$$\max \sum_{t=0}^{\infty} \beta^t \left[\frac{(c_t m_t^{b})^{1-\phi}}{1-\phi} - \psi \frac{l_t^{1-\eta}}{1-\eta} \right]$$

$$c_t + k_t + b_t + m_t = y_t + (1-\delta) k_{t-1} + \tau_t + \frac{1+i_{t-1}}{1+\Pi_t} b_{t-1} + \frac{m_{t-1}}{1+\Pi_t} \; ; \; \tau_t = \frac{\theta_t}{1+\Pi_t} m_t$$

$$m_t = \frac{1+\theta_t}{1+\Pi_t} m_{t-1}$$

$$\theta_t = \theta^{ss} + u_t \; ; \; u_t = \rho_u u_{t-1} + \rho_z z_{t-1} + \varphi_t$$

$$z_t = \rho z_{t-1} + e_t$$

$$1 - l_t = n_t \qquad y = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha}$$

State $k_{t-1}, b_{t-1}, m_{t-1}$

control c_t, k_t, m_t, n_t, b_t

F.O.C

$$c_t: \quad \frac{(c_t m_t^{b})^{1-\phi}}{c_t} = \lambda_t \qquad \underline{1}$$

$$n_t: \quad \psi (1-n_t)^{-\eta} = \lambda_t (1-\alpha) k_{t-1}^{\alpha} n_t^{-\alpha} e^{z_t}$$

$$m_t: \quad \frac{(c_t m_t^{b})^{1-\phi} b}{m_t} = +\lambda_t - E_t \beta \left[\lambda_{t+1} \cdot \frac{1}{1+\Pi_{t+1}} \right]$$

$$b_t: \quad \lambda_t = \beta E_t \left[\lambda_{t+1} \frac{1+i_t}{1+\Pi_{t+1}} \right]$$

$$k_t \quad \lambda_t = \beta E_t \left(\alpha \cdot e^{z} k_t^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta \right) \lambda_{t+1}$$

$$\lambda_t \quad c_t + k_t + b_t + m_t = y_t + (1-\delta) k_{t-1} + \tau_t + \frac{1+i_{t-1}}{1+\Pi_t} b_{t-1} + \frac{m_{t-1}}{1+\Pi_t}.$$

$$m_t = \frac{1+\theta_t}{1+\Pi_t} m_{t-1}$$

+

$u_t = ..$

$z_t = ..$

define

$$R_t = E_t\left(\alpha \frac{k_{t+1}^{\alpha} n_{t+1}^{1-\alpha}}{k_{t+1}} + 1-\delta\right) = \alpha E_t \frac{y_t}{k_t} + 1-\delta$$ real rate

$$x_t = k_t - (1-\delta)k_{t-1}$$ investment

$$c_t + k_t - (1-\delta)k_{t-1} = y_t$$ physical capital resource contraint

Fisher equat.

$$(1+i_t) = (1+\pi_t)(1+r_t)$$

Steady state. $n, R, \frac{y}{k}, y, k, x, c, m, \Pi, \lambda, i$ 10 variables

$c^{-\phi} m^{b(1-\phi)} = \lambda$

$\psi (1-n)^{-\eta} = \lambda (1-\alpha) k^{\alpha} n^{-\alpha}$

$\lambda = \beta \lambda \frac{1+i}{1+\Pi} \Rightarrow 1 = \beta \frac{1+i}{1+\Pi} \Rightarrow i = \frac{1+\Pi-\beta}{\beta}$

$\lambda = \beta [\underbrace{\alpha k^{\alpha-1} n^{1-\alpha} + 1-\delta}_{R}]\lambda \Rightarrow R = 1/\beta$

$m = \frac{1+\theta}{1+\Pi} m \Rightarrow \theta = \Pi$

n - given

$R = \underbrace{\alpha k^{\alpha-1} n^{1-\alpha}}_{\alpha \frac{y}{k}} + 1-\delta \Rightarrow \frac{y}{k} = \frac{1}{\alpha}(R - 1+\delta)$

