Notes on state variables choice and timing in the money-in-utility models[*]

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Abstract

In this note we demonstrate how minor changes in the timing of markets and an improper choice of a household problem state variable in the money-in-the-utility function models can dramatically change the equilibrium behavior of the model, its impulse response functions to shocks and simulated second-order moments. We also show how these minor changes in set-ups lead to different Fisher rules and money demands.

Abstrakt

V této práci ukazujeme, jak drobné změny v časování trhů nebo nesprávná volba stavové proměnné v optimalizačním problému domácnosti v monetárních modelech s penězi v preferenční funkci mohou dramaticky změnit rovnovážné chování modelu, jeho funkce impulzních odezev na šokové proměnné a simulované momenty druhého řádu. Také předvádíme, jak tyto drobné změny v uspořádání modelu vedou k odlišným formám tzv. Fisherova pravidla a poptávky po penézích.

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1   Introduction

Despite the long-lasting and broad use of the money-in-the-utility function general equilibrium models in monetary economics, the importance of the issues related to the timing of markets, trading, and money transfers, as well as the choice of state variables, has not still be fully appreciated and continues to be a source of confusion in monetary research.

In this note, as we said above, we focus on the money in the utility models. However, the similar issues appear to be present in the cash-in-advance models as well. We will demonstrate the issues related with differences in:

1. the choice of real wealth as a state variable

2. the timing of trading

3. the timing of markets and the timing of transfers.

4. the effect of alternative approaches to the log-linearization.

Ad 1) We will show that an improper choice of a state variable can significantly change the behavior of model equilibria.[1] As an example of this we will document the choice of real wealth as a state variable as in Walsh (2003).

Ad 2) Here we follow Carlstrom and Fuerst (2001) to show the implications of a model when the relevant money balances are those held at the beginning of period rather than those at the end of period, i.e. the "cash-in-advance" money balances are used instead of the "cash-when-I'm-done" balances as is standard in the most of the literature. Additionally to the result obtained by Carlstrom and Fuerst

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[1]While the improper choice, which violates elementary assumptions imposed on state variables, can easily happen during the derivation of a model, it is quite difficult to implement it when we solve the model numerically on the computer. For example, both the Uhlig toolbox - due to its explicit structure of the model equations - or Dynare - where state variables are chosen by the toolbox - do not allow to solve such an incorrectly specified model. So we demonstrate the properties of the model with the improper chosen state variables by the use of a general solution approach like in Blanchard and Kahn (1980) and Klein (2000).

(2001) that money demand under a proper trading arrangement is related just to the nominal interest rate, i_t, rather than to the "interest rate factor", $i_t/(1 + i_t)$. We also show that the Fisher equation takes more intuitive form and demonstrate significant quantitative implications of such changed trading on the model impulse response functions and the second-order moments of model variables. We also note that without an explicit sequencing of the asset and goods markets, which is often neglected in the specification of models, will lead to a misspecified money demand and the Fisher equation. The quantitative implications of such lack of the specification for the numerical solution are only marginal.

Ad 3) We replicate here and quantify the largely neglected Salyer (1991) result that not the timing of markets - Lucas (1982) versus Svensson (1985) - but the timing of money transfers matters in a representative agent framework.

Ad 4) We compare quantitatively different approaches to the log-linearization of the money in utility model. First, we replicate the results of Walsh (2003) by using the analytically derived "mixed log-linearization" often used in the New Keynesian literature [e.g. Gali (2008)], which uses linearization for rate variables in otherwise log-linearized model. Second, we solve the model by the use of the analytical version of a fully log-linearized model. And we also use the numerical log-linearization. We found significant differences in the size of the second-order moments between the mixed log-linearization and the proper log-linearization. The numerical log-linearization closely followed the results obtained in the properly log-linearized model.

The Walsh's (2003) book became one of the most used textbooks on the monetary economics. This book is widely used as the support material in the courses focused on the monetary theory and it provides an overview of the many topics in monetary policies. Despite the excellent overview provided, some sections of the book are missing more details in derivations. We believe, that some of the short-

cuts used in the book can confuse readers by omitting and not clearly defining the assumptions that are crucial for the following derivations.

