Worked Examples in Dynamic Optimization: Analytic and Numeric Methods

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Abstract

Economists are accustomed to think about economic growth models in continuous time. However, applied models require numerical methods because of the absence of tractable analytical solutions. Since these methods operate by essence in discrete time, models involve discrete formulation. We demonstrate the usefulness of two off-the-shelf algorithms to solve these problems : nonlinear programming and mixed complementarity. We then show the advantage of the latter for approximating infinite-horizon models.

JEL classification: C69; D58; D91

Keywords: Dynamic optimization; Mathematical methods; Infinite-horizon models

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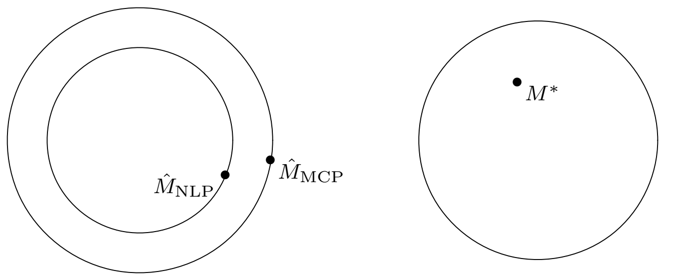
1 Introduction

Dynamic optimization in economics appeared in the 1920s with the work of Hotelling and Ramsey. In the 1960s dynamic mathematical techniques became then more familiar to economists mainly due to the work of neoclassical growth theorists. These techniques involve most of the time formulation of models in continuous time. When closed form solutions do not exist they are then formulated in discrete time. The purpose of this document is to provide some sample solutions of a collection of dynamic optimization problems in two settings, using analytical methods in continuous time and numerical methods in discrete time.

Formulation of infinite-horizon models are not possible with numerical methods. Therefore approximation issues are crucial in finite-horizon models. We consider two classes of off-the-shelf algorithms to solve these dynamic models. The first is nonlinear programming (NLP) developed originally for optimal planning models. The second class is the mixed complementarity problem (MCP) approach. The MCP formulation is represented by the first-order conditions for nonlinear programming. Hence any NLP problem can be solved as an MCP, not necessarily as efficient as using NLP-specific methods. It is then not so surprising that an MCP formulation can provide a better approximation to infinite-horizon problems than does conventional finite NLP.

Figure 1: Approximating infinite-horizon models



Set of finite-horizon models Set of infinite-horizon models

The outline of the paper is as follows. Starting from the classical mathematical technique to solve dynamic economizing problems in continuous time, the next section shows how to derive the NLP and MCP formulation to solve these problems. Section 3 presents in detail analytical solutions to economic planning problems and shows how to formulate them in off-the-shelf softwares. The following section moves on to the neoclassical growth model. The last section explains how to use the optimal neoclassical growth model in applied economics.

2 Mathematical Methods

The dynamic economizing problem may be solved in three different approaches. The first approach going back up to Bernoulli in the very late 1600s is the calculus of variations. The second is the maximum principal developed in the 1950s by Pontryagin and his co-workers. The third approach is dynamic programming developed by Bellman about the same time.

Early applications of dynamic optimization to economics are due to Ramsey and Hotelling in the 1920s. At that time the mathematical technique used to solve dynamic problems was the calculus of variations. Therefore in the following section we first state in a concise way the calculus of variations problem. Then we move on to the maximum principle which can be considered a dynamic generalization of the method of Lagrange multiplier. This method is well-known among economists and is especially suited to the formulation in discrete time. Regarding dynamic programming it is usually applied to stochastic models and then will not be covered here.

2.1 Continuous time approach

The classical calculus of variations problem may be written as

$$\max_{\{x(t)\}} J = \int_{t_0}^{t_1} I(x(t), x'(t), t)dt$$
$$\text{subject to various initial and endpoint conditions}$$

where these conditions are defined as follow:

a. Euler equation: $F_x = dF_{x'}/dt$, $t_0 \leq t \leq t_1$.

b. Legendre condition: $F_{x'x'} \leq 0$, $t_0 \leq t \leq t_1$.

c. Boundary conditions:

- Initial conditions always apply: $x(t_0) = x_0$.
- The terminal time and terminal value may be fixed exogenously or free.

d. Transversality conditions apply when the terminal value and time are free:

- If only the terminal value is free, then $F_{x'} = 0$ at t_1.
- If only the terminal time is free, then $F - x'F_{x'} = 0$ at t_1.
- If both the terminal value and time are free, then $F = 0$ and $F_{x'} = 0$ at t_1.

These necessary conditions of the calculus of variations can be derived from the maximum principle. Intuitively it remains to let the rates of change of the state variables to be the control variables in the maximum principle, which means $u(t) = x'(t)$. Assuming that the terminal time value is fixed, which is always the case in numerical problems, the corresponding maximum principle may be defined as

$$\max_{\{u(t)\}} J = \int_{t_0}^{t_1} I(x(t), u(t), t)dt + F\left(x\left(t_1\right), t_1\right)$$
$$\begin{aligned} \text{subject to} \quad & x'(t) = f(x(t), u(t), t) \\ & t_0, t_1 \text{ and } x\left(t_0\right) = x_0 \text{ fixed} \\ & x\left(t_1\right) = g\left(x\left(t_1\right), t_1\right) \text{ or free} \end{aligned}$$

where $I(\cdot)$ is the intermediate function, $F(\cdot)$ is the final function, $f(\cdot)$ is the state equation function and $g(\cdot)$ is the terminal constraint function.

In a concise way the maximum principle technique involves adding costate variables $\lambda(t)$ to the problem, defining a new function called the Hamiltonian,

$$H(x(t), u(t), \lambda(t), t) = I(x, u, t) + \lambda(t)f(x, u, t)$$

and solving for trajectories $\{u(t)\}$, $\{\lambda(t)\}$, and $\{x(t)\}$ satisfying the following conditions

$$\begin{array}{ll}
\text{optimality condition} & \dfrac{\partial H}{\partial u} = I_u + \lambda f_u = 0 \\
\text{costate equation} & \lambda' = -\dfrac{\partial H}{\partial x} = -\left(I_x + \lambda f_x\right) \\
\text{state equation} & x' = \dfrac{\partial H}{\partial \lambda} = f \quad \text{with} \quad x\left(t_0\right) = x_0 \\
\text{terminal conditions} & \cdot \quad x\left(t_1\right) \geq 0 \quad \perp \quad \lambda\left(t_1\right) \geq \dfrac{\partial F}{\partial x} \\
 & \cdot \quad \lambda\left(t_1\right) = \dfrac{\partial F}{\partial x} + \tilde{\lambda}\dfrac{\partial g}{\partial x} \\
 & \quad\;\; \text{with } \tilde{\lambda} \geq 0 \perp x\left(t_1\right) = g\left(x\left(t_1\right), t_1\right)
\end{array}$$

which are necessary for a local maximum.

2.2 Discrete time formulation

The formulation of the discrete time version of the maximum principle is straightforward. Forming the Hamiltonian,

$$H(x_t, u_t, \lambda_{t+1}, t) = I(x, u, t) + \lambda_{t+1} f(x, u, t)$$

the necessary conditions are as follow:

$$\begin{array}{ll}
\text{optimality condition} & \dfrac{\partial H}{\partial u_t} = I_u + \lambda_{t+1} f_u = 0 \\
\text{costate equation} & \lambda_{t+1} - \lambda_t = -\dfrac{\partial H}{\partial x_t} = -\left(I_x + \lambda_{t+1} f_x\right) \\
\text{state equation} & x_{t+1} - x_t = \dfrac{\partial H}{\partial \lambda_{t+1}} = f \quad \text{with} \quad x_{t_0} = x_0 \\
\text{terminal conditions} & \cdot \quad x_{t_1+1} \geq 0 \quad \perp \quad \lambda_{t_1+1} \geq \dfrac{\partial F}{\partial x} \\
 & \cdot \quad \lambda_{t_1+1} = \dfrac{\partial F}{\partial x_{t_1+1}} + \tilde{\lambda}\dfrac{\partial g}{\partial x_{t_1+1}} \\
 & \quad\;\; \text{with } \tilde{\lambda} \geq 0 \perp x_{t_1+1} = g\left(x_{t_1+1}, t_1+1\right)
\end{array}$$

As mentioned earlier, the maximum principle can be considered the extension of the method of Lagrange multipliers to dynamic optimization problems. This method allows us to state problems in the same way they would be written in off-the-shelf softwares. Write L for the Lagrangian of the full intertemporal problem. The NLP formulation is then

$$\begin{aligned}
\max_{\{u(t)\},\{\lambda(t)\},\{x(t)\}} L = \sum_{t=t_0}^{t_1} & I\left(x_t, u_t, t\right) + \lambda_{t+1}\left[x_t + f\left(x_t, u_t, t\right) - x_{t+1}\right] \\
& + \lambda_{t_0}\left[x_{t_0} - x_0\right] + \tilde{\lambda}_{t_1+1}\left[g\left(x_{t_1+1}, t_1+1\right) - x_{t_1+1}\right] + F\left(x_{t_1+1}, t_1+1\right)
\end{aligned}$$

and the MCP formulation follows from the first-order conditions

$$\begin{aligned}
\frac{\partial L}{\partial u_t} &= I_u\left(x_t, u_t, t\right) + \lambda_{t+1} f_u\left(x_t, u_t, t\right) = 0 && t = t_0, \ldots, t_1 \\
\frac{\partial L}{\partial x_t} &= I_x\left(x_t, u_t, t\right) + \lambda_{t+1} f_x\left(x_t, u_t, t\right) + \lambda_{t+1} - \lambda_t = 0 && t = t_0, \ldots, t_1 \\
\frac{\partial L}{\partial x_{t_1+1}} &= -\lambda_{t_1+1} + \tilde{\lambda}_{t_1+1}\left(g_x - 1\right) + F_x = 0 \\
\frac{\partial L}{\partial \lambda_{t+1}} &= f\left(x_t, u_t, t\right) - \left(x_{t+1} - x_t\right) = 0 && t = t_0, \ldots, t_1 \\
\frac{\partial L}{\partial \lambda_{t_0}} &= x_{t_0} - x_0 = 0 \\
\frac{\partial L}{\partial \tilde{\lambda}_{t_1+1}} &= g\left(x_{t_1+1}, t_1 + 1\right) - x_{t_1+1} = 0
\end{aligned}$$

which are the necessary conditions of the maximum principle.

The method of Lagrange multipliers shows clearly that when the system has a fixed final state, as here, there are two constraints for the terminal period t_1+1: the state variable x_{t_1+1} must satisfy the terminal constraint and still satisfies the state equation. This explains the two Lagrange multipliers associated with x_{t_1+1}: λ_{t_1+1} for the state equation, and $\tilde{\lambda}_{t_1+1}$ for the terminal constraint. When the system has a free final state, which means that the terminal constraint is not specified, the Lagrange multiplier λ_{t_1+1} is equal to zero if the value of the final function is zero.

3 Economic Planning Models

3.1 Optimal consumption plan

[1]Find the consumption plan $C(t)$, $0 \leq t \leq T$, over a fixed period to maximize the discounted utility stream

$$\int_0^T e^{-rt} C^a(t) dt \quad \text{subject to} \quad C(t) = iK(t) - K'(t), \quad K(0) = K_0, \quad K(T) = 0$$

where $0 < a < 1$ and K represents the capital stock.

Analytic solution

We have a variational problem based on the function:

$$F(t, k, k') = e^{-rt} U(ik - k')$$

Taking derivatives, we have:

$$F_k = ie^{-rt} U'$$

and

$$F_{k'} = -e^{-rt} U'$$

The Euler equation is then:

$$\frac{d}{dt}\left[-e^{-rt} U'\right] = ie^{-rt} U'$$

[1]Problem 4.5 in Kamien and Schwartz (2000).