$\frac{y}{k} = \frac{k^{\alpha-1}}{n^{\alpha-1}} = \left(\frac{k}{n}\right)^{\alpha-1} \Rightarrow \frac{k}{n} = \left(\frac{y}{k}\right)^{1/\alpha-1}$

$k = \left(\frac{y}{k}\right)^{\frac{1}{\alpha-1}} \cdot n$; $y = k^{\alpha} n^{1-\alpha}$

$x = \delta k$

$c = y - x =$

$b c^{1-\phi} m^{b(1-\phi)-1} = \lambda - \beta \lambda \frac{1}{1+\Pi}$ ← plug in $\lambda = c^{-\phi} m^{b(1-\phi)}$

$\beta \frac{1}{1+\Pi} = \frac{1}{1+i}$

$b c m^{-1} = \frac{i}{1+i}$

$m = \frac{1+i}{i} bc$

Log. linear. 10 equations + 2 ~~stat~~ exogenous eqn. are needed.

1: $c_t^{-\phi} m_t^{b(1-\phi)} = \lambda_t \Rightarrow c^{-\phi} m^{b(1-\phi)} (1-\phi \tilde{c}_t)(1+b(1-\phi)\tilde{m}_t) = \lambda (1+\tilde{\lambda}_t)$

$$1-\phi\tilde{c}_t + b(1-\phi)\tilde{m}_t = 1+\tilde{\lambda}_t$$

$$\tilde{\lambda}_t = -\phi\tilde{c}_t + b(1-\phi)\tilde{m}_t$$

2) $\psi (l_t)^{-\eta} = \lambda_t (1-\alpha) k_{t-1}^{\alpha} n_t^{-\alpha} e^{z_t} = \lambda_t (1-\alpha) \frac{y_t}{n_t}$

$$1-n_t = l_t$$
$$1-n(1+\tilde{n}_t) = l(1+\tilde{l}_t)$$
$$\tilde{l}_t = -\frac{n}{l}\tilde{n}_t$$

$$\psi l^{-\eta}(1-\eta\tilde{l}_t) = (1-\alpha)\lambda\frac{y}{n}(1+\tilde{\lambda}_t)(1+\tilde{y}_t)(1-\tilde{n}_t)$$

$$1-\eta\tilde{l}_t = 1+\tilde{\lambda}_t+\tilde{y}_t-\tilde{n}_t$$

$$-\eta\cdot\frac{-n}{l}\tilde{n}_t = \tilde{\lambda}_t+\tilde{y}_t-\tilde{n}_t$$

$$\left(1+\eta\frac{n}{1-n}\right)\tilde{n}_t = \tilde{\lambda}_t+\tilde{y}_t$$

3) $y_t = e^{z_t} k_{t-1}^{\alpha} n_t^{1-\alpha} \Rightarrow y(1+\tilde{y}_t) = k^{\alpha} n^{1-\alpha}(1+\alpha\tilde{k}_{t-1})(1+(1-\alpha)\tilde{n}_t)(1+\tilde{z}_t)$

$$1+\tilde{y}_t = 1+\alpha\tilde{k}_{t-1}+(1-\alpha)\tilde{n}_t+\tilde{z}_t$$

$$\tilde{y}_t = \alpha\tilde{k}_{t-1}+(1-\alpha)\tilde{n}_t+\tilde{z}_t$$

4) $y_t = c_t + k_t - (1-\delta)k_{t-1} \Rightarrow y(1+\tilde{y}_t) = c(1+\tilde{c}_t) + k(1+\tilde{k}_t) - k(1+\tilde{k}_{t-1})(1-\delta)$

$$y+\tilde{y}_t y = c + c\tilde{c}_t + k + k\tilde{k}_t + k - k(1-\delta)\tilde{k}_{t-1}$$

$$\tilde{y}_t y = c\tilde{c}_t + k\tilde{k}_t - k(1-\delta)\tilde{k}_{t-1}$$

5) $R_t = \alpha E_t \frac{y_t}{k_t} + 1-\delta \Rightarrow R(1+\tilde{R}_t) = \alpha \frac{E_t y_{t+1}}{k_t} + 1-\delta =$

$$\tilde{R}_t = \frac{\alpha y}{R k} E(\tilde{y}_{t+1} - \tilde{k}_t)$$

See more derivations ~~attached~~ at the end.