Walsh (2003) tries to follows derivations in an illustrative form when he specifies the utility and production function so he is able to present impulse response functions. Walsh (2003) is also specific about the processes for distortion to technology process and money growth. However, he deviates in formulation of the money in utility model and its definition when searching for solution. Also, we find Walsh's (2003) reasoning for selecting wealth as the state variable misleading.

In the flexible price economy, a shock in the rate of growth of the money stock does not only changes the inflation rate. This shock also causes an immediate jump in the price level at the moment that households become realize that the rate of money growth has changed. In Walsh's (2003) statement of the problem, he choose the real wealth in as the state variable. However, when a shock hits this economy in such way that the current price level changes as a response to the resource reallocation then the state variable also changes. But this is inconsistent with the use of the real wealth as a state variable because induces inconsistency between the "state" variable value before and after the shock. This error appears in the following publications Walsh (1998) and Walsh (2003) (page 81).

Our criticism also focuses on the assumption (page 81) that is used to couple with the problem of the state variable choice and makes the money growth known. We object against the used of the state variable caries the the information on the shock realized in the same period. Walsh's (2003) choice of state variable assume not only knowledge of money growth but also of price level. Though, the price level is decided in the decision process.

The mentioned inconsistency in the monetary model implies that an impulse response to shocks may lead to jump responses in the present and the period after the shock realization. However, this contradicts rational expectation principle that

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allows for jump adjustment only as a response to unexpected shock and in the period of new information arrival. Therefore, there should be no further jumps in response.

Therefore, this note should be provide a correct solution to the model presented in the textbook. We explain the derivation of the model in more details. This note is organized as follows. First, the model is stated and solved.

2   Literature overview

Sidrauski's (1967b) model of money in the utility function in a neoclassical growth model is very popular. It is often the first monetary model in courses and textbooks [i.e, Walsh (2003) and Canova (2007)], and often it is the starting point for the monetary models that feature capital accumulation.

Sidrauski's (1967b) has extended the Ramsey model with the motive for holding money while with a full set of Arrow-Debreu claims, money is a redundant asses [Canova (2007)]. The household now has to decide on consuming, holding money and investing. It is assumed that agents derive utility from holding real money balances, otherwise money are not needed or their holdings are dominated by holdings of other assets. However, in this model money is super-neutral (money growth does not affect output growth). So the classical dichotomy holds.

Svensson (1985) proposes a simple extension of the ?) cash-in-advance model. Svensson's (1985) assumption is that consumers have to choose how much cash to hold before they observe the actual state of the world, that means that household is ignoring the current money supply or productivity shock in its decision process. As a result of this uncertainty/ignorance, household will chose more money to hold that it needs for purchase of its consumption.

Gali (2008) show the Fischer equation that implies that the nominal interest rate is responding one for one with the expected inflation, given that real interest

rate is determined by the real factors. However, Gali (2008) is not explicit about the fact that the real is based on the expectations about the real factors. Contrary to this, Obstfeld and Rogoff (1996) directly states that the Fischer equation is based on the expected real interest rate.

Obstfeld and Rogoff (1996) describe money in utility model in the context of the open economy. They assume that Fisher parity equates bond returns regardless their currency denomination and pay the same real interest rate. They point out, that after introduction of stochastic factor, the Fischer parity relating future inflation, actual nominal and real interest rate does not hold. The expected value of real return on the riskless bond should be used in analog to deterministic Fischer parity while the bond riskless in the nominal terms is not necessarily riskless in real terms. They also formulate condition when the stochastic Fischer equation can be reduced to its certainty equivalent. Further, the alternative motivations for money introduction are discussed by Obstfeld and Rogoff (1996). They use the store-of-value function of the money to introduce the model of dollarization. In this model, penalties for domestic transaction in foreign currency are not sufficient to discourage its use entirely.

