of human and physical capital

$$(\alpha-1)\gamma_K + \cancel{\eta\gamma_u} + (\eta+\epsilon)\gamma_H = 0$$

note that γ_u is zero in SS

$$\boxed{\gamma_K = \frac{\eta+\epsilon}{1-\alpha}\gamma_H} \qquad (10)$$

- thus the human capital grows at the same rate as physical capital only if there is no externality (i.e. $\epsilon=0$)

- differentiating (6) also gives $\gamma_C = \gamma_K$ and from prod. fcn. $\gamma_Y = \gamma_K$

- differentiating (2) gives us the growth condition

$$\frac{\dot{\lambda}}{\lambda} + \alpha\gamma_K + (\eta+\epsilon)\gamma_H = \frac{\dot{\mu}}{\mu} + \gamma_H \qquad (8)$$

- combining (2) and (4) gives

$$\dot{\mu} = -\mu u B - \mu\left[B(1-u) - \delta\right]$$

$$\rightarrow \frac{\dot{\mu}}{\mu} = \delta - B \qquad (9)$$

- using (1) and (9) in (8) we get another condition

$$(\alpha-\theta)\gamma_K = \delta - B + \rho + (1-\eta-\epsilon)\gamma_H \qquad (11)$$

- we thus have five conditions which can be solved for the steady state values of $\gamma_K, \gamma_H, u, \frac{C}{K}, \frac{K^{\alpha-1}}{H^{-(\eta+\epsilon)}}$

(5) $\gamma_C = \frac{1}{\theta}\left[\alpha A K^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \delta\right] \rightarrow \frac{K^{\alpha-1}}{H^{-(\eta+\epsilon)}}$

(6) $\gamma_K = A K^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \frac{C}{K} - \delta \rightarrow \frac{C}{K}$

(7) $\gamma_H = B(1-u) - \delta \rightarrow u$

(10) $\gamma_H = \frac{1-\alpha}{\eta+\epsilon}\gamma_K \rightarrow \gamma_H$

(11) $(\alpha-\theta)\gamma_K = \rho + \delta - B + (1-\eta-\epsilon)\gamma_H \rightarrow \gamma_K$

- differentiating (2) gives us the growth condition

$$\frac{\dot{\lambda}}{\lambda} + \alpha\gamma_K + (\eta+\epsilon)\gamma_H = \frac{\dot{\mu}}{\mu} + \gamma_H \qquad (8)$$

- combining (2) and (4) gives

$$\dot{\mu} = -\mu u B - \mu[B(1-u)-\delta]$$

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- we thus have five conditions which can be solved for the steady state values of $\gamma_K, \gamma_H, u, \frac{c}{K}, \frac{K^{\alpha-1}}{H^{-(\eta+\epsilon)}}$

(5) $\gamma_c = \frac{1}{\theta}\left[\alpha A K^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \delta\right]$ $\rightarrow \frac{K^{\alpha-1}}{H^{-(\eta+\epsilon)}}$

(6) $\gamma_K = A K^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \frac{c}{K} - \delta$ $\rightarrow \frac{c}{K}$

(7) $\gamma_H = B(1-u) - \delta$ $\rightarrow u$

(10) $\gamma_H = \frac{1-\alpha}{\eta+\epsilon}\gamma_K$ $\rightarrow \gamma_H$

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Social Planner

$$\mathcal{H} = \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} + \lambda\left[AK^{\alpha} u^{\eta} H^{\eta+\epsilon} - c - \delta K\right] + \mu\left[B(1-u)H - \delta H\right]$$