6) $\lambda_t = \beta E_t(R_t \cdot \lambda_{t+1}) \Rightarrow \lambda(1+\tilde{\lambda}_t) = \beta R \lambda E_t(1+\tilde{R}_t + \tilde{\lambda}_{t+1})$

$$\tilde{\lambda}_t = E_t[R_t + \tilde{\lambda}_{t+1})$$

7) $\lambda_t = \beta E_t\left[\lambda_{t+1} \frac{1+i_t}{1+\Pi_{t+1}}\right]$ plug for λ_{t+1} into

$$\frac{b(c_t m_t^b)^{1-\phi}}{m_t} = \lambda_t - E_t \beta\left[\lambda_{t+1} \frac{1}{1+\Pi_{t+1}}\right]$$

$$\frac{b(C_t m_t^b)^{1-\phi}}{m_t} = \lambda_t - \frac{\lambda_t}{1+i_t} = \lambda_t \frac{i_t}{1+i_t}$$, plug $\lambda_t = \frac{(c_t m_t^b)^{1-\phi}}{c_t}$ for λ_t

$$\frac{b c_t}{m_t} = \frac{i_t}{1+i_t}$$ $\Leftarrow$ def $q_t = 1+i_t$, log lin $\tilde{q}_t = \frac{1}{q} i \tilde{i}_t = \frac{i}{1+i} \tilde{q}_t$

$$\frac{bc}{m}(1+\tilde{c}_t)(1-\tilde{m}_t) = \frac{i}{q}(1+\tilde{i}_t)(1-\tilde{q}_t)$$

$$1 + b\tilde{c}_t - \tilde{m}_t = 1 + \tilde{i}_t - \tilde{q}_t$$

$$b\tilde{c}_t - \tilde{m}_t = \frac{1}{1+i} \tilde{i}_t$$

8) $\tilde{i}_t = \tilde{R}_t + E_t \tilde{\Pi}_{t+1}$

9) $\tilde{m}_t = \tilde{m}_{t-1} - \tilde{\Pi}_t + u_t$ $\qquad u_t = \theta_t - \bar{\theta} = \theta \cdot \frac{\theta_t - \theta}{\theta} = \theta \cdot \tilde{\theta}_t$

10) $x_t = k_t - (1-\delta) k_{t-1} \Rightarrow x \tilde{x}_t = k \tilde{k}_t - (1-\delta) k \tilde{k}_{t-1}$

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5) continued eliminate Expectation operator

$E \tilde{z}_{t+1} = \rho \tilde{z}_t$

plugging $\underbrace{\left(1+\eta \frac{n}{1-n}\right) \tilde{n}_t = \tilde{\lambda}_t + \tilde{y}_t}_{②}$ for $\tilde{n}_t$ into $\underbrace{\tilde{y}_t = \alpha \tilde{k}_{t-1} + (1-\alpha) \tilde{n}_t + \tilde{z}_t}_{3}$

gives $\tilde{y}_t = \frac{A\alpha}{A-1+\alpha} \tilde{k}_{t-1} + \frac{A}{A-1+\alpha} \frac{(1-\alpha)}{A} \tilde{\lambda}_t + \tilde{z}_t \cdot \frac{A}{A-1+\alpha}$ where $A = \left(1+\eta \frac{n}{1-n}\right)$