3   The benchmark model

The basic money-in-the-utility function model that we present here is a stochastic discrete time version of the original deterministic continuous time Sidrauski (1967b) and Sidrauski (1967a) model.[2] Otherwise, we follow the model presented in Walsh (2003), so the household cares about consumption, leisure and money holdings at the end of period, i.e. $U(c, m', 1 - n)$ where c is consumption, m' is the relevant money holdings, n is working time, and $U_c, U_m > 0$ and $U_n < 0$. We assume that

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[2]However, in our derivation, we ignore the population growth and we will consider a problem of representative household. We ignore the population growth while a simple normalization that uses the size of population transforms the problem in to the per capita version.

there are two shocks: a technology shock, - which affects the productivity of the production of goods, z, and a monetary shock - which influences the growth of the aggregate monetary supply, u. The representative household, after observing the aggregate shocks at the beginning of period, decides on the current consumption, labor, and real money and bond holdings in order to solve the following maximization problem:

$$V(m, b, k, s) = \max_{c,n,m',b',k'} U(c, m', 1 - n) + \beta EV(m', b', k', s') \tag{1}$$

subject to the household budget constraint, where $0 < \beta < 1$ is subjective rate of discount. In here, the values with prime represent the values at the end of period, the state of the world in the current period is captured by $s = (z, u)$ with the technology shock and monetary growth shock given by z, and u, respectively.

The household budget constraint in the real terms is given as follows:

$$c + k' + \frac{M'}{P} + \frac{B'}{P} = f(z, k, n) + (1 - \delta)k + \frac{M}{P} + \frac{(1 + i)B}{P} + \tau \tag{2}$$

where P is the money price of goods in current period after shock s has been observed, M' are money held by household at the end of the period, B' are nominal private bonds held by the household at the end of period. In this formulation it is assumed that the bonds bought in previous period deliver nominal return, $1 + i$, at the current period so i is the nominal interest rate. The capital depreciates at rate of δ, and τ is the real value of lump-sum transfers to the household received before the household decision. These transfers are equal to the aggregate change in the money supply, so $P\tau = M' - M$, and the money supply of money grows at the rate θ', so it follows $1 + \theta' = \frac{M'}{M}$.

The aggregate output is produced according to the production function $y =$

$f(z, k, n)$ with the following form:

$$y = exp(z)k^\alpha n^{1-\alpha}, \tag{3}$$

where $0 < \alpha < 1$ and the total factor productivity process is defined as

$$z' = \rho_z z + \varepsilon_z' \tag{4}$$

with the serially uncorrelated productivity shock $\varepsilon_z \sim N(0, \sigma_z)$ and $0 \le \rho_z < 1$.

Defining the real money and bond holdings by $M/P \equiv m$ and $B/P \equiv b$, respectively, the budget constraint can be rewritten as follows:

$$c + k' + m' + b' = f(z, k, n) + (1 - \delta)k + \frac{m}{1 + \pi} + \frac{(1 + i)b}{1 + \pi} + \tau, \tag{5}$$

where the inflation rate, π, is the inflation rate between the current and past period. The law of motion for the aggregate real money supply is

$$m' = \frac{1 + \theta'}{1 + \pi'}m, \tag{6}$$

where the stochastic growth rate of money supply is $\theta' = \bar{\theta} + u'$, with the steady state growth of money $\bar{\theta}$. The disturbance to the growth rate of money u is given by:

$$u' = \gamma_u u + \gamma_z z + \varepsilon_u', \tag{7}$$

where the serially uncorrelated money growth shock $\varepsilon_u \sim N(0, \sigma_u^2)$ and the correlation parameters $0 \le \gamma_z, \gamma_u < 1$.

3.1 First order conditions

Taking the derivatives with respect to the control variables in the household problem above we get the following set of the first order conditions:

$$U_c - \lambda = 0 \tag{8}$$

$$U_{m'} + \beta E V_m' - \lambda = 0 \tag{9}$$

$$U_x - \lambda f_n = 0 \tag{10}$$

$$\beta E V_b' - \lambda = 0 \tag{11}$$

$$\beta E V_k' - \lambda = 0 \tag{12}$$

Further, the envelope theorem implies the following conditions:

$$
\begin{aligned}
V_k &= \lambda\left(f_k + 1 - \delta\right) \\
V_m &= \lambda\frac{1}{1+\pi} \\
V_b &= \lambda\frac{1+i}{1+\pi}
\end{aligned}
$$