Although the underlying problem is defined in terms of the capital stock, it is convenient at this point to use consumption as the decision variable, when we form the time derivative we have:

$$re^{-rt}U' - e^{-rt}U''c' = ie^{-rt}U'$$

which reduces to:

$$-\frac{U''c'}{U'} = i - r$$

In the case of constant elasticity utility,

$$U(c) = c^a,$$

the Euler equation becomes:

$$(1-a)\frac{c'}{c} = i - r$$

and, integrating we determine the growth rate of consumption:

$$c = c_0 e^{\frac{i-r}{1-a}t}$$

In this expression, initial consumption level is a constant of integration. In order to determine the consumption level, we need to focus on the initial and terminal conditions for the capital stock. To determine the time path of the capital stock, we consider the equation which relates consumption to capital earnings and investment:

$$ik - k' = c_0 e^{\frac{i-r}{1-a}t}$$

In order to solve an equation of this form, it is necessary to use a standard method for solving this sort of an equation, multiplying by an integrating factor:

$$e^{-it}\left[k' - ik\right] = -c_0 e^{\theta t}$$

in which we define:

$$\theta = \frac{i-r}{1-a} - i = \frac{ai - r}{1-a}$$

This equation can be written:

$$d\left[e^{-it}k(t)\right] = -c_0 e^{\theta t} dt$$

which integrates to:

$$e^{-it}k(t) = \gamma - c_0 \frac{e^{\theta t}}{\theta}$$

so:

$$k(t) = \gamma e^{it} - \frac{c_0}{\theta} e^{\frac{i-r}{1-a}t}$$

We then have two boundary conditions to determine the constants of integration:

$$k(0) = k_0, \quad k(T) = 0$$

The initial condition produces:

$$\gamma = k_0 + \frac{c_0}{\theta}$$

Substituting into the terminal condition, we have:

$$c_0 = \frac{\theta k_0}{e^{\theta T} - 1}$$

Finally, substitute the integrating constants back into the expression for the capital stock to obtain:

$$k(t) = k_0 e^{it}\left[\frac{e^{\theta T} - e^{\theta t}}{e^{\theta T} - 1}\right]$$

Numeric solution

Working in discrete time, the following code sets up the model as a nonlinear optimization problem. The first solution is used to compare results from the analytic and numeric models. The second set of solutions evaluate the qualitative properties of the consumption path for alternative elasticities parameters, a. The final calculation in this program presents an alternative representation of the choice problem as budget-constrained welfare maximization.

```
$title Kamien and Schwartz, problem 4.5 - NLP formulation

sets                t         time periods                 / 0*60 /
                    decade(t) decades                      / 10, 20, 30, 40, 50 /
                    tfirst(t) first period of time
                    tlast(t)  last period of time;

tfirst(t) = yes$(ord(t) eq 1);
tlast(t) = yes$(ord(t) eq card(t));

scalars             r         discount rate                / 0.03 /
                    i         interest rate                / 0.04 /
                    a         utility coefficient          / 0.5 /;

variables           c(t)      consumption level
                    k(t)      capital stock
                    kt        terminal capital stock
                    u         utility function;

equations           market(t) market clearance in period t
                    market_t  terminal market clearance
                    const_kt  terminal capital constraint
                    utility   objective function definition;

market(t)..         k(t) - k(t-1) =e= 1$tfirst(t) + i*k(t-1) - c(t-1);

market_t..          (kt - sum(tlast, k(tlast))) =e= sum(tlast, i*k(tlast) - c(tlast));

const_kt..          kt =e= 0;

utility..           u =e= sum(t, (1/(1+r))**((ord(t)-1)) * c(t)**a);

model ramsey        / all /;

*         Lower bound to avoid domain errors

c.lo(t) = 0.001;

*         Numeric solution

parameters          compare   comparison of analytic and numeric solution;

solve ramsey using nlp maximizing u;

compare(t,'numeric c') = c.l(t);
compare(t,'numeric k') = k.l(t);

*         Analytic solution

scalars             theta, c0;

theta = (a * i - r) / (1 - a);
c0 = theta / (exp(theta * (card(t)-1)) - 1);

compare(t,'analytic c') = c0 * exp((theta+i)*(ord(t)-1));
compare(t,'analytic k') = exp(i*(ord(t)-1)) * (exp(theta*(card(t)-1))
                          -exp(theta*(ord(t)-1))) / (exp(theta*(card(t)-1))-1);

```

```
*         Alternative elasticities parameters

sets                elasval   alternative elasticity values / '0.3','0.6','0.9'/;

parameters          consum    consumption path for alternative elasticities;

a = 0;
loop(elasval,
          a = 0.3 + a;
          solve ramsey using nlp maximizing u;
          consum(t,elasval) = c.l(t);
);

*         Alternative representation of the choice problem

parameter           p(t)      present value of consumption in period t;

p(t) = (1/(1 + i))**(ord(t)-1);

equations           budget    present-value budget constraint;

budget..            sum(t, p(t) * c(t)) =e= 1 + i;

model altmodel      /utility, budget/;

a = 0.5;
solve altmodel using nlp maximizing u;

compare(t,"altmodel c") = c.l(t);

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$setglobal domain t
$setglobal labels decade

$if %batch%==yes $setglobal gp_opt3 "set output 'ks45a.eps'"
$libinclude plot compare

$if %batch%==yes $setglobal gp_opt3 "set output 'ks45b.eps'"
$setglobal gp_opt4 "set key left"
$libinclude plot consum
```

Figure 2: Analytic and Numeric Solutions

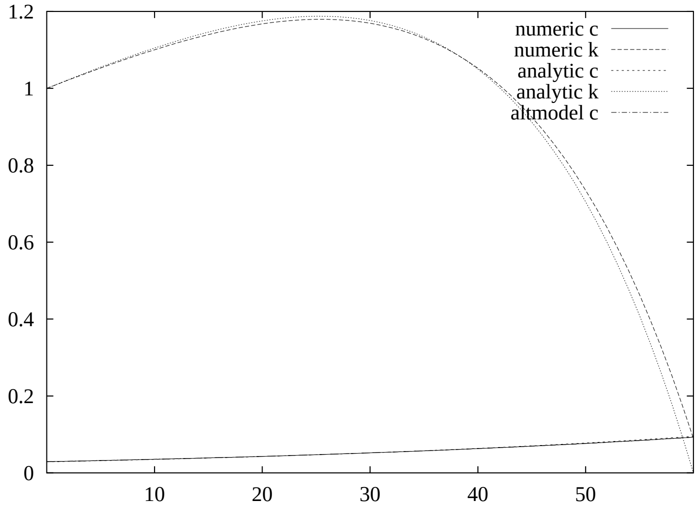


Figure 3: Consumption Paths

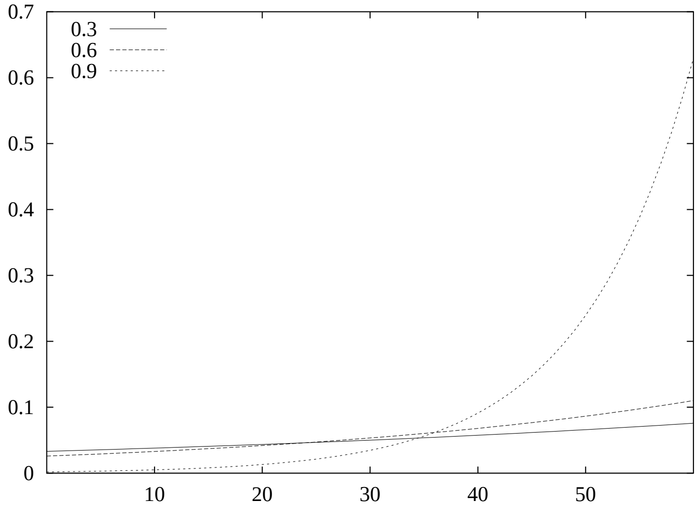
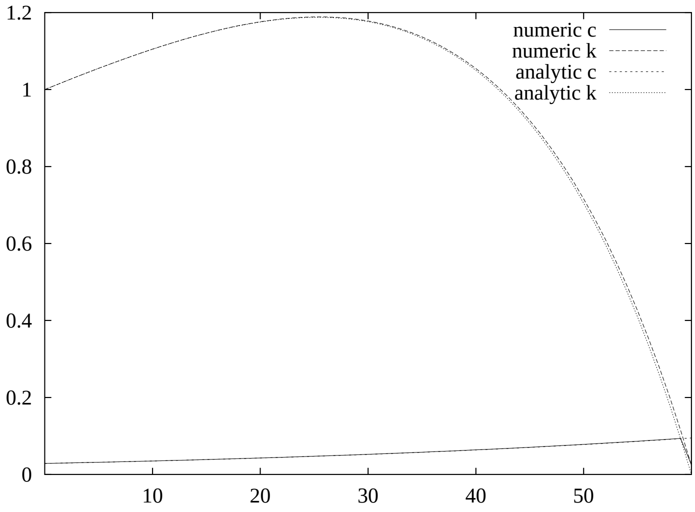


Figure 1 compare the state variable time path between analytic and numeric models. They don't seem to be identical especially the value of capital at the final period. The reason comes from the treatment of time being a discrete succession of periods. It results that, the final period in continuous-time model corresponds to the end of the final period in discrete-time model, which is the beginning of period $t_1 + 1$. Since the continuous time is the limit of discrete periods shrinking to zero, differences between the two approaches are reduced when smaller periods of time are considered. An illustration is given below.

```
$title Kamien and Schwartz, problem 4.5 - Smaller time periods

sets                t         time periods                 / 0*120 /
                    m(t)      main time periods            / 0 /
                    tfirst(t) first period of time
                    tlast(t)  last period of time;

tfirst(t) = yes$(ord(t) eq 1);
tlast(t) = yes$(ord(t) eq card(t));

scalars             r         discount rate                / 0.03 /
                    i         interest rate                / 0.04 /
                    a         utility coefficient          / 0.5 /
                    dt        increment of time subperiod  / 2 /;

loop(t$m(t), m(t+dt)=yes; );

variables           c(t)      consumption level
                    k(t)      capital stock
                    kt        terminal capital stock
                    u         utility function;

equations           market(t) market clearance in period t
                    market_t  terminal market clearance
                    const_kt  terminal capital constraint
                    utility   objective function definition;

market(t)..         dt * (k(t) - k(t-1)) =e= (1*dt)$tfirst(t) + i*k(t-1) - c(t-1);

market_t..          (kt - sum(tlast, k(tlast))) =e= sum(tlast, i*k(tlast) - c(tlast));

const_kt..          kt =e= 0;

utility..           u =e= sum(t, (1/(1+r))**((ord(t)-1)/dt) * c(t)**a / dt);

model ramsey        / all /;

*         Lower bound to avoid domain errors

c.lo(t) = 0.001;

*         Do a comparison of numeric and analytic solutions

parameters          compare   comparison of analytic and numeric solution;

solve ramsey using nlp maximizing u;

compare(m,'numeric c') = c.l(m);
compare(m,'numeric k') = k.l(m);

scalars             theta, c0;

theta = (a * i - r) / (1 - a);
c0 = theta / (exp(theta * (card(t)-1)/dt) - 1);

compare(m(t),'analytic c') = c0 * exp((theta+i)*(ord(t)-1)/dt);
compare(m(t),'analytic k') = exp(i*(ord(t)-1)/dt) * (exp(theta*(card(t)-1)/dt)
```

```
                              -exp(theta*(ord(t)-1)/dt)) / (exp(theta*(card(t)-1)/dt)-1);

option decimals = 8;
display compare;

sets                decade    / 20 '10', 40 '20', 60 '30', 80 '40', 100 '50' /;

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$setglobal labels decade

$if %batch%==yes $setglobal gp_opt3 "set output 'ks45dt.eps'"
$libinclude plot compare m
```

Figure 4: Solution for 60 years with 120 time periods



MCP formulation

It may be useful to represent prices explicitly in the model. Below are therefore the MPSGE ((Rutherford, 1999)) and algebraic models using the MCP formulation.

```
$title Kamien and Schwartz, problem 4.5 - MPSGE and MCP formulation

sets                t         time periods                / 0*60 /
                    tfirst(t) first period of time
                    tlast(t)  last period of time;

tfirst(t) = yes$(ord(t) eq 1);
tlast(t) = yes$(ord(t) eq card(t));

scalars             r         discount rate               / 0.03 /
                    i         interest rate               / 0.04 /;

variables           c(t)      consumption level
                    u         utility function;

positive variables  k(t)      capital stock
                    kt        terminal capital stock;

equations           market(t) market clearance in period t
```

```
                    market_t  terminal market clearance
                    const_kt  terminal capital constraint
                    utility   objective function definition;

market(t)..         k(t) - k(t-1) =e= 1$tfirst(t) + i*k(t-1) - c(t-1);

market_t..          (kt - sum(tlast, k(tlast))) =e= sum(tlast, i*k(tlast) - c(tlast));

const_kt..          kt =e= 0;

utility..           u =e= sum(t, (1/(1+r))**((ord(t)-1)) * log(c(t)));

model ks_nlp        / all /;

*         Lower bound to avoid domain errors

c.lo(t) = 0.001;

*         NLP solution

solve ks_nlp using nlp maximizing u;

*         MPSGE formulation

alias (t,t1);

parameters          theta     budget share over time
                    epsilon   budget share over time (lagged)
                    k_nlp     capital value from NLP solution;

theta(t) = (1/(1+r))**(ord(t)-1)/sum(t1,(1/(1+r))**(ord(t1)-1));
epsilon(t+1) = theta(t);
k_nlp(t) = k.l(t);

$ontext

$model:ks_mge

$sectors:
          k(t)      ! capital stock

$commodities:
          pk(t)     ! price of capital stock
          pkt       ! price of terminal capital stock

$consumers:
          ra        ! representative agent

$auxiliary:
          kt        ! terminal capital stock
          pktc      ! price of constraint terminal capital stock

$prod:k(t)
          o:pk(t+1)           q:(1+i)
          o:pkt$tlast(t)      q:(1+i)
          i:pk(t)             q:1

$demand:ra          s:1.0
          e:pk(t)$tfirst(t)             q:1
          d:pk(t)$(not tfirst(t))       q:epsilon(t)
          d:pkt                         q:(sum(tlast,theta(tlast)))

$constraint:kt
          pkt =e= pktc;

$constraint:pktc
          kt =e= 0;
```

```
$report:
          v:cons(t)           d:pk(t)             demand:ra
          v:cons_t            d:pkt               demand:ra

$offtext
$sysinclude mpsgeset ks_mge

*         NLP values to initialize the model

k.l(t) = k_nlp(t);
pk.l(t) = market.m(t);
pkt.l = market_t.m;
pktc.l = -const_kt.m;

*         MPSGE solution

ks_mge.iterlim = 0;
$include ks_mge.gen
solve ks_mge using mcp;

*         MCP formulation

equations           pr_k(t)   zero profit condition for capital stock
                    pr_kt     zero profit condition for terminal capital stock
                    demand(t) demand function;

pr_k(t)..           (1+i) * (pk(t+1) + pkt$tlast(t)) =e= pk(t);

pr_kt..             pkt =e= pktc;

demand(t)..         c(t) * (pk(t+1) + pkt$tlast(t)) =e=
                    theta(t) * sum(tfirst, pk(tfirst)*k(tfirst));

MODEL ks_mcp        / pr_k.k, pr_kt.kt, market.pk, market_t.pkt, const_kt.pktc, demand.c /;

*         MCP solution

ks_mcp.iterlim = 0;
solve ks_mcp using mcp;
```

