Sketch of HW 5 solution

a)

i-th producer of final good solves

$$\max \Pi = Y_i - wL_i - \sum_j R_j X_{ij} = A L_i^{1-\alpha} \sum_j X_{ij}^{\alpha} - wL_i - \sum_j R_j X_{ij}$$

FOC
$$\frac{\partial \Pi}{\partial x_{ij}}: \quad A\alpha L_i^{1-\alpha} x_{ij}^{\alpha-1} = R_j \qquad (\square)$$

$$\frac{\partial \Pi}{\partial L_i}: \quad A(1-\alpha) L_i^{-\alpha} \sum_j x_{ij}^{\alpha} = (1-\alpha)\frac{Y_i}{L_i} = w$$

<u>intermediate good j</u>:

cost of producing x_j = φ units of Y (omit index "i")

intermediate good producer solves

$$\max \Pi = -\eta - \varphi X_j(0) + \int_0^{\infty} [R_j(t) x_j(t) - \dot{X}_j(t)] e^{-\int_0^t r(s)ds} dt$$

η: fixed cost; $\varphi X_j(0)$: installation cost; $\dot{X}_j(t)$: production $dt > 0$

s.t. demand for good x_j

~~assume that~~

a) From ($\square$) follows

$$\circledast \quad A L^{1-\alpha} \alpha x_j^{\alpha-1} = R_j \; \forall j \Rightarrow X_j^{\alpha-1} = \frac{R_j}{\alpha A} L^{\alpha-1}$$

demand for good x_j:
$$X_j = \left(\frac{R_j}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$$

assuming that in steady state $r(t) = r$

intermediate firm's ~~problem so~~ maximization problem collapses to

$$\max_{x_j} -\eta + \int_0^{\infty} \left(R_j(t) x_j(t) - r\varphi X_j(t)\right) e^{-rt} dt$$

s.t.
$$X_j = \left(\frac{R_j(t)}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$$

because $\int_0^\infty \dot{x}_i(t) e^{-rt} = -x_j(0) + \int_0^\infty r x_j(t) e^{-rt} dt$

~~[illegible]~~

this reduces it to time t optimization

$$\max_{x_j} \; R_j x_j - \varphi r x_j$$

s.t.

$$x_j = \left(\frac{R_j}{\alpha A}\right)^{\frac{1}{\alpha-1}} L$$

F.O.C x_j:

$$\frac{\partial R_j(x_j)}{\partial x_j} x_i + R_j \underbrace{\frac{\partial x_j}{\partial x_j}}_{1} - r\varphi = 0$$

implicit derivation $= \frac{1}{\alpha-1} \frac{R_j}{x_j}$

$$\Rightarrow R_j = \frac{r\varphi}{1 + \frac{1}{\alpha-1}} = \frac{r\varphi}{\alpha}$$

$\Rightarrow$ R_j is the same for all goods

b) assuming that in s.s. r is constant from (*) follows

$$A\alpha L_i^{1-\alpha} x_{ij}^{\alpha-1} = R_j = \frac{r\varphi}{\alpha}$$

$$x_{ij}^{\alpha-1} = \frac{\alpha}{r\varphi} A \alpha L_i^{1-\alpha}$$

$$x_{ij} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{A}{r\varphi}\right)^{\frac{1}{1-\alpha}} L_i$$

omit index i $\quad x_j = \alpha^{\frac{2}{1-\alpha}} \left(\frac{A}{r\varphi}\right)^{\frac{1}{1-\alpha}} L \quad (0)$

$$x_j = \left(\frac{\alpha^2 A}{r\varphi}\right)^{\frac{1}{1-\alpha}} L$$

c) substitute for R_j, x_j into profit function of intermediate firm. in steady state

$$\Pi = -\eta - \varphi x_j + \int_0^\infty R_j x_j e^{-rt}\, dt$$

$$\Pi = -\eta - \varphi x_j + R_j x_j \frac{1}{r}$$

assuming free entry and perfect competition drives profit to 0

$$\Rightarrow \Pi = 0$$

$$\eta = x_j \left(\frac{R_j}{r} - \varphi\right) \qquad R_j = \frac{r\varphi}{\alpha}$$

because $x_j = x \ \forall j$

$$\eta = x\left(\frac{\varphi}{\alpha} - \varphi\right) = x\left(\frac{\varphi - \alpha\varphi}{\alpha}\right) = x\varphi\left(\frac{1-\alpha}{\alpha}\right)$$

substitute for x from (a)

$$\eta = \left(\frac{\alpha^2 A}{r\varphi}\right)^{\frac{1}{1-\alpha}} L \cdot \varphi \left(\frac{1-\alpha}{\alpha}\right)$$

$$\eta = \alpha^{\frac{2}{1-\alpha}} A^{\frac{1}{1-\alpha}} \cdot L \cdot r^{\frac{1}{\alpha-1}} \cdot \varphi^{\frac{1}{\alpha-1}} \cdot \varphi \left(\frac{1-\alpha}{\alpha}\right)$$

$$\eta = \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \cdot L \, r^{\frac{1}{\alpha-1}} \varphi^{\frac{\alpha}{\alpha-1}} (1-\alpha)$$

$$r^{\frac{1}{1-\alpha}} = \eta^{-1} \alpha^{\frac{1+\alpha}{1-\alpha}} A^{\frac{1}{1-\alpha}} \cdot \varphi^{\frac{\alpha}{\alpha-1}} (1-\alpha) L$$

$$r = \eta^{\alpha-1} \alpha^{1+\alpha} A \cdot \varphi^{-\alpha} \cdot (1-\alpha)^{1-\alpha} L^{1-\alpha}$$

we know that in Ramsey model from Household maximization problem we get $\gamma_c = \frac{\dot{c}}{c} = \frac{1}{\theta}(r - \rho)$

therefore we can plug steady state r into this formula to get $\gamma_c^{durable} = \frac{1}{\theta}\left[\eta^{-1+\alpha}\alpha^{1+\alpha} A \varphi^{-\alpha}(1-\alpha)^{+1-\alpha} L^{1-\alpha} - \rho\right]$

$$= \frac{1}{\theta}\left[\left(A^{\frac{1}{1-\alpha}} \cdot \alpha^{\frac{2}{1-\alpha}} \frac{1-\alpha}{\alpha} \frac{L}{\eta}\right)^{1-\alpha} \varphi^{-\alpha} - \rho\right]$$

$$\gamma_c^{perishable} = \frac{1}{\theta}\left[A^{\frac{1}{1-\alpha}} \cdot \alpha^{\frac{2}{1-\alpha}} \cdot \frac{1-\alpha}{\alpha} \frac{L}{\eta} - \rho\right]$$

in case of durables the interest rate ~~is~~

~~of interest rate~~

the $r^{durable} = (r^{perishable})^{1-\alpha} \cdot \varphi^{-\alpha}$

↑ cost of installing in terms of goods.

assuming $\varphi = 1$ the increase in α ($\alpha \in (0,1)$) leads to increase (decrease) in case $r < 1$ ($r > 1$) therefore to increase (decrease) in steady state growth rate.

d) due to durability of goods, households can not build up and destroy capital without transition dynamics.

There will be also transitional dynamics when households will ~~approach~~ approach steady state. or as reaction to shocks.

In case of perishables there is no transitional dynamics as reaction to exogenous shocks.