After plugging the envelope conditions into the equations (8)–(12) we obtain the following first order conditions:

$$
\begin{aligned}
\frac{U_x}{U_c} &= f_n \\
U_c &= \beta E\left[U_c'\left(f_k' + 1 - \delta\right)\right] \\
U_c &= \beta E\left[\frac{U_c'(1 + i')}{1 + \pi'}\right] \\
\frac{U_{m'}}{U_c} &= 1 - \beta E\left[\frac{U_c'}{(1 + \pi')\,U_c}\right].
\end{aligned}
$$

Additionally the last set of the first order conditions is given by the transver-

sality conditions

$$\lim_{t \to \infty} \beta^t U_{ct} m_t = 0$$

$$\lim_{t \to \infty} \beta^t U_{ct} b_t = 0$$

$$\lim_{t \to \infty} \beta^t U_{ct} k_t = 0.$$

Let us denote by Ξ the quadruplet of the state variables, i.e. $\Xi \equiv (m, b, k, s)$, then the general equilibrium is a set of policy functions $(c(\Xi), m'(\Xi), n(\Xi), b'(\Xi), k'(\Xi))$, pricing functions $(P(\Xi), i(\Xi))$, and the value function $V(\Xi)$ contingent on the the state of the economy, Ξ.

Using these functions the first order conditions can be rewritten as

$$\frac{U_x [c(\Xi), m'(\Xi), 1 - n(\Xi)]}{U_c [c(\Xi), m'(\Xi), 1 - n(\Xi)]} = f_n [z, k, n(\Xi)]$$

$$U_c [c(\Xi), m'(\Xi), 1 - n(\Xi)] = \beta E \{ U_c [c(\Xi'), m'(\Xi'), 1 - n(\Xi')]$$

$$\times (f_k [z', k'(\Xi), n(\Xi')] + 1 - \delta) \}$$

$$U_c [c(\Xi), m'(\Xi), 1 - n(\Xi)] = \beta E \left\{ \frac{U_c [c(\Xi'), m'(\Xi'), 1 - n(\Xi')] [1 + i'(\Xi)]}{1 + \pi(\Xi')} \right\}$$

$$(13)$$

$$\frac{U_{m'} [c(\Xi), m'(\Xi), 1 - n(\Xi)]}{U_c [c(\Xi), m'(\Xi), 1 - n(\Xi)]} = 1 - \beta E \left\{ \frac{U_c [c(\Xi'), m'(\Xi'), 1 - n(\Xi')]}{[1 + \pi(\Xi')] U_c [c(\Xi), m'(\Xi), 1 - n(\Xi)]} \right\}$$

$$(14)$$

Equation (13) can be further rearranged and the Euler equation takes following form:

$$\beta E \left\{ \frac{U_c [c(\Xi'), m'(\Xi'), 1 - n(\Xi')]}{[1 + \pi(\Xi')] U_c [c(\Xi), m'(\Xi), 1 - n(\Xi)]} \right\} = \frac{1}{1 + i'(\Xi)} \qquad (15)$$

and using equation (14) we get the money demand function depending on the

interest rate factor, $\frac{i'(\Xi)}{1+i'(\Xi)}$, prevailing between the current and next period:

$$\frac{U_{m'}\left[c(\Xi), m'(\Xi), 1 - n(\Xi)\right]}{U_c\left[c(\Xi), m'(\Xi), 1 - n(\Xi)\right]} = \frac{i'(\Xi)}{1 + i'(\Xi)}. \tag{16}$$

Using this notation, the first order conditions can be rewritten:

$$U_x(\Xi) = U_c(\Xi)f_n(\Xi)$$

$$y(\Xi) = f(\Xi)$$

$$y(\Xi) = c(\Xi) + k'(\Xi) - (1 - \delta)k$$

$$r(\Xi) = f_k(z', k, n(\Xi')) - \delta$$

$$E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}r(\Xi')\right\} = 1$$

$$E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}r(\Xi')\right\} = E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}\frac{1}{1 + \pi(\Xi')}\right\}[1 + i'(\Xi)] \tag{17}$$

$$\frac{U_{m'}(\Xi)}{U_c(\Xi)} = \frac{i'(\Xi)}{1 + i'(\Xi)}$$

$$m'(\Xi) = m\frac{1 + \theta'}{1 + \pi(\Xi)}$$

where the marginal products are as:

$$f_n(\Xi) = (1 - \alpha)\frac{y(\Xi)}{n(\Xi)} \tag{18}$$

$$f_k(\Xi) = \alpha\frac{y(\Xi)}{k}. \tag{19}$$

In the equations above we used the fact that the amount of private bonds held in equilibrium, $b = 0$. Additionally, if we substitute this fact into the household budget constraint together with money transfers we get the resource constraint of the economy $y = c + k' - (1 - \delta)k$.