3.2 The monopolist

[2]The demand function for a monopolist depends on both the product price and the rate of change of the product price, according to:

$$x = a_0 p + b_0 + c_0 p'$$

Assume that the cost of production at rate x is:

$$C(x) = a_1 x2 + b_1 x + c_1$$

Given the initial price, $p(0) = p_0$, and the required ending price, $p(T) = p_T$, find the price policy over $0 \leq t \leq T$ which maximizes profits:

$$\int_0^T [px - C(x)]dt$$

Analytic solution

Notice that because the time period is fixed, the fixed term in the cost function, c_1, is irrelevant if the firm is committed to produce; so we will ignore that term to

[2]Problem 5.4 in Kamien and Schwartz (2000).

conserve on algebra.
Substituting with the demand function, we see that this problem corresponds to a calculus of variations problem in which the function depends only on the price and the gradient of price, but not on the time path, i.e.

$$F(p, p') = px(p, p') - C(p, p')$$

Neglecting constants, this reduces to:

$$F(p, p') = a_0(1 - a_0a_1)p2 - a_1c_02p'^2 + c0(1 - 2a_0a_1)pp' + (b_0 - 2a_0a_1b_0 - a_0b_1)p$$

Some differentiation:

$$F_p = 2a_0(1 - a_0a_1)p + c_0(1 - 2a_0a_1)p' + b0 - 2a_0a_1b_0 - a_0b_1$$

and

$$\frac{dF_{p'}}{dt} = c_0(1 - 2a_0a_1)p' - 2a_1c_02p''$$

The Euler equation is then a second-order, linear differential equation with constant coefficients:

$$p'' + Bp = R$$

where

$$B = \frac{a_0(1 - a_0a_1)}{a_1c_02}$$

and

$$R = \frac{a_0b_1 + 2a_0a_1b_0 - b_0}{2a_1c_02}$$

Notice that in steady-state, where $p'' = 0$, the Euler condition implies that:

$$p^* = \frac{R}{B} = \frac{a_0b_1 + 2a_0a_1b_0 - b_0}{2a_0(1 - a_0a_1)}$$

which is equivalent the optimal monopoly price in the static equilibrium.
If we are to assume that the static equilibrium model is based on a downward sloping demand function and a convex technology, then:

$$a_0 < 0, \quad b_0 > 0, \quad a_1 > 0, \text{ and } b_1 > 0$$

Hence, we have may conclude:

$$p^* > 0, \quad B < 0$$

In order to solve the differential equation, we begin with the adjacent homogeneous system:

$$p'' + Bp = R$$

We know that the solution of this equation has the form:

$$p(t) = ce^{rt}$$

Hence:

$$ce^{rt}(r2 + B) = 0$$

Defining:

$$\hat{r} = \sqrt{\frac{a_0(a_0a_1 - 1)}{a_1c_02}}$$

The solution to the non-homogeneous equation therefore has the form:

$$p(t) = c_1 e^{\hat{r}t} + c_2 e^{-\hat{r}t} + c_3$$

and follows from the definition of $\hat{r}$ that

$$c_3 = \frac{R}{B} = p^*$$

And boundary conditions determine c_1 and c_2 as solutions to the following system of equations:

$$c_1 + c_2 = p_0 - p^*, \quad c_1 e^{\hat{r}T} + c_2 e^{-\hat{r}T} = p_T - p^*$$

When the initial and final prices are both equal to the static monopoly price, $c_1 = c_2 = 0$ and the optimal policy is to keep the price fixed over the time horizon.

If the terminal price equals the optimal static value, then over the horizon the price moves monotonically from the initial value to the terminal value (when the terminal price equals the p^*, then c_1 and c_2 are of opposite sign).

Numeric solution

Working in discrete time, we can formulation this model as a nonlinear optimization problem and solve it using GAMS/MINOS, as illustrated in the following code:

```
$title Kamien and Schwartz, problem 5.4 - NLP formulation

set     t               /1*100/,
        decade(t)       /10,20,30,40,50,60,70,80,90/;

scalar  sigma           elasticity of demand /4/
        eta             elasticity of supply /0.25/
        c0              multiplier           /-20/
        a0,a1,b1, r,coef1,coef2;

parameter
        compare       comparison of numerical approximation with analytic solution,
        pricepath     approach path for prices from various starting points,
        turnpike      illustrating turnpike property of the optimal price path;

*       Impute a1 and b1 so that we have a steady-state with
*       the price and quantity both equal to unity:

b1 = (1 - 1/sigma) * (1 - 1/eta);
a1 = (1/2) * (1 - 1/sigma - b1);
a0 = -sigma;

*       Declare the model:

variables       p(t)    price
                x(t)    quantity
                c(t)    cost
                profit  maximand;

equations       demand, cost, objdef;

*       By declaring equations over t+1, we omit equations for the
*       first period in which the price is fixed exogenously:

demand(t)..     x(t) =e= (1+sigma) - sigma * p(t) + c0 * (p(t)-p(t-1));

cost(t)..       c(t) =e= a1 * x(t)*x(t) + b1*x(t);

objdef..        profit =e= sum(t, x(t) * p(t) - c(t));

*       Create a model with all of these equations:
```

```

model dynamic /all/;

*       Fix terminal period price at the equilibrium price:

p.fx("100") = 1;

*       Fix initial period values:

p.fx("1") = 0.5;
solve dynamic using nlp maximizing profit;

compare(t,"numeric") = p.l(t);

r = sqrt( a0 * (a0 * a1 - 1) / (a1*c0*c0) );
coef2 = (p.l("1") - 1) / (1 - exp(-2 * r * 99));
coef1 = - coef2 * exp(-2 * r * 99);
compare(t,"analytic") = 1 + coef1 * exp(r * (ord(t)-1) ) + coef2 * exp(-r * (ord(t)-1));

*       Create a set defined by either the initial or terminal period price:

set     p0              /"0.1","0.3","0.5","0.7","0.9"/;

loop(p0,
  p.fx("1") = 0.1 + 0.2 * (ord(p0)-1);
  solve dynamic using nlp maximizing profit;
  pricepath(t,p0) = p.l(t);
);

*       Now illustrate the turnpike property:

p.fx("1") = 0.6;
loop(p0,
  p.fx("100") = 0.1 + 0.2 * (ord(p0)-1);
  solve dynamic using nlp maximizing profit;
  turnpike(t,p0) = p.l(t);
);

*       Display the results using GNUPLOT:

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$setglobal domain t
$setglobal labels decade

$if %batch%==yes $setglobal gp_opt3 "set output 'ks54a.eps'"
$libinclude plot compare

$if %batch%==yes $setglobal gp_opt3 "set output 'ks54b.eps'"
$libinclude plot pricepath

$if %batch%==yes $setglobal gp_opt3 "set output 'ks54c.eps'"
$setglobal gp_opt4 "set key bottom left"
$libinclude plot turnpike
```

Figure 5: Analytic and Numeric Solutions

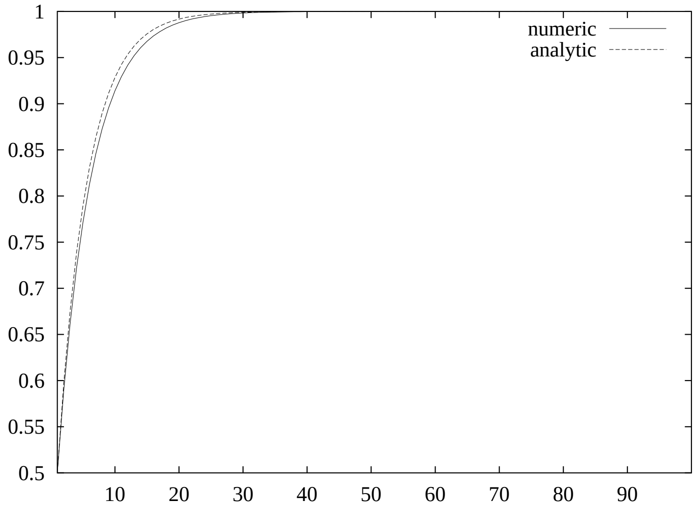


Figure 6: Approach Paths for Various Starting Points

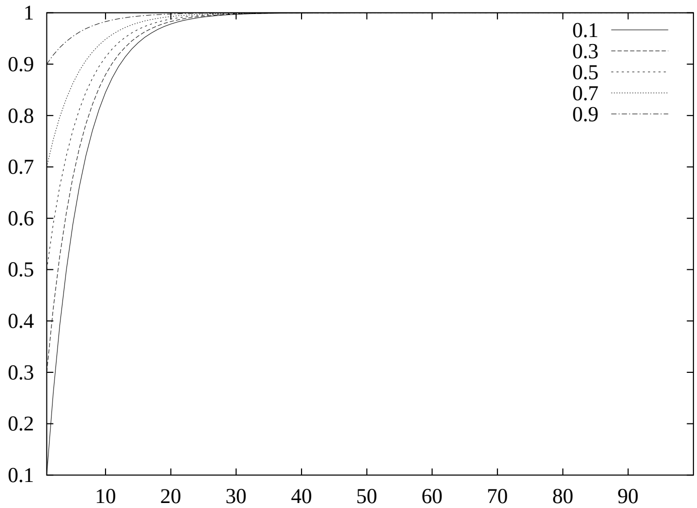
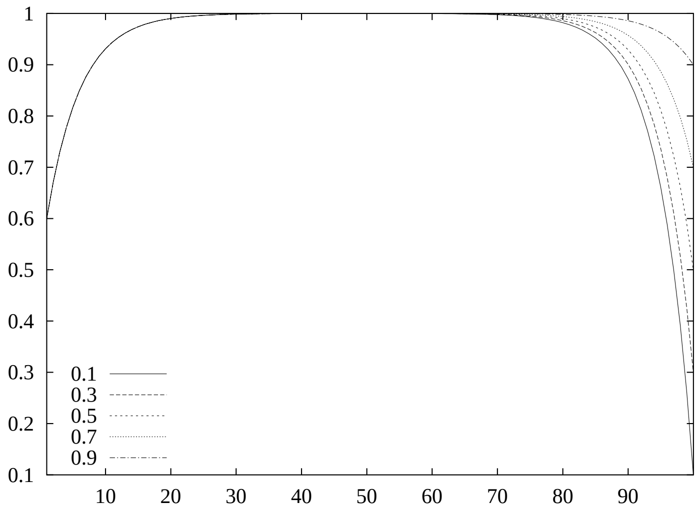


Figure 7: Turnpike Property for Optimal Price Path



3.3 Non-renewable resource

[3]Suppose a mine contains an amount B of a mineral resource (like coal, copper or oil). The profit rate that can be earned from selling the resource at rate x is $\ln x$. Find the rate at which the resource should be sold over the fixed period [0,T] to maximize the present value of profits from the mine. Assume the discount rate a constant r. Assume the resource has no value beyond time T.

Analytic solution

Following the hint, define $y(t)$ as the cumulative sales by time t. Then $y'(t)$ is the sales rate at time t Find $y(t)$ to:

$$\max \int_0^T e^{-rt} \ln y'(t) dt$$

subject to:

$$y(0) = 0, \quad y(T) = B$$

We therefore have:

$$F(t, y, y') = e^{-rt} \ln y'(t), \quad F_y = 0 \quad \text{and} \quad \frac{dF_{y'}}{dt} = \frac{-e^{-rt}}{y'(t)} \left(\frac{y''}{y'} + r \right)$$

The Euler equation then gives us the differential equation:

$$\frac{y''}{y'} = -r$$

Integrating, we have:

$$y(t) = c_1 e^{-rt} + c_2$$

Then applying the boundary conditions, we have:

$$y(t) = B \frac{e^{-rt} - 1}{e^{-rT} - 1}$$

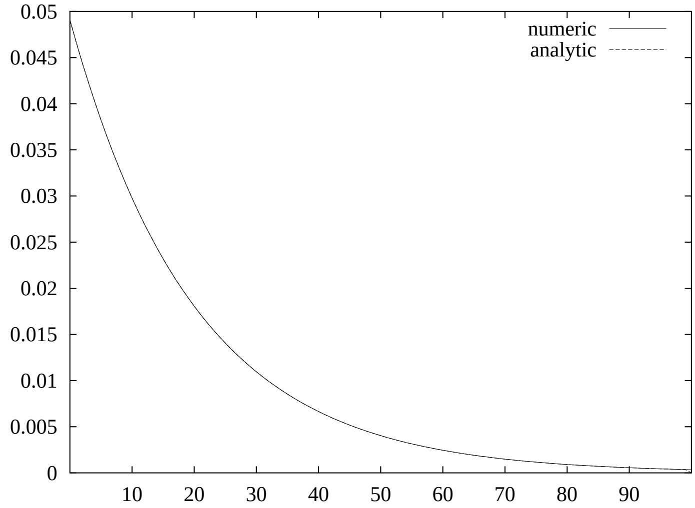
[3]Problem 5.5 in Kamien and Schwartz (2000).