FOC: $\frac{\partial \mathcal{H}}{\partial c} = 0$: $\lambda = c^{-\theta} e^{-\rho t}$; $\dot{\lambda} = -\lambda\left[\theta \gamma_c + \rho\right]$

$$\frac{\partial \mathcal{H}}{\partial u} = 0: \quad \lambda \eta A K^{\alpha} u^{\eta-1} H^{\eta+\epsilon} - \mu B H = 0$$

$$\dot{\lambda} = -\frac{\partial \mathcal{H}}{\partial K} = -\lambda\left[A\alpha K^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \delta\right]$$

$$\dot{\mu} = -\frac{\partial \mathcal{H}}{\partial H} = -\underbrace{\lambda\left[AK^{\alpha}(\eta+\epsilon) u^{\eta} H^{\eta+\epsilon-1}\right]}_{\mu u B \frac{\eta+\epsilon}{\eta}} - \mu\left[B(1-u) - \delta\right]$$

$$\gamma_c = \frac{1}{\theta}\left[\alpha A K^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \delta - \rho\right]$$

$$\gamma_K = AK^{\alpha-1} u^{\eta} H^{\eta+\epsilon} - \frac{c}{K} - \delta$$

$$\gamma_H = B(1-u) - \delta$$

again $\gamma_K = \frac{\eta+\epsilon}{1-\alpha} \gamma_H$

$$(\alpha-\theta)\gamma_K - \rho + (\eta+\epsilon)\gamma_H = \frac{\dot{\mu}}{\mu} + \gamma_H$$

$$\frac{\dot{\mu}}{\mu} = -uB\frac{\eta+\epsilon}{\eta} - B + Bu + \delta = -uB\frac{\epsilon}{\eta} - B + \delta$$

$$\rightarrow (\alpha-\theta)\gamma_K - (1-\eta-\epsilon)\gamma_H = \rho + \delta - B - uB\frac{\epsilon}{\eta}$$

- plugging for u and γ_H we get

$$(\alpha-\theta)\gamma_K - \frac{(1-\eta-\epsilon)(1-\alpha)}{\eta+\epsilon}\gamma_K = \rho+\delta-B-\frac{\epsilon}{\eta}\left(B-\delta-\frac{1-\alpha}{\eta+\epsilon}\gamma_K\right)$$

$$\gamma_K\left[\frac{\alpha(\eta+\epsilon)-\theta(\eta+\epsilon)-(1-\eta-\epsilon)+\alpha-\alpha(\eta+\epsilon)-\frac{\epsilon}{\eta}(1-\alpha)}{\eta+\epsilon}\right]=\rho+(\delta-B)\frac{\eta+\epsilon}{\eta}$$

$$\gamma_K=\frac{\eta+\epsilon}{(1-\theta)(\eta+\epsilon)-(1-\alpha)-\frac{\epsilon}{\eta}(1-\alpha)}\left[\rho+(\delta-B)\left(1+\frac{\epsilon}{\eta}\right)\right]$$

- using $\eta=1-\alpha$ this can be simplified to

$$\gamma_K=\frac{\cancel{\eta+\epsilon}}{\cancel{(1-\theta)(\eta+\epsilon)-(1-\alpha)\frac{\eta+\epsilon}{\eta}}}\left[\rho+(\delta-B\left(1+\frac{\epsilon}{\eta}\right)\right]$$

$$\gamma_K^*=\frac{1}{\theta}\left[(B-\delta)\frac{\eta+\epsilon}{\eta}-\rho\right]$$

$$\gamma_H=\frac{\cancel{1-\alpha}}{\cancel{\eta+\epsilon}}\,\frac{1}{\theta}\,\frac{\cancel{\eta+\epsilon}}{\cancel{\eta}}\left[(B-\delta)-\frac{\eta}{\eta+\epsilon}\rho\right]$$

$$\gamma_H^*=\frac{1}{\theta}\left[B-\delta-\frac{\eta}{\eta+\epsilon}\rho\right]$$

- for $\theta=1$ $\quad \gamma_H^{*\,SP}-\gamma_H^{*\,CE}=B-\delta-\frac{\eta}{\eta+\epsilon}\rho+\rho+\delta-B=\frac{\epsilon}{\eta+\epsilon}\rho>0$

- the externality causes too low growth of human capital if agents are risk neutral