Notice that equation (17) is a general form of the Fisher equation (or Fisher parity, as in Obstfeld and Rogoff (1996)) which states that the expected real return

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on capital must be equal to the expected real return on nominal bonds properly evaluated, as there is no arbitrage on the asset market. Note the odd result which says that the next period expected real return on nominal bonds is derived from the current period nominal interest rate rather than from the next period one albeit the relevant inflation rate is the next period one.

3.2 Functional form specifications

While in this note the model from the Walsh (2003) is followed, an utility function non-separable in money holdings and consumption is assumed in the following form:

$$U(c', m', 1 - n') \;=\; \frac{[ac'^{1-b} + (1-a)m'^{1-b}]^{\frac{1-\Phi}{1-b}}}{1 - \Phi} + \Psi \frac{(1 - n')^{1-\eta}}{1 - \eta},$$

where $0 < a < 1$; $0 < b, \eta, \Phi, \Psi$; and $b, \eta, \Phi \neq 1$. It can be shown that for $\Phi = b = 1$ the utility function becomes logarithmic. This form of utility function was for the first time used by Lucas (2000). In here, b is closely related to the inverse of interest rate elasticity of money demand while the interest rate elasticity of money demand is given $\frac{\partial m}{\partial i} \frac{i}{m} = -\frac{1}{b} \frac{1}{1+m}$

As in the Walsh (2003), we define

$$\chi(\Xi) = [c(\Xi)]^{1-b} \left\{ a + (1-a) \left[\frac{m'(\Xi)}{c(\Xi)} \right]^{1-b} \right\}$$

to simplify the following derivations. So, for the aforementioned specification of the utility function form follows:

$$U_x(\Xi) = \Psi \left[1 - n(\Xi) \right]^{-\eta}$$

$$U_c(\Xi) = a \left[X(\Xi) \right]^{\frac{b-\phi}{1-\phi}} \left[c(\Xi) \right]^{-b}$$

$$\frac{U_{m'}(\Xi)}{U_c(\Xi)} = \left(\frac{1-a}{a} \right) \left[\frac{m'(\Xi)}{c(\Xi)} \right]^{-b}$$

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3.3 Steady state

To assess the properties of this model, we consider the economy in the steady state equilibrium, where we assume that shocks are not present. Therefore, to solve for the steady state we set $\varepsilon_z = \varepsilon_u = 0$. Moreover, in the steady state equilibrium technology is constant $(exp(\bar{z}) = 1)$ and the nominal money grows at the rate $\bar{\theta}$. For the steady state of transfers, from the law of motion of the real money supply it follows:

$$\bar{m} = \frac{1+\bar{\theta}}{1+\bar{\pi}}\bar{m}.$$

While money demand by household implies non-zero holdings of money, it follows that $\bar{\pi} = \bar{\theta}$. So the steady state inflation is pure monetary phenomena. The steady state of transfers is given by $\bar{\tau} = \bar{m} - \bar{m}/(1+\bar{\theta}) = \bar{\theta}\bar{m}/(1+\bar{\theta})$.

To solve for the equilibrium, we assume that bond market clears, and $\bar{b} = 0$ hence in steady steady For the real side of economy, we assume $\bar{y} = \bar{k}^\alpha \bar{n}^{1-\alpha}$, the budget constraint reduces to $\bar{c} = \bar{y} + \delta\bar{k}$. And from the first order condition (11), for the real rate of return follows that $1 + \bar{r} = \beta^{-1}$. Therefore the Fisher equation implies for the nominal interest rate that $\bar{i} = (1+\bar{\theta})\beta^{-1} - 1$.