Numeric solution

Working in discrete time, we can formulation this model as a nonlinear optimization problem and solve it using GAMS/MINOS, as illustrated in the following code:

```
$title Kamien and Schwartz, problem 5.5 - NLP formulation

set     t               /0*100/,
        decade(t)       /10,20,30,40,50,60,70,80,90/;

scalar    r         interest rate /0.05/;

variables profit    present value of extraction
                    x(t)      production at time t;

equations objdef    defines profit
                    supply    defines cumulative extraction;

objdef..  profit =e= sum(t, exp(-r * (ord(t)-1)) * log(x(t)));

supply..  sum(t, x(t)) =e= 1;

model hotelling /all/;

x.lo(t) = 0.00001;
x.l(t) = 1/card(t);

solve hotelling using nlp maximizing profit;

parameter compare   comparison of numeric and analytic solutions
                    y(t)      cumulative extraction in the analytic solution;

y(t) = (exp(-r * (ord(t)-1)) - 1) / (exp(-r * 100) - 1);

compare(t,"numeric") = x.l(t);
compare(t,"analytic") = y(t+1) - y(t);
compare("100","analytic") = 0;
display compare;

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$setglobal domain t
$setglobal labels decade

$if %batch%==yes $setglobal gp_opt3 "set output 'ks55.eps'"
$libinclude plot compare
```

Figure 8: Analytic and Numeric Solutions



3.4 Non-renewable resource (general case)

[4]Reconsider the previous problem but suppose that the profit rate is $P(x)$ when the resource is sold at rate x, where $P'(0) > 0$ and $P'' < 0$.

1. Show that the present value of the marginal profit from extraction is constant over the planning period (otherwise it would be worthwhile to shift the time of sale of a unit of the resource from a less profitable moment to a more profitable one). Marginal profit, $P'(t)$ therefore grows exponentially at the discount rate r.

2. Show that the optimal extraction rate declines through time.

Analytic solution

We have a calculus of variations problem in which:

$$F(t, y, y') = e^{-rt}P(y'(t)), \quad F_y = 0 \quad \text{and} \quad F_{y'} = e^{-rt}P'(y').$$

The Euler condition therefore implies:

$$\frac{dF_{y'}}{dt} = \frac{de^{-rt}P'(y')}{dt} = 0$$

or, in answer to question 1:

$$e^{-rt}P'(y') = \text{constant}$$

Then if $P'' < 0$, then only way that $P'(y')$ increases at an exponential rate r over time is that the extraction rate, y', must be declining through time.

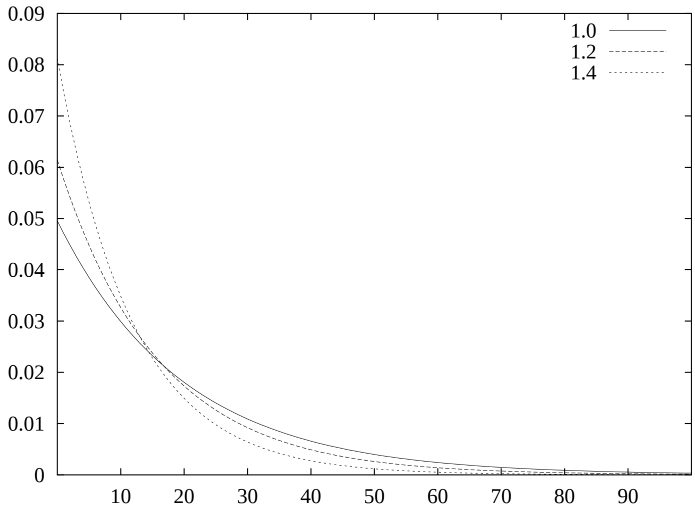
[4]Problem 5.6 in Kamien and Schwartz (2000).

Numeric solution

As the demand curve becomes more elasticity, the production profile must decline at a faster rate so that the present value of the marginal from extraction remains constant over the planning period.

```
$title Kamien and Schwartz, problem 5.6 - NLP formulation

set     t                     /0*100/,
        decade(t)             /10,20,30,40,50,60,70,80,90/;

scalar              r         interest rate /0.05/,
                    sigma     elasticity of demand /0.5/;

variables           profit    present   value of extraction
                    x(t)      production at time t;

equations           objdef    defines   profit
                    supply    defines cumulative extraction;


objdef..  profit =e= sum(t, exp(-r * (ord(t)-1)) * x(t)**(sigma-1)/sigma );

supply..  sum(t, x(t)) =e= 1;

model hotelling /all/;

x.lo(t) = 0.00001;
x.l(t) = 1/card(t);

set       sigval /"1.0","1.2","1.4"/

parameter           extract   Extraction profile over time;

loop(sigval,
          sigma = 0.81 + 0.2 * ord(sigval);
          solve hotelling using nlp maximizing profit;
          extract(t,sigval) = x.l(t);
);

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$setglobal domain t
$setglobal labels decade

$if %batch%==yes $setglobal gp_opt3 "set output 'ks56.eps'"
$libinclude plot extract
```

Figure 9: Extraction Values for Alternative Values of σ



3.5 Pollution control

Utility $U(C, X)$ increases with the consumption rate C and decreases with the stock of pollution, X. For $C > 0$, $P > 0$,

$$U_C > 0, \qquad U_{CC} < 0, \qquad \lim_{C \to 0} U_C = \infty;$$
$$U_X < 0, \qquad U_{XX} < 0, \qquad \lim_{X \to 0} U_X = 0; \qquad U_{CX} = 0.$$

The constant rate of output Y is to be divided between consumption and pollution control. Consumption contributes to pollution, while pollution control reduces it; $Z(C)$ is the net contribution to the pollution flow, with $Z' > 0, Z'' > 0$. For small C, little pollution is created and much abated; thus net pollution declines: $Z(C) < 0$. But for large C, considerable pollution is created and few resources remain for pollution control, therefore on net pollution increases: $Z(C) > 0$. Let C^* be the consumption rate that satisfies $Z(C^*) = 0$. In addition, the environment absorbs pollution at a constant proportionate rate b. Characterize the consumption path $C(t)$ that maximizes the discounted utility stream:

$$\int_0^\infty e^{-rt} U(C, X) dt$$

subject to

$$X' = Z(C) - bX, \qquad X(0) = X_0, \qquad 0 \le C \le Y, \qquad 0 \le X$$

Also characterize the corresponding optimal pollution path and the steady state.

This kind of problems are typically the ones which are much more convenient to solve with numerical methods rather than analytically.

```
$title Kamien and Schwartz, problem II.8.5 - NLP formulation

sets                t         time periods                  / 0*10 /
                    tfirst(t) first period of time
                    tlast(t)  last period of time;

tfirst(t) = yes$(ord(t) eq 1);
```

```
tlast(t) = yes$(ord(t) eq card(t));

scalars             r        discount rate                 / 0.03 /
                    b        rate of pollution decay       / 0.05 /
                    psi      disutility rate of pollution  / 0.15 /
                    alpha    pollution control parameter   / 5 /
                    beta     function curvature parameter  / 16 /
                    y        constant rate of output       / 8 /
                    x0       initial stock of pollution    / 30 /;

variables           u        utility function;

positive variables  c(t)     consumption level
                    x(t)     pollution stock
                    xt       terminal capital stock;

equations           steq_x(t) state equation of pollution
                    steq_xt   state equation of terminal pollution
                    appr_xt   approximation of terminal pollution
                    utility   objective function definition;

steq_x(t)..         x(t) - x(t-1) =e= x0$tfirst(t)
                    + (-alpha + beta / (y - c(t-1)) - b * x(t-1))$(not tfirst(t));

steq_xt..           (xt - sum(tlast, x(tlast))) =e=
                    - alpha + sum(tlast, beta / (y - c(tlast)) - b * x(tlast));

appr_xt..           xt =e= 150;

utility..           u =e= sum(t, (1/(1+r))**((ord(t)-1)) * (log(c(t)) - psi * log(x(t))));

model pollution     / all /;

*         Lower bound to avoid domain errors

c.lo(t) = 0.001;
c.up(t) = y-0.001;
x.lo(t) = 0.001;

*         NLP solution

solve pollution using nlp maximizing u;
```

4 The Neoclassical Growth Model

4.1 Factor shares

[5] For a neoclassical function, show that each factor of production earns its marginal product. Show that if owners if capital save all their income and workers consume all their income, then the economy reaches the golden rule of capital accumulation. Explain the results.

Analytic solution

The neoclassical function and its properties:

$$Y = F(K, L)$$

Non-negative and diminishing marginal products:

$$F_K \geq 0, \quad F_{KK} < 0, \quad F_L \geq 0, \quad F_{LL} < 0$$

[5] Problem 1.5 in Barro and Sala-I-Martin (2004).

Constant returns to scale:

$$F(\lambda K, \lambda L) = \lambda F(K, L)$$

Inada conditions assuring an interior solution:

$$\lim_{K \to 0} F_K = \infty, \quad \lim_{K \to \infty} F_K = 0$$

$$\lim_{L \to 0} F_L = \infty, \quad \lim_{L \to \infty} F_L = 0$$

The production function may then be expressed in *intensive form*:

$$Y = LF(K/L, 1) \equiv Lf(k)$$

where $k = K/L$, or $y = f(k)$ where $y = Y/L$. The marginal production of capital:

$$\frac{\partial Y}{\partial K} = \frac{\partial yL}{\partial kL} = \frac{\partial y}{\partial k} = f'(k)$$

The marginal product of labour:

$$\frac{\partial Y}{\partial L} = \frac{\partial yL}{\partial L} = y + L\frac{\partial y}{\partial L} = f(k) + Lf'(k)\frac{\partial k}{\partial L} = f(k) - f'(k)(K/L) = f(k) - kf'(k)$$

The firm's objective is to maximize profits defined as:

$$\max \Pi \equiv F(K, L) - wL - rK$$

Dividing this expression by L, we have:

$$\max \pi \equiv f(k) - w - rk$$

The first order condition for k is:

$$f'(k) = r$$

Under constant returns to scale, all revenue is returned to capital and labour:

$$f(k^*) = w + rk^*$$

Substituting for r, we determine the wage rates which results in zero profit:

$$w^* = f(k^*) - k^* f'(k^*)$$

We see that this wage is precisely the marginal product of labour. Hence, when firms maximize profits constant returns to scale assures that profits are driven to zero.
Assume now that all capital income is fully reinvested, so:

$$I^* = r^* K^*$$

Also assume that all labour is consumed:

$$C^* = w^* L$$

We therefore have:

$$\frac{\dot{C}}{C} = \frac{\dot{L}}{L} = n$$

If we define $c = C/L$, then we have:

$$\frac{\dot{c}}{c} = 0$$

The laws of motion for capital in the Solow-Swann model are defined as:

$$\dot{K} = I - \delta K = r^* K - \delta K$$

so it follows that:

$$\dot{k} = r^* k - (\delta + n)k$$

On a steady-state growth path, we have:

$$\frac{\dot{K}}{K} = \frac{\dot{C}}{C} = \frac{\dot{L}}{L} = n$$

hence, $\dot{k} = 0$, and

$$r^* - (\delta + n) = 0$$

Substituting for the marginal product of capital, we recover the Golden Rule condition:

$$f'(k) = \delta + n$$

Numeric solution

```
$title Barro and Sala-i-Margin, problem 1.5 - Solow-Swann Growth Model

*         Declare the time horizon here.  It is important to
*         declare this set as ordered (using the "*") so that
*         we can reference  "t+1" from inside the loop over t:

set       t /0*200/, t0 /0/;

*         Data describing the base year are given here:

scalar              alpha     base year capital value share           /0.6/
                    r0        base year gross return to capital       /0.12/
                    delta     capital depreciation rate               /0.07/
                    n         labor growth rate                       /0.02/
                    s_L       savings rate of workers                 /0.10/
                    s_K       savings rate of capital owners          /0.90/

*         The elasticity of substitution between labor and capital is a free
*         parameter. For the initial simulation, we set it to unity (almost).
*         (We use 1.01 rather than 1 in order to avoid having to change the
*         functional form from Cobb-Douglas to CES in the model):

                    sigma     elasticity of substitution              /1.01/

*         Calibrated parameters:

                    c0        base year consumption (calibrated)
                    rho       primal CES exponent
                    k0        base year capital stock (calibrated)
                    l0        base year labor supply  (calibrated);

*         The following parameters hold an equilibrium time path:

parameter k                   Time path of capital stock
                    l         Time path of labor
                    c         Time path of total consumption
                    y         Time path of output
                    r         Time path of return to capital
                    w         Time path of wage rate
```

```

*         The following parameters hold output to be plotted:

                    timepath  Time path of per-capita variables,
                    output    Time path of output (alternative sigma values),
                    return    Time path of return (alternative sigma values),
                    consum    Time path of consumption (alternative sigma values),
                    wage      Time path of wage (alternative sigma values);

*         Compute the primal elasticity exponent:

rho = 1 - 1/sigma;

*         Calibrate the base year capital stock and labor supply
*         in efficiency units, taking base year output equal to unity
*         and measuring labor in efficiency units:

k0 = alpha / r0;
l0 = (1-alpha);

*         Base year consumption is based on capital and labor earnings
*         shares and the marginal propensity to save out of those income
*         sources:

c0 = alpha * (1-s_K) + (1-alpha) * (1-s_L);

*         Initialize base year (time 0) output, capital and labor stock:

y(t0) = 1;
k(t0) = k0;
l(t0) = l0;

*         Do an initial simulation with the specified value of simga
*         (1.01 = Cobb Douglas).

loop(t,

*         Entering period t the values of capital and labor are known, so the
*         output is known:

          y(t) = ( alpha * (k(t)/k0)**rho + (1-alpha) * (l(t)/l0)**rho)**(1/rho);

*         The return to capital and labor are computed as marginal products:

          r(t) = (y(t)*k0/k(t))**(1/sigma) * alpha / k0;
          w(t) = (y(t)*l0/l(t))**(1/sigma) * (1-alpha) / l0;

*         Consumption is the sum of consumption levels by capital owners and
*         workers:

          c(t) = (1-s_K) * r(t) * k(t) + (1-s_L) * w(t) * l(t);

*         Capital evolves through depreciation and investment:

          k(t+1) = k(t) * (1 - delta) + y(t) - c(t);

*         Labor growth is exogenous at rate n:

          l(t+1) = l(t) * (1 + n);
);

*         Store the time path of key values for plotting:

timepath(t,"output") = y(t) * l0 / l(t);
timepath(t,"consum") = (c(t)/c0) * l0 / l(t);
timepath(t,"return") = r(t) * k0 / alpha;
timepath(t,"wage")   = w(t) * l0 / (1-alpha);
```

```
*         Declare a set over values of the elasticity of substitution (sigma)
*         to be compared:

set       sigval    /"0.5", "1.0", "2.0" /;

parameter sigvalue(sigval) / "0.5" 0.5, "1.0" 1.01, "2.0" 2.0 /;

loop(sigval,

*         Assign the elasticity:

          sigma = sigvalue(sigval);
          rho = 1 - 1/sigma;

*         Compute the equilibrium time path (period 0 values are the same in all
*         simulations):

          loop(t,
            y(t) = ( alpha * (k(t)/k0)**rho + (1-alpha) * (l(t)/l0)**rho)**(1/rho);
            r(t) = (y(t)*k0/k(t))**(1/sigma) * alpha / k0;
            w(t) = (y(t)*l0/l(t))**(1/sigma) * (1-alpha) / l0;
            c(t) = (1-s_K) * r(t) * k(t) + (1-s_L) * w(t) * l(t);
            k(t+1) = k(t) * (1 - delta) + y(t) - c(t);
            l(t+1) = l(t) * (1 + n);
          );

*         Save some values to plot comparisons:

          consum(t,sigval) = (c(t)/c0) * l0 / l(t);
          output(t,sigval) = y(t) * l0 / l(t);
          return(t,sigval) = r(t) * k0 / alpha;
          wage(t,sigval)   = w(t) * l0 / (1-alpha);

);

*         Generate some labeled plots:

set       tics(t) / 0, 25, 50, 75, 100, 125, 150, 175, 200 /

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome ’Times-Roman’ 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$setglobal gp_xl tics
$setglobal gp_xlabel years
$setglobal domain t
$setglobal labels tics

$if %batch%==yes $setglobal gp_opt3 "set output ’bs15a.eps’"
$setglobal gp_opt4 "set key outside"
$setglobal gp_opt5 "set xlabel ’years’"
$setglobal gp_opt6 "set ylabel ’% change’"
$libinclude plot timepath

$if %batch%==yes $setglobal gp_opt3 "set output ’bs15b.eps’"
$libinclude plot output

$if %batch%==yes $setglobal gp_opt3 "set output ’bs15c.eps’"
$libinclude plot consum

$if %batch%==yes $setglobal gp_opt3 "set output ’bs15d.eps’"
$libinclude plot return

$if %batch%==yes $setglobal gp_opt3 "set output ’bs15e.eps’"
$libinclude plot wage
```

Figure 10: Key Variables

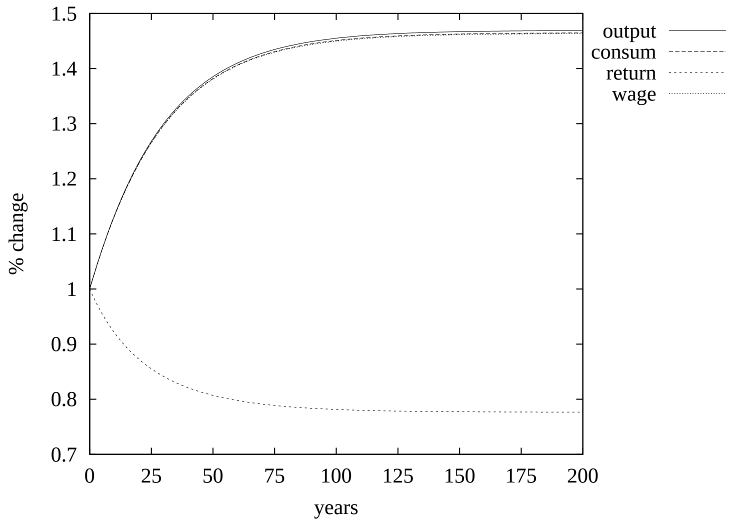


Figure 11: Output

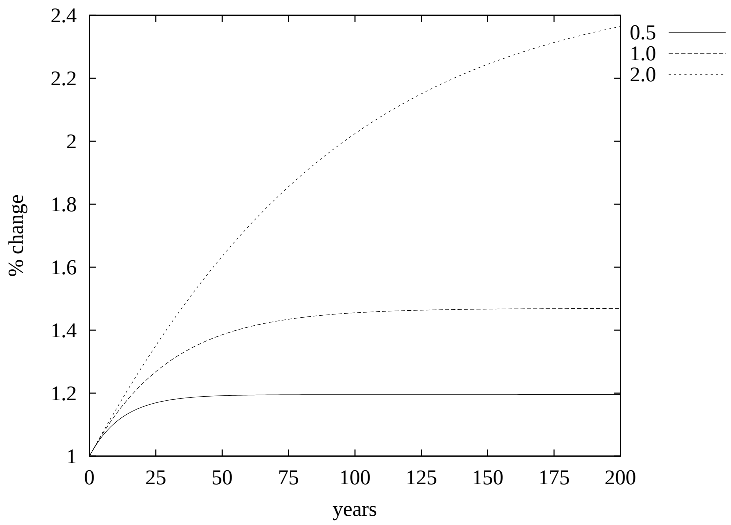


Figure 12: Return to Capital

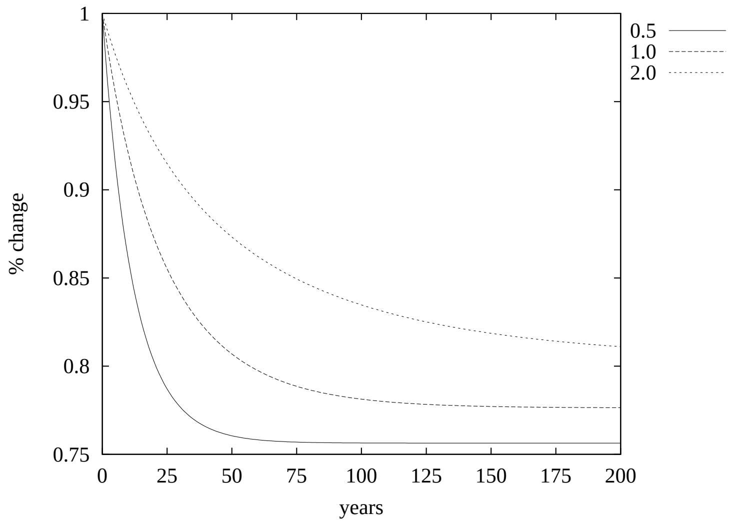
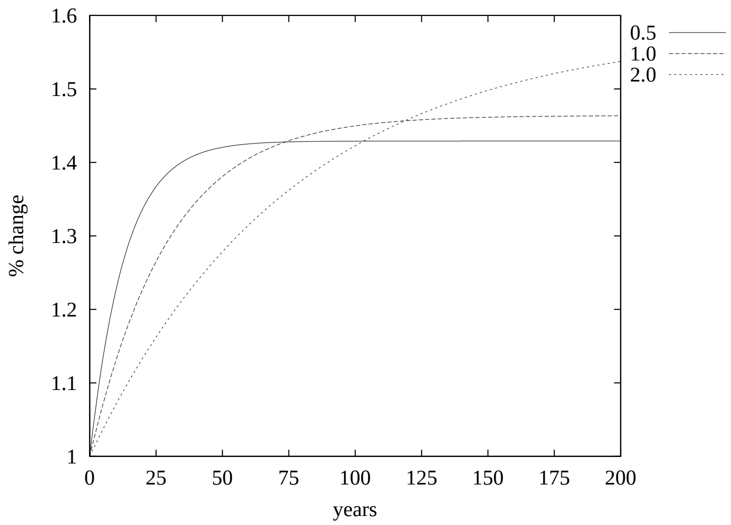


Figure 13: Wage Rate



4.2 Distortions in the Solow-Swan model

[6]Assume that output is produced by the CES production function,

$$Y = [(a_F K_F^\eta + a_I K_I^\eta)^{\phi/\eta} + a_G K_G^\phi]^{1/\phi}$$

where Y is output; K_F is formal capital, which is subject to taxation; K_I is informal capital, which evades taxation; K_G is public capital, provided by government and used freely by all producers; $a_F, a_I, a_G > 0$; $\eta < 1$, and $\phi < 1$. Installed formal and informal capital differ in their location and form of ownership and, therefore, in their productivity.

Output can be used on a one-for-one basis for consumption or gross investment in the three types of capital. All three types of capital depreciate at the rate δ. Population is constant, and technology progress is nil.

Formal capital is subject to tax at the rate τ at the moment of its installation. Thus, the price of formal capital (in units of output) is $1+\tau$. The price of a unit of informal capital is one. Gross investment in public capital is the fixed fraction s_G of tax revenues. Any unused tax receipts are rebated to households in a lump-sum manner. The sum of investment in the the two forms of private capital is the faction s of income net of taxes and transfers. Existing private capital can be converted on a one-to-one basis in either direction between formal and informal capital.

a. Derive the ratio of informal to formal capital used by profit-maximizing producers.

b. In the steady-state, the three forms of capital grow at the same rate. What is the ratio of output to formal capital in the steady-state?

c. What is the steady-state growth rate of the economy?

d. Numerical simulations show that, for reasonable parameter values, the graph of the growth rate against the tax rate, τ, initially increases rapidly, then reaches a peak, and finally decreases steadily. Explain this nonmonotonic relation between the growth rate and the tax rate.

Analytic solution

The problem states that output is given by the CES production function

$$\begin{aligned} y &= f(k_F, k_I, k_G) \\ &= \left[(a_F k_F^\eta + a_I k_I^\eta)^{\psi/\eta} + a_G k_G^\psi\right]^{1/\psi} \\ &= k_F \left[\left(a_F + a_I \left(\frac{k_I}{k_F}\right)^\eta\right)^{\psi/\eta} + a_G \left(\frac{k_G}{k_F}\right)^\psi\right]^{1/\psi} \quad , \end{aligned}$$

where k denotes capital and subscripts F, I and G denote formal, informal and government, respectively; population growth is constant and technological progress is nil; depreciation is the same for all forms of capital, implying that

$$\begin{aligned} \dot{k}_F &= i_F - \delta k_F \ , \\ \dot{k}_I &= i_I - \delta k_I \ , \quad \text{and} \\ \dot{k}_G &= i_G - \delta k_G \ , \end{aligned}$$

where i denotes the investment at time t for the kind of capital specified by the subscript; taxes are collected as a fixed fraction of formal investment,

$$T = \tau i_F \ ;$$

[6]Problem 1.7 in Barro and Sala-I-Martin (2004) based on Easterly (1993).

gross investment in public capital is a fixed fraction of taxes,

$$i_G \quad = \quad s_G T = s_G \tau i_F \ ;$$

unused taxes,

$$T_U \quad = \quad (1 - s_G)T \quad = \quad (1 - s_G)\tau i_F \ ,$$

are rebated to households in a lump-sum manner; prices (in units of output) are

$$P_F = (1 + \tau) \qquad \text{and} \qquad P_I = 1 \ ;$$

the sum of private investments in formal and informal capital is given by

$$i_F + i_I = s(Y - T + T_U) = s(Y - s_G \tau i_F) \ ,$$

which implies

$$i_I = s(Y - (1 + s_G \tau) i_F) \ ;$$

Treating public capital as an externality, a profit maximizing producer chooses formal and informal investments to solve

$$\max_{i_F, i_I} \{P_y y - P_F i_F - P_I i_I\} \qquad \equiv \qquad \max_{i_F, i_I} \{y - (1+\tau) i_F - i_I\}$$

subject to

$$\begin{aligned} g_F &\stackrel{\text{def}}{=} \frac{\dot{k}_F}{k_F} = \frac{i_F}{k_F} - \delta \ , \quad \text{and} \\ g_I &\stackrel{\text{def}}{=} \frac{\dot{k}_I}{k_I} = \frac{i_I}{k_I} - \delta \ , \end{aligned}$$

or equivalently,

$$k_F = \frac{i_F}{g_F + \delta} \ , \qquad \text{and} \qquad k_I = \frac{i_I}{g_I + \delta} \ .$$