Shifting by one period

$$\tilde{y}_{t+1} = \frac{A\alpha}{A-1+\alpha} \tilde{k}_t + \frac{1-\alpha}{A-1+\alpha} \cdot \tilde{\lambda}_{t+1} + \tilde{z}_{t+1} \cdot \frac{A}{A-1+\alpha}$$

taking expectations

$$E_t \tilde{y}_{t+1} = E_t \left[\frac{A\alpha}{A-1+\alpha} \cdot \tilde{k}_t + \frac{1-\alpha}{A-1+\alpha} \tilde{\lambda}_{t+1} + \tilde{z}_{t+1} \frac{A}{A-1+\alpha} \right]$$

plugging ⑥ for $E_t \tilde{\lambda}_{t+1}$ and for $\tilde{z}_{t+1}$

$$E_t \tilde{y}_{t+1} = \frac{A\alpha}{A-1+\alpha} \tilde{k}_t + \frac{1-\alpha}{A-1+\alpha} \tilde{\lambda}_t + \frac{1-\alpha}{A-1+\alpha} \tilde{R}_t + \rho \tilde{z}_t \cdot \frac{A}{A-1+\alpha}$$

plug this into 5 for $E_t \tilde{y}_{t+1}$

$$\tilde{R}_t = \frac{\alpha}{R} \frac{y}{k} \left[\frac{A\alpha \tilde{k}_t}{A-1+\alpha} + \frac{1-\alpha}{A-1+\alpha} \tilde{\lambda}_t + \frac{1-\alpha}{A-1+\alpha} \tilde{R}_t + \frac{\rho A}{A-1+\alpha} \tilde{z}_t - \tilde{k}_t \right]$$

$$\left(1+\frac{\alpha}{R}\frac{y}{k}\frac{1-\lambda}{A-1+\alpha}\right)\tilde{R}_t = \frac{\alpha y}{Rk}\left(\frac{A\alpha-A+1-\lambda}{A-1+\alpha}\tilde{k}_t + \frac{1-\lambda}{A-1+\alpha}\tilde{\lambda}_t + \frac{\rho A}{A-1+\alpha}\tilde{z}_t\right)$$

$a_1 \quad a_2 \quad a_3 \quad a_4$

---

c)

follows from F.O.C. w.r.t. m_t

$$\lambda_t = b c_t^{1-\Phi} m^{b(1-\Phi)-1} + E_t\beta\left[\frac{\lambda_{t+1}}{1+\Pi_{t+1}}\right] \qquad \tilde{q}_{t+1} = \frac{\Pi}{1+\Pi}\tilde{\Pi}_{t+1}$$

q_{t+1}

$$\lambda(1+\tilde{\lambda}_t) = b c^{1-\Phi} m^{b(1-\Phi)-1}\cdot\left(1+\tilde{c}_t(1-\Phi)\right)\cdot\left(1+\left(b(1-\Phi)-1\right)\tilde{m}_t\right) + E_t\beta\left[\frac{\lambda}{1+\Pi}\cdot(1+\tilde{\lambda}_{t+1})(1-\tilde{q}_{t+1})\right]$$

$$\lambda\tilde{\lambda}_t = b c^{1-\Phi} m^{b(1-\Phi)-1}\cdot\left(\tilde{c}_t(1-\Phi) + [b(1-\Phi)-1]\tilde{m}_t\right) + E_t\beta\left[\frac{\lambda}{1+\Pi}\left(\tilde{\lambda}_{t+1} - \frac{\Pi}{1+\Pi}\tilde{\Pi}_{t+1}\right)\right]$$

$$\tilde{\lambda}_t = \frac{1}{m}\left((1-\Phi)\tilde{c}_t + [b(1-\Phi)-1]m_t\right) + \beta E_t\left[\frac{1}{1+\Pi}\left(\tilde{\lambda}_{t+1} - \frac{\Pi}{1+\Pi}\tilde{\Pi}_{t+1}\right)\right]$$