So the evaluation of the first order conditions at steady state gives following equations:

$$\bar{x} = [a\bar{c}^{1-b} + (1-a)\bar{m}^{1-b}] \tag{21}$$

$$0 = a\bar{c}^{-b}\bar{x}^{\frac{b-\Phi}{1-b}} - \bar{\lambda} \tag{22}$$

$$0 = \frac{1}{1+\bar{i}} + \frac{1-a}{a}\left(\frac{\bar{m}}{\bar{c}}\right)^{-b} - 1 \tag{23}$$

$$0 = -\Psi(1-\bar{n})^{-\eta} + \bar{\lambda}(1-\alpha)\frac{\bar{y}}{\bar{n}} \tag{24}$$

$$0 = \beta \frac{1 + \bar{i}}{1 + \bar{\pi}} - 1 \tag{25}$$

$$0 = \beta \alpha \bar{k}^{\alpha-1} \bar{n}^{\alpha} - \delta + 1 \tag{26}$$

$$0 = \bar{y} - \bar{c} - \delta \bar{k} \tag{27}$$

Due to the non-separability of the utility function, we follow Walsh (2003) in use of the ratios to describe the steady state of the model, therefore we present derivations of these ratios. These ratios are also used to derive the log-linear approximation. In these derivations, we use the fact that in the steady state inflation equals money growth rate and that for the real return on capital is inverse of household's discount factor. From production function definition $y'/k = (n'/k)^{1-\alpha}$. From equation (12) and knowing that $f_k(z', k, n') = \alpha f(z', k, n')/k$; and $y/k = (n/k)^{1-\alpha}$ in steady state, for the output-capital and labor-capital ratio it follows:

$$
\begin{aligned}
\frac{\bar{y}}{\bar{k}} &= \frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right), \\
\frac{\bar{n}}{\bar{k}} &= \left(\frac{1}{\alpha}\left(\frac{1}{\beta} - 1 + \delta\right)\right)^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

By use of the equation (24) and aforementioned ratios, the equation for the steady state of labor supply can be derived as follows:

$$
\bar{n} = \left(\frac{1}{\Psi}(1-\alpha)\frac{\bar{y}}{\bar{k}}\left(\frac{\bar{n}}{\bar{k}}\right)^{-1}\right)^{\frac{-1}{\eta}}.
$$

Further, equation (11) can be rewritten $\beta E[\lambda' \frac{1}{1+\pi'}] = \frac{\lambda_t}{1+i}$, substituting this into the equation (9) and replacing λ with $U_c(c', m', 1 - n')$ as it follows from equation (8), we get $U_c(c', m', 1 - n')\frac{-i}{1+i} = -U_m(c', m', 1 - n')$. Substituting the functional forms, dividing by x' and rearranging, it follows for the money-consumption ratio

in the steady state:

$$\frac{\bar{i}}{1+\bar{i}} = \frac{1-a}{a}\left(\frac{\bar{m}}{\bar{c}}\right)^{-b}.$$

(28)

When evaluated at steady state and nominal interest rate is plugged in, the equation (28) implies:

$$\frac{\bar{m}}{\bar{c}} = \left(\frac{(1+\theta)(1+\beta^{-1})-1}{(1+\theta)(1+\beta^{-1})}\right)^{-\frac{1}{b}}\left(\frac{a}{1-a}\right)^{-\frac{1}{b}}.$$

(29)

The budget constraint evaluated at steady state gives $\bar{y}/\bar{k} = \bar{c}/\bar{k} - \delta$. Plugging for consumption-capital ratio into the steady state money-consumption ratio, gives the following formula for money-capital ratio:

$$\frac{\bar{m}}{\bar{k}} = \left(\frac{(1+\theta)(1+\beta^{-1})-1}{(1+\theta)(1+\beta^{-1})}\right)^{-\frac{1}{b}}\left(\frac{a}{1-a}\right)^{-\frac{1}{b}}\frac{\bar{c}}{\bar{k}}.$$

(30)

3.4 Calibration

To replicate the results of model by Walsh (2003) and to create a benchmark model, we use the following structural parameters for all the analyzed models:

α	β	γ_u	γ_z	δ	η	Φ	θ	ρ	σ^u	σ^z	a	b
0.36	0.989	0.5	0.0	0.019	1	2	0.0125	0.95	0.007	0.0089	0.95	2.56

Table 1: Baseline Parameters Values

Further, in the log-linearized models the setting of weight of leisure in the utility Ψ can be replaced by setting the steady state fraction of time devoted to work. In here, the steady state value of employment n is set to $1/3$. For the model of numerical approximation the Ψ is set to 1.447 that leads to steady state value of labor $n = 0.3333$.