The first order conditions for this problem are

$$\frac{\partial y}{\partial k_F} \frac{dk_F}{di_F} = (1+\tau) \qquad \text{and} \qquad \frac{\partial y}{\partial k_I} \frac{dk_I}{di_I} = 1 \ ,$$

which implies

$$\left(\frac{\partial y}{\partial k_I} \frac{dk_I}{di_I}\right) \left(\frac{\partial y}{\partial k_F} \frac{dk_F}{di_F}\right)^{-1} \quad = \quad \left(\frac{k_I}{k_F}\right)^{\eta - 1} \left(\frac{a_I (g_F + \delta)}{a_F (g_I + \delta)}\right) \quad = \quad \frac{1}{1+\tau} \ .$$

a. From the first order conditions, the ratio of informal to formal capital used by profit-maximizing producers can be computed to be

$$\frac{k_I}{k_F} \quad = \quad \left[\frac{a_F (g_I + \delta)}{(1+\tau) a_I (g_F + \delta)}\right]^{1/(\eta - 1)} \quad = \quad \left[\frac{(1+\tau) a_I (g_F + \delta)}{a_F (g_I + \delta)}\right]^{1/(1-\eta)} \ .$$

b. If the three forms of capital grow at the same rate so that

$$g_F \quad = \quad g_I \quad = \quad g_G \ ,$$

then

$$\frac{k_I}{k_F} \quad = \quad \left[(1+\tau) \left(\frac{a_I}{a_F}\right)\right]^{1/(1-\eta)} \qquad \text{and} \qquad \frac{k_G}{k_F} \quad = \quad \frac{i_G}{i_F} \quad = \quad s_G \tau \ ,$$

and thus

$$y \quad = \quad k_F \left[\left(\beta_F + \beta_I (1+\tau)^{\eta/(1-\eta)} \right)^{\psi/\eta} + \beta_G \tau^\psi \right]^{1/\psi} ,$$

where

$$\beta_F \stackrel{\text{def}}{=} a_F \ , \qquad \beta_I \stackrel{\text{def}}{=} a_I \left(\frac{a_I}{a_F} \right)^{\eta/(1-\eta)} \qquad \text{and} \qquad \beta_G \stackrel{\text{def}}{=} a_G s_G^\psi \ .$$

The ratio of output to formal capital in the steady state is therefore

$$\frac{y}{k_F} = \left[\left(\beta_F + \beta_I (1+\tau)^{\eta/(1-\eta)} \right)^{\psi/\eta} + \beta_G \tau^\psi \right]^{1/\psi} .$$

c. Since the ratio of output to formal capital in the steady state is constant, it must be the case that the growth rate of the economy is equal to the growth rate of formal capital, which is given by

$$g \quad = \quad s \left(\frac{y}{k_F} \right) - \delta \ .$$

The growth rate for the economy is therefore

$$g(\tau) \quad = \quad sG(\tau)^{1/\psi} - \delta$$

where

$$G(\tau) \quad \stackrel{\text{def}}{=} \quad \left[\left(\beta_F + \beta_I (1+\tau)^{\eta/(1-\eta)} \right)^{\psi/\eta} + \beta_G \tau^\psi \right] \ > \ 0 \ .$$

d. An economic explanation for simulated nonmonotonic behaviour for the change in growth as a function of taxes is as follows:

> As taxes grow from zero, the public good becomes available but capital is also moved from the formal to the informal sector. For reasonable parameter values, the increased productivity due to the public good dominates the loss of productivity due to the transfer from formal capital to informal capital, and so the economy grows.
> At some point, the growth with respect to taxes is maximized, indicating that the loss of productivity due to movement away from formal capital to informal capital is exactly offset by the increase in the productivity due to public capital.
> As the tax rate increases beyond the maximal point and approaches unity, the steady state growth will decrease as the low productivity informal capital dominates the increase in public sector productivity.

However,

$$g'(\tau) \quad = \quad \left(\frac{s}{\psi} \right) G(\tau)^{(-1+1/\psi)} G'(\tau)$$

where

$$G'(\tau) \quad = \quad \psi \left[\left(\frac{\beta_I}{1-\eta} \right) \left(\beta_F + \beta_I (1+\tau)^{\eta/(1-\eta)} \right)^{-1+\psi/\eta} (1+\tau)^{-1+\eta/(1-\eta)} + \beta_G \tau^{\psi-1} \right] .$$

The sign of g' is always positive, since the sign of G is always positive and the sign of G' is the same as the ratio of savings rate to the elasticity of substitution between private and public capital. This analytic result thus indicates that growth is always increasing in the tax rate, thereby contradicting the simulated nonmonotonic behaviour discussed above.

Numeric solution

```
$title Barro and Sala-i-Margin, problem 1.7 - Easterly model

*         This programs illustrates how to calibrate the Easterly
*         model, evaluate how the steady-state growth rate depends on
*         the tax rate, and then evaluates the transition path for a
*         change in the tax rate


set       taxrate   Alternative tax rates to evaluate (%)   / 1*200 /,
          taxlabel(taxrate)           /10,30,50,70,90,110,130,150,170,190 /,
          t         Years to simulate                     /1997*2041/,
          return    Assumed returns to public capital     /low, medium, high/,
          tplot     Time periods to plot                  /1997*2040/,
          decade(tplot)               / 2000, 2010, 2020, 2030, 2040/;

scalar

*===============================================================================
*         Base year data are specified here:

tau0      benchmark tax rate on formal sector investment             /0.50/
tau_s     tax rate on formal sector in simulated adjustment          /0.90/
r_g       benchmark relative return to public sector capital         /1.0/
g         baseline growth rate                                       /0.03/
s_g       public savings rate                                        /0.75/
delta     depreciation rate                                          /0.07/
iratio    ratio of informal to formal capital                        /0.15/
sigmag    substitution elasticity between private and public capital  /0.5/
sigmaf    substitution elasticity between formal and informal capital /4.0/

*===============================================================================
*         Calibrated or temporary parameters:

s         private savings rate
thetag    implicit benchmark value share of public capital
thetaf    share of private capital in the formal sector
tau       tax rate in counter-factual
k_f       baseline formal capital
k_i       baseline informal capital
k_g       baseline public capital
k_p       private sector capital stock (state variable)
y0        scale parameter in benchmarking
a_f       CES share parameter for formal capital
a_i       CES share parameter for informal capital
a_g       CES share parameter for public capital
eta       CES exponent (inner nest)
psi       CES exponent (outer nest)

ki_ratio  ratio of informal to formal capital
yf_ratio  ratio of output to formal capital;

parameter

r0(return)          Implicit baseyear relative return to public capital

growth    Steady-state growth rate (sensitivity to return on public capital)

y         Output level in simulated transition
kg        Public capital in simulated transition
kf        Formal capital in simulated transition
ki        Informal capital  in simulated transition

transition          Growth rates through the transition;


*===============================================================================
```

```
*         Benchmarking steps are explained here:

eta = 1 - 1/sigmaf;
psi = 1 - 1/sigmag;

*         For purpose of deriving coefficients, set magnitude of formal
*         capital to unity:

k_f = 1;
k_i = iratio;

thetaf = k_f / (k_f + k_i);

*         Given formal capital, we know the public capital stock from the
*         tax rate and public sector savings rate:

k_g = s_g * tau0 * k_f;

*         Value share of public capital depends on assumed shadow return
*         to public sector capital:

thetag = r_g * k_g / (k_f + (1+tau0)*k_i + r_g * k_g);

*         Calibrate public capital coefficient from the base year quantity,
*         the value share and the elastiicty:

a_g = k_g**(-psi) * thetag / (1 - thetag);

*         Calibrate relative size of formal and informal coefficients,
*         based on relative size of the capital stock and the elasticity:

a_f = 1;
a_i = (1 / (1 + tau0)) * (k_i/k_f)**(1-eta);

*         Now compute the implicit output level and rescale capital stocks to
*         be consistent with benchmark output equal to unity:

y0 = ( (a_f * k_f**eta + a_i * k_i**eta)**(psi/eta) + a_g * k_g**psi )**(1/psi);

k_f = k_f / y0;
k_i = k_i / y0;
k_g = k_g / y0;

*         Calibrate private savings to be consistent with steady-state:

s = (g + delta) * (1 + tau0 + iratio) / (1/k_f);


*===============================================================================
*  1)     Simulate the transitional dynamics associated with a change
*         in the tax rate.

*         Apply the new tax rate:

tau = tau_s;

*         Given the tax rate, we know the formal share of private capital use:

thetaf  = 1 / ((a_i * (1 + tau) / a_f)**(1/(1-eta)) + 1);

*         Initialize the state variable:

k_p = k_f + k_i;

loop(t,

*         Record current capital stocks:
```

```

          kg(t) = k_g;
          kf(t) = k_f;
          ki(t) = k_i;

*         Compute output:

          y(t) = ( (a_f * k_f**eta + a_i * k_i**eta)**(psi/eta)
                   + a_g * k_g**psi )**(1/psi);

*         Update capital stocks for the next period.

*         Note: the price of a unit of new capital is given by a share-weighted
*         average of the informal price (1) and the formal sector price (1+tau).
*         This explains the termin the the denominator of the k_p expression:

          k_p = k_p * (1 - delta) + s * y(t) / (1 - thetaf + (1+tau)  * thetaf);
          k_g = k_g * (1 - delta) + s_g * tau * thetaf * s * y(t) / (1 - thetaf
               + (1+tau) * thetaf);
          k_f = k_p * thetaf;
          k_i = k_p * (1 - thetaf);
);

*         Compute growth rates:

transition(t,"g0")$y(t+1) = 100 * g;
transition(t,"y")$y(t+1) = 100 * (y(t+1)-y(t)) / y(t);
transition(t,"kg")$kg(t+1) = 100 * (kg(t+1)-kg(t)) / kg(t);
transition(t,"kf")$kf(t+1) = 100 * (kf(t+1)-kf(t)) / kf(t);
transition(t,"ki")$ki(t+1) = 100 * (ki(t+1)-ki(t)) / ki(t);

display transition;


*=============================================================================
*  2)     Perform a sensitivity analysis: growth as a function of the
*         tax rate, accounting for alternative assumptions regarding
*         the base year shadow price on public capital:

*         Assign base year relative returns to public capital
*         (base year return to informal capital = 1):

r0("low") = 0.5;
r0("medium") = r_g;
r0("high") = 1 + tau0 + 0.5;

*         For each alternative base year return to public capital,
*         recalibrate the model:

loop(return,

          r_g = r0(return);
          eta = 1 - 1/sigmaf;
          psi = 1 - 1/sigmag;
          k_f = 1;
          k_i = iratio;
          k_g = s_g * tau0 * k_f;
          a_f = 1;
          a_i = (1 / (1 + tau0)) * (k_i/k_f)**(1-eta);
          thetag = r_g * k_g / (k_f + (1+tau0)*k_i + r_g * k_g);
          a_g = k_g**(-psi) * thetag / (1 - thetag);
          y0 = ( (a_f * k_f**eta + a_i * k_i**eta)**(psi/eta) + a_g * k_g**psi )
               **(1/psi);
          k_f = k_f / y0;
          k_i = k_i / y0;
          k_g = k_g / y0;
          s = (g + delta) * (1 + tau0 + iratio) / (1/k_f);
```

```
          display s;

*         Then for each model, evaluate how the growth rate
*         changes with the tax rate:

          loop(taxrate,
                   tau = 0.01 * ord(taxrate);
                   ki_ratio = (a_i * (1 + tau) / a_f)**(1/(1-eta));
                   yf_ratio = ( (a_f + a_i * (ki_ratio)**eta )**(psi/eta)
                              + a_g * (s_g * tau)**psi )**(1/psi);
                   growth(taxrate,return) = 100 * (s * yf_ratio
                                         / (1 + tau + ki_ratio) - delta);
  );

);

display growth;


*===============================================================================
*         Generate some plots:

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$if %batch%==yes $setglobal gp_opt3 "set output 'bs17a.eps'"
$setglobal gp_opt4 "set key outside width 4"
$setglobal gp_opt5 "set xlabel 'Tax rate on formal capital (%)'"
$setglobal gp_opt6 "set ylabel 'Economic growth rate (%)'"
$setglobal gp_opt7 "set grid"
$setglobal gp_opt8 "set yrange [-2:4]"

$setglobal domain taxrate
$setglobal labels taxlabel
$libinclude plot growth

$if %batch%==yes $setglobal gp_opt3 "set output 'bs17b.eps'"
$setglobal gp_opt5 "set xlabel 'Year'"
$setglobal gp_opt6 "set ylabel 'Economic growth rate (%)'"
$setglobal gp_opt8 "set yrange [-1:5]"

$setglobal domain tplot
$setglobal labels decade
$libinclude plot transition
```

Figure 14: Capital Taxes and Steady-State Growth

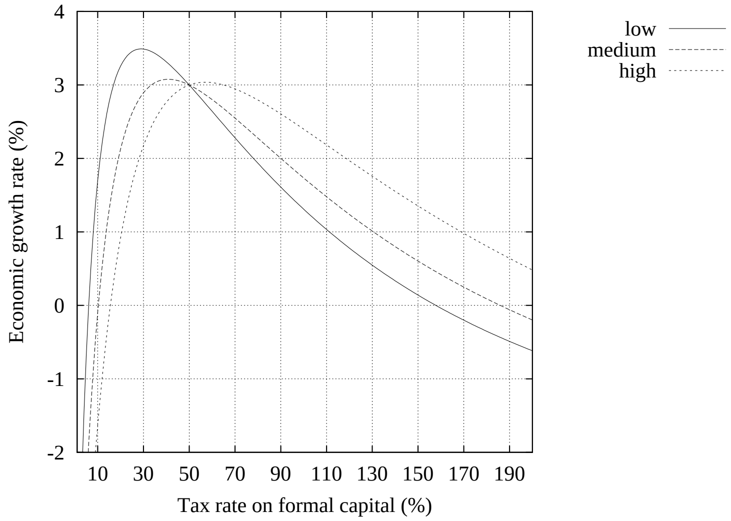
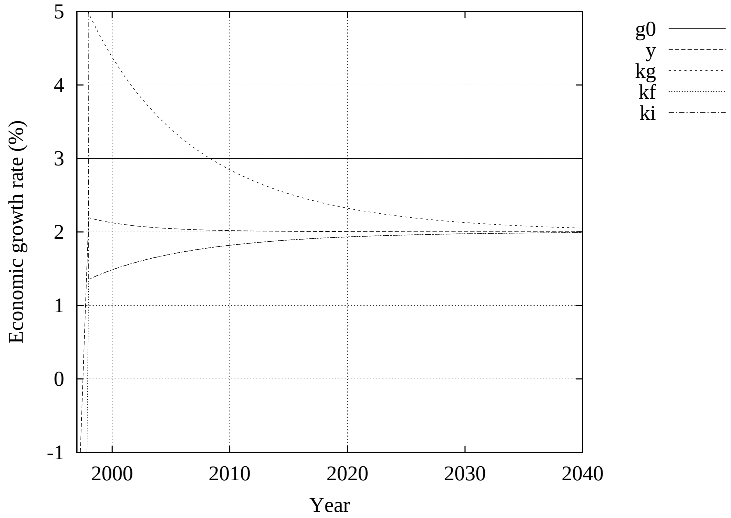


Figure 15: Growth Rates through the Transition



5 The Neoclassical Optimal Growth Model

This section lays down the basics for developing applied dynamic CGE models. We begin by going through the logic of the Ramsey model which is often presented as a dynamic optimization problem[7]:

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \frac{C_t^{1-\theta}-1}{1-\theta}$$

s.t.

$$C_t = f(K_t) - I_t$$
$$K_{t+1} = (1-\delta)K_t + I_t$$
$$K_0 = \bar{K}_0$$

The maximand in this problem is often called constant-elasticity-of-intertemporal substitution (CEIS) utility function. As will be shown below, it simply represents a monotonic transformation of conventional CES utility function.

Here, as in many macroeconomics textbooks, aggregate output is expressed as a function of the capital stock alone, i.e.:

$$Y_t = f(K_t)$$

In the MPSGE representation of the Ramsey model[8], it is convenient to work with a constant-returns production function in which we have inputs of both labour and capital:

$$Y_t = F(\bar{L}_t, K_t)$$

When labour is in fixed supply, the production function exhibits diminishing returns to capital. There is therefore no loss of generality by formulating the model on the basis of a constant returns to scale technology.

In writing down a model it is helpful to employ the *unit cost function* associated with the production function $F(\cdot)$:[9]

$$c(p_t^L, r_t^K) \equiv \min p_t^L a_L + r_t^K a_K$$

s.t.

$$F(a_L, a_K) = 1$$

Shephard's lemma tells us that the *compensated demand functions* for labour and capital are the partial derivatives of the unit cost function:

$$a_K(r^K, p^L) = \frac{\partial c(p_t^L, r_t^K)}{\partial r_t^K}$$

and

$$a_L(r^K, p^L) = \frac{\partial c(p_t^L, r_t^K)}{\partial p_t^L}$$

The representative agent model can be formulated as a general equilibrium model which is completely routine, apart from the fact that there are an infinite-number of variables. Following the conventional GAMS/MPSGE framework, equilibrium in the model is characterized by three classes of equations:

[7] For simplicity it is assumed that there is no population growth.

[8] See the last part in section 3.1 for a simple model represented in three equivalent formulations, i.e. NLP, algebraic MCP and MPSGE.

[9] Note that the lower-case function $c(\cdot)$ represents unit cost, while the upper case C_t represents consumption in year t. In the equilibrium model $C_t(p, M)$ represents the demand for output in year t as a function of output prices and aggregate present value of income.

1. Market clearance conditions and associated market prices are as follows:[10]

   - Output market (market price p_t):

   $$Y_t = C_t(p, M) + I_t$$

   - Labour market (wage rate p_t^L):

   $$\bar{L}_t = a_L(r_t^K, p_t^L)\ Y_t$$

   - Market for capital services (capital rental rate r_t^K):

   $$K_t = a_K(r_t^K, p_t^L)\ Y_t$$

   - Capital stock (capital purchase price p_t^K):

   $$K_{t+1} = (1-\delta)K_t + I_t$$

2. Zero profit conditions and associated activities are:[11]

   - Output (Y_t):

   $$p_t = c(p_t^L, r_t^K)$$

   - Investment ($I_t \geq 0$):

   $$p_t \geq p_{t+1}^K$$

   - Capital stock (K_t):

   $$p_t^K = r_t^K + (1-\delta)p_{t+1}^K$$

3. Income balance:

   $$M = p_0^K \bar{K}_0 + \sum_{t=0}^{\infty} p_t^L \bar{L}_t$$

Two questions might arise for an MPSGE modeler looking at this equilibrium model. First, the careful observer might note that the demand functions, $C_t(p, M)$, have not been specified, and because these arise from CIES preferences so there may be some details to work out. This problem is considerably easier than the second issue, namely *how do we solve an infinite-dimensional system of nonlinear equations*. Let's first look at this latter issue. The issue of CEIS preferences will be considered in the calibration section below.

In order to solve a finite approximation of the model with a T-period model horizon, we need to *decompose* the consumer's problem. Consider the infinite-horizon problem of the representative agent in Ramsey's model:

$$\max \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(c_t)$$

s.t.

$$\sum_{t=0}^{\infty} p_c c_t = p_0^K \bar{K}_0 + \sum_{t=0}^{\infty} p_t^L \bar{L}_t$$

[10] The demand functions employed in this model assure that all prices will be nonzero in equilibrium. There is no formal need, therefore, to associated prices with market clearance conditions, as would be required in a conventional complementarity problem. We provide an associated here in order to help understand how the model might be extended with demand functions which would admit zero prices.

[11] The only activity level which could possibly fall to zero would be investment, and that would only happen in a policy scenario which resulted in a substantial reduction in the return to capital.

in which $u(c) = \frac{c^{1-\theta}-1}{1-\theta}$, and define a value of terminal assets to be:

$$A_T^* = \sum_{t=T+1}^{\infty} \left(p_c c_t^* - p_t^L \bar{L}_t\right)$$

Then consider the equivalent model:

$$\max \sum_{t=0}^{T} \left(\frac{1}{1+\rho}\right)^t u(c_t) + \sum_{t=T+t}^{\infty} \left(\frac{1}{1+\rho}\right)^t u(c_t)$$

s.t.

$$\sum_{t=0}^{T} p_c c_t = p_0^K \bar{K}_0 + \sum_{t=0}^{T} p_t^L \bar{L}_t - A_T$$

$$\sum_{t=T+1}^{\infty} p_c c_t = A_T + \sum_{t=T+1}^{\infty} p_t^L \bar{L}_t$$

If A_T is fixed then this can be posed as two separate optimization problems, one running through time period T and another for the post-terminal period. When terminal assets are assigned a value of A_T^*, corresponding to the infinite-horizon solution, then the finite horizon model will then produce consumption levels for years 0 through T which are identical to the ∞-horizon model. The question is how do we find A_T^*?

Terminal assets in the closed economy model are simply equal to the value of the capital stock at the start of period $T + 1$. The model running through year T then produces a good approximation to the consumer problem when we have a good approximation to the terminal capital stock. The key insight provided by Lau, Pahlke, and Rutherford (2002) is that the state variable K_{T+1} can be determined as part of the equilibrium calculation by *targeting* the associated control variable, I_T. In the present model this could be based on any of the following *primal constraints*:

- Terminal investment growth rate set equal to the long-run steady-state growth rate:
$$I_T/I_{T-1} = 1 + g$$
- Terminal investment growth rate set equal to the growth rate of aggregate output:
$$I_T/I_{T-1} = Y_T/Y_{T-1}$$
- Terminal investment growth rate set equal to the growth rate of consumption:
$$I_T/I_{T-1} = C_T/C_{T-1}$$

State-variable targeting provides a very compact means of determining the terminal capital stock. In models with multiple consumers living beyond period T, it would be necessary to account for which of these agents owns the assets. Note that some agents may have *negative* asset positions at the end of the model – particularly in overlapping generations models where young households accumulate debt which is repaid in middle age.

The final detail involved in implementing a dynamic model in MPSGE is *calibration.* The simplest approach is to set up the model along a steady-state growth rate in which the interest rate ($\bar{r}$) and growth rate ($\bar{g}$) are given. The first thing to work out is to determine the structure of the benchmark equilibrium.

Here are the steps involved in sorting out the steady-state conditions which related investment and capital earnings in a static data set which is consistent with a steady-state growth path:

1. The zero-profit condition for I_t reveals the price level for capital:

$$p^K_{t+1} = \frac{p^K_t}{1+\bar{r}} = p_t$$

hence

$$p^K_t = (1+\bar{r})p_t$$

The base year price of capital is then:

$$\bar{p}^K = 1 + \bar{r}$$

2. The zero profit condition for K_t determines the price level for r^K_t:

$$p^K_t = r^K_t + (1-\delta)p^K_{t+1}$$

Substituting the values of p^K_t and p^K_{t+1} reveals that the base year rental price of capital is sufficient to cover interest plus depreciation:

$$\bar{r}^K = \bar{r} + \delta$$

3. The main challenge involved in calibrating a dynamic model centers on the reconciliation of base year capital earnings, investment, the steady-state interest rate and the capital depreciation rate. To see how this works, consider the market clearance condition for capital in the first period:

$$K_1 = \bar{K}_0(1-\delta) + \bar{I} = (1+\bar{g})\bar{K}_0$$

This implies that base year investment can be calculated on the basis of growth and depreciation of the base year capital stock:

$$\bar{I} = \bar{K}_0(\bar{g} + \delta)$$

Finally, we can use $\bar{r}^K$ to determine $\bar{K}_0$ on the basis of the value of capital earnings in the base year, $\overline{VK}$, hence:

$$\bar{I} = \overline{VK}\ \frac{\bar{g}+\delta}{\bar{r}+\delta}$$

The problem that arises in applied models is that $\bar{I}$ and $\overline{VK}$ will not satisfy this relation for arbitrary values of $\bar{g}$, $\bar{r}$ and δ. Something typically has to be adjusted to match up the dataset with the baseline growth path.

The second issue to work out is the representation of CEIS preferences in a MPSGE model. Consider the following *equivalent* representations of intertemporal preferences:

1. Additively separable utility:

$$U(C) = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \frac{C_t^{1-\theta} - 1}{1-\theta}$$

2. Linearly homogeneous utility:

$$\hat{U}(C) = \left[\sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t C_t^{1-\theta}\right]^{\frac{1}{1-\theta}}$$

It is possible to determine the equivalence of U and $\hat{U}$ by recalling that a monotonic transformation of utility does not alter the underlying preference ordering. Observe that:

$$\hat{U} = V(U) = [aU + \kappa]^{1/a}$$

where

$$\kappa = \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t = \frac{1+\rho}{\rho},$$

and

$$a = 1 - \theta.$$

$V(\cdot)$ is a monotonic transformation ($V' > 0$), hence optimization of U and $\hat{U}$ yield identical demand functions.

Alternatively, recall that preference orderings are defined by the *marginal rate of substitution.* In both of these models we have:

$$\frac{\partial U/\partial C_{t+1}}{\partial U/\partial C_t} = \frac{1}{1+\rho}\left(\frac{C_t}{C_{t+1}}\right)^\theta$$

There are several advantages associated with the use of linearly homogeneous representation. First of all, these preferences can be represented in MSPGE. Second, the reporting of welfare changes as Hicksian-equivalent variations is trivial with $\hat{U}$: a 1% change in $\hat{U}$ corresponds to a 1% equivalent variation in income.

CEIS preferences *over a finite horizon* can be represented in MPSGE as follows (lines 110 to 112 in the code given below):

```
$PROD:U        s:sigma
       O:PU           Q:(c0*sum(t, pref(t)*qref(t)))
       I:P(t)         Q:(qref(t)*C0)  P:pref(t)
```

Intertemporal preferences in an MPSGE model are typically based on the following parameters:

- `c0` is the base year consumption level,
- `qref(t) = (1+g0)**(ord(t)-1)` is the *baseline equilibrium index of economic activity*, calculated on the basis of a steady-state growth rate equal to `g0`,
- `pref(t) = (1/(1+r0))**(ord(t)-1)` is the *baseline present value price path*, calculated on the basis of a steady-state interest rate equal to `r0`, and
- `sigma` is the intertemporal elasticity of substitution.

Figure 5 shows how the utility function is calibrated using these parameters. Benchmark quantities determine an anchor point for the set of indifference curves. Benchmark prices fix the slope of the indifference curve at that point, and the elasticity describes the curvature of the indifference curve.

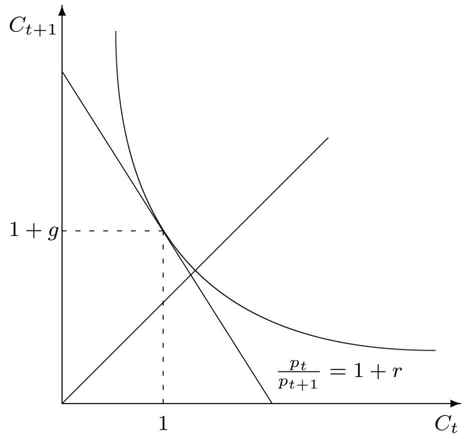
The MPSGE representation includes a discount rate which is define *implicitly* as:

$$\rho = \frac{1+r}{(1+g)^\theta} - 1$$

Numeric implementation

The code on the following pages presents a GAMS/MPSGE model which has been formulated following these ideas. Lines 1 to 72 reads base year data describing a steady-state equilibrium. Investment levels are imputed from the base year capital stock which is in-turn inferred from the assumed capital value share. Lines 73 to 150

Figure 16: Calibrated intertemporal preferences



declares the GAMS/MPSGE model, assigns steady-state values for activity levels and price, and then checks consistency of the resulting model. Lines 151 to 157 runs a policy experiment. It assigns a tax on capital earnings beginning in year 6. The resulting equilibrium is computed assuming that economic agents anticipate the application of the tax, resulting in a sharp response in investment and other economic variables to the new economic environment. Over time the tax leads to a reduction in the steady-state capital stock and the real wage. Finally, lines 158 to the end show how to present output in graphs using GNUPLOT, both to the windows screen and as encapsulated postscript.

```
1  $TITLE Ramsey Model - MPSGE formulation
2
3  $ontext
4
5  Calibrate to the steady-state condition:
6
7  I0 = KD0 * (g + delta) / (r + delta)
8
9  where  g=2, delta=7, r=5, so
10
11 I0 = 48 * 9 / 12 = 36
12
13         Y        I        FD
14 P       100     -36       -64
15 PL      -52               52
16 RK      -48               48
17 PS              36        -36
18
19 $offtext
20
21 SET       tt        Time horizon (including the first year of the post-terminal period)
22                     /2004*2081/,
23           t(tt)     Time period over the model horizon
24                     /2004*2080/;
25
26 SET       t0(t), tl(t), tterm(tt);
27
28 PARAMETER g         Growth rate                   /0.02/
29           r         Interest rate                 /0.05/
30           delta     Depreciation rate             /0.07/
31           kvs       Capital value share           /0.48/
32           sigma     Elasticity of substitution    /1.00/
33
34
35           y0        Base year output
```

```
          kd0       Base year rental value of capital

          k0        Base year capital stock
          i0        Base year investment
          c0        Base year consumption
          l0        Base year labor input

          kstock    Base year capital stock multiplier /1/

          taxk(t)   Capital tax rate in period T

          qref(t)   Reference quantity path
          pref(tt)  Reference price path;


*         Use the GAMS ORD (ordinality) and CARD (cardinality)
*         functions to automate the identification of the first
*         and last periods of the model horizon:

t0(t)     = yes$(ord(t) eq 1);
tl(t)     = yes$(ord(t) eq card(t));
tterm(tt) = yes$(ord(tt) eq card(tt));

*         Calibrate the model to the baseline growth path:

y0 = 100;
kd0 = kvs * y0;
l0 = y0 - kd0;
k0 = kd0 / (r + delta);
i0 = (g + delta) * k0;
c0 = y0 - i0;
taxk(t) = 0;
qref(t) = (1+g)**(ord(t)-1);
pref(tt) = (1/(1+r))**(ord(tt)-1);

DISPLAY y0, kd0, l0, k0, i0, c0, g, r, delta;

$ONTEXT

$MODEL:RAMSEY

$SECTORS:
        U       !       Intertemporal utility index
        Y(t)    !       Output
        I(t)    !       Investment
        K(t)    !       Capital stock

$COMMODITIES:
        PU      !       Intertemporal utility price index
        P(t)    !       Output price
        RK(t)   !       Return to capital
        PK(tt)  !       Capital price
        PL(t)   !       Wage rate

$CONSUMERS:
        RA      !       Representative agent

$AUXILIARY:
        TK      !       Post-terminal capital stock

$PROD:Y(t) s:1
        O:P(t)          Q:Y0
        I:PL(t)         Q:L0
        I:RK(t)         Q:KD0           A:RA      T:TaxK(t)

$PROD:K(tt)$T(tt)
        O:PK(TT+1)      Q:(K0*(1-DELTA))
```

```
        O:RK(tt)        Q:KD0
        I:PK(tt)        Q:K0

$PROD:I(tt)$T(tt)
        O:PK(TT+1)      Q:I0
        I:P(tt)         Q:I0

$PROD:U         s:sigma
        O:PU            Q:(c0*sum(t, pref(t)*qref(t)))
        I:P(t)          Q:(qref(t)*c0)  P:pref(t)

$DEMAND:RA
        D:PU
        E:PL(t)         Q:(L0*qref(t))
        E:PK(T0)        Q:(K0*KSTOCK)
        E:PK(TTERM)     Q:-1            R:TK

$REPORT:
        V:C(t)          I:P(t)          PROD:U
        V:W             W:RA

$CONSTRAINT:TK
        SUM(T$TL(T+1),  I(T+1)/I(t) - Y(T+1)/Y(t)) =E= 0;

$OFFTEXT
$SYSINCLUDE mpsgeset RAMSEY

*         Assign steady-state equilibrium values for quantities and prices:

Y.L(t)  = qref(t);
I.L(t)  = qref(t);
K.L(t)  = qref(t);

P.L(t)  = pref(t);
RK.L(t) = pref(t);
PL.L(t) = pref(t);

*         The steady-state price of capital is the output price
*         times one plus the interest rate:

PK.L(tt) = (1+r) * pref(tt);
TK.L     = k0 * (1+g)**card(t);

RAMSEY.ITERLIM = 0;
$INCLUDE RAMSEY.GEN
SOLVE RAMSEY USING MCP;
RAMSEY.ITERLIM = 1000;

*         Apply a tax on capital inputs of 25% beginning in year 6:

TAXK(t)$(ORD(t) > 5) = 0.25;

$INCLUDE RAMSEY.GEN
SOLVE RAMSEY USING MCP;

*         Generate some reports with graphs:

PARAMETER       indices        "Consumption, Investment and Capital Stock Indices";

indices(t,"C") = C.L(t)/(c0*qref(t));
indices(t,"I") = I.L(t)/qref(t);
indices(t,"K") = K.L(t)/qref(t);

DISPLAY INDICES;

*         Define the domain over which the X-axis will be defined:

```

```
$setglobal domain t

*          Define the labels to be printed along the X axis:

set         tlbl(t) Time periods to be labelled in output plots /2010,2030,2050,2070/;

$setglobal labels tlbl

*          Place the key to the figures outside the graph (see GNUPLOT 3.7 Help File):

$setglobal gp_opt0 "set key outside"

*          Plot the graph with horizonal and vertical grid lines (see GNUPLOT 3.7 Help File):

$setglobal gp_opt1 "set grid"

*          Generate the plot to the screen -- it can subsequently be copied to the clipboard
*          using a right-click of the mouse, and then pasted into a separate program for
*          publication:

$if %batch%==yes $setglobal batch yes
$if %batch%==yes $setglobal gp_opt1 "set term postscript eps monochrome 'Times-Roman' 20"
$if %batch%==yes $setglobal gp_opt2 "set title"

$if %batch%==yes $setglobal gp_opt3 "set output 'ramsey1.eps'"
$setglobal gp_opt4 "set key outside width 4"

$libinclude plot indices

*          Repeat the report generation process a couple more times:

parameter       price       "Capital prices and wage rate";
price(t,"RK") = RK.L(t)/P.L(t);
price(t,"PK") = PK.L(t)/((1+r)*P.L(t));
price(t,"PL") = PL.L(t)/P.L(t);

$if %batch%==yes $setglobal gp_opt3 "set output 'ramsey2.eps'"
$libinclude plot price

parameter grates(t,*)  Growth rates through the transition;

grates(t,"c") = 100 * (C.l(t+1)/C.l(t) - 1);
grates(t,"i") = 100 * (I.l(t+1)/I.l(t) - 1);
grates(t,"k") = 100 * (K.l(t+1)/K.l(t) - 1);

set gs /c,i,k/;
grates(tl,gs) = na;

$if %batch%==yes $setglobal gp_opt3 "set output 'ramsey3.eps'"
$libinclude plot grates
```

Figure 17: Consumption, Investment and Capital Stock Indices

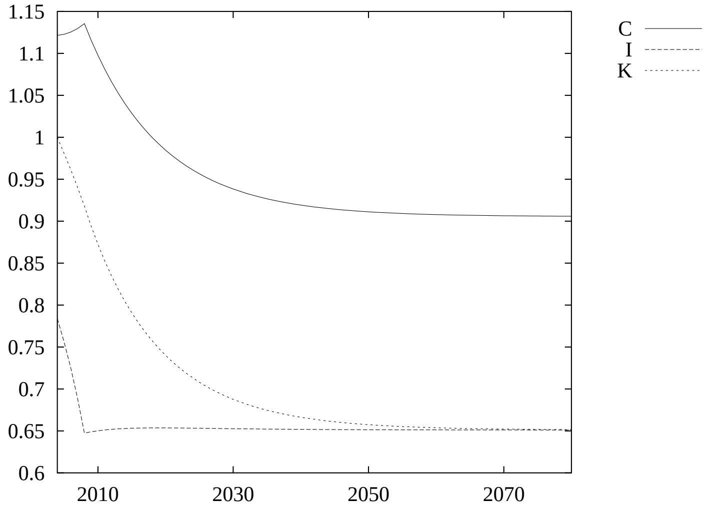


Figure 18: Capital Prices and Wage Rates

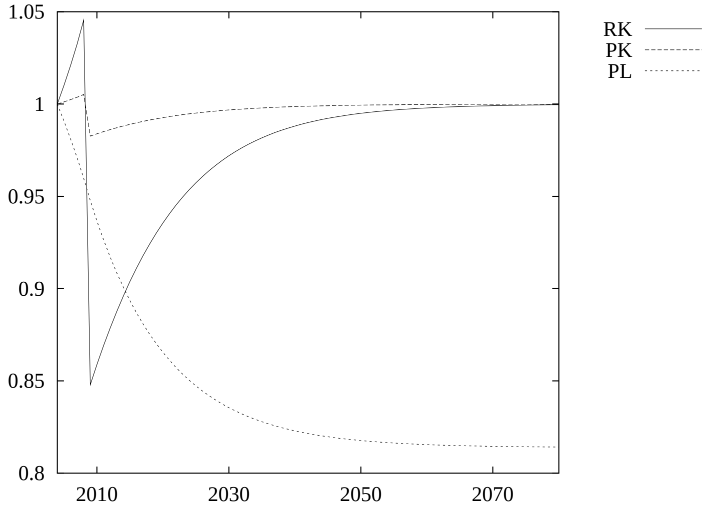
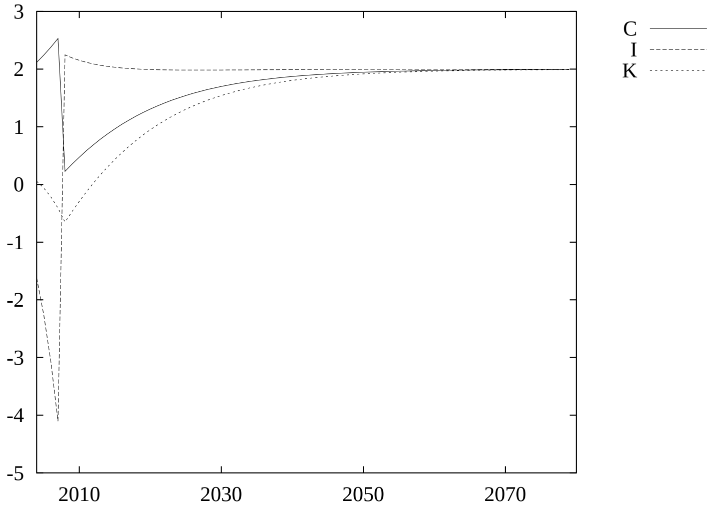


Figure 19: Growth Rates through the Transition



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