The linearized solutions rely on the ratios of variables, these ratios are reported in the table (2). The values presented coincides with values reported by Walsh

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(2003) in the Table 2.3.

$\frac{\bar{y}}{\bar{k}}$	$\frac{\bar{n}}{\bar{k}}$	$\frac{\bar{m}}{\bar{c}}$	$\frac{\bar{m}}{\bar{k}}$	$\frac{\bar{c}}{\bar{k}}$
0.0837	0.0207	1.3762	0.0890	0.0647

Table 2: Steady state ratios

3.5 Simulation results

In this section we compare quantitatively different approaches to the log-linearization of the money in utility model. First, we replicate the results of Walsh (2003) by using the analytically derived "mixed log-linearization" used often in the literature.[3] This approach, often used in the New Keynesian literature (see Gali), uses linearization for rate variables in otherwise log-linearized model. Second, we solve the model by the use of the analytical version of a fully log-linearized model. And we also use the numerical log-linearization of the model by using the Dynare toolbox developed by Juillard (1996). We solve these three versions of the model and compare their impulse response functions to monetary shock and we also the differences between the implied second-order moments.

Because we also solved model by method presented Juillard (1996), we are able to obtain the steady state values of the endogenous variables.

$\bar{c}$	$\bar{k}$	$\bar{\lambda}$	$\bar{m}$	$\bar{n}$	$\bar{i}$	$\bar{\pi}$	$\bar{r}$	$\bar{y}$
1.0399	16.0798	0.8401	1.4311	0.3333	0.0238	0.0125	0.0111	1.3454

Table 3: Steady state values

In here, we also compare the differences that occur when relative deviations for all variables are computed with the approach where inflation and interest rates approximations are just percentage deviations. The derivation of log-linearized version of the model that uses relative approximations for all variables is presented

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[3] The Matlab code for such model is available on the web as a supplement to the book. This code uses the method described by Uhlig (1995) to solve the model.

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in the Appendix (A). The presented numerical approximation in here is also based on relative deviations for all variables.

Table (4) compares the standard deviations of the models. As can be seen the model presented by Walsh (2003) (denoted as Original) delivers low volatility of money holdings and inflation in comparison with the results obtained by numerical approximation (denoted as Numerical). Also, it delivers much higher volatility of nominal interest rate than the other presented models. As the reason for this difference, the typo in money demand equation is identified, so the original code is corrected (denoted Typo).

Further, we log-linearize the model using the presented first order conditions (Replication) and the results coincides with those obtained by corrected model. Finally, the log-linearized model (Relative) where all variables are computed as relative deviations from their steady states. The derivation of the relative approximation is presented in the Appendix A. The version with relative approximation of all variables reveals lower values for inflation, nominal and real interest rate. However, the pattern of steady state returns is in accordance with numerical solution

	Original	Typo correct.	Replication	Relative	Numerical
Capital	0.270	0.270	0.270	0.273	0.273
Money holdings	0.262	0.744	0.736	1.027	0.736
Output	1.088	1.087	1.088	1.088	1.090
Consumption	0.234	0.235	0.235	0.234	0.235
Employment	0.315	0.314	0.315	0.313	0.315
Nominal rate	0.334	0.044	0.044	0.024	0.043
Real rate	0.030	0.030	0.034	0.027	0.030
Inflation	0.899	1.407	1.407	1.260	1.390
Investment	4.039	4.037	4.038	4.043	4.045

Table 4: Standard deviations

The differences in the standard deviations originates from the from the change of the impulse response function to monetary shock. The following figures shows the impulse response of the inflation and nominal interest rate in these models. Figures (1) and (2) show the response of the original model (blue dashed line), corrected

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model (green dash-dotted line), relative (black dotted line) and numerical solution (red solid line). In here, we omit response from model replication while it coincides with the model after correction. The results presented in Figure (1) show that up to the first period, the responses of inflation are significantly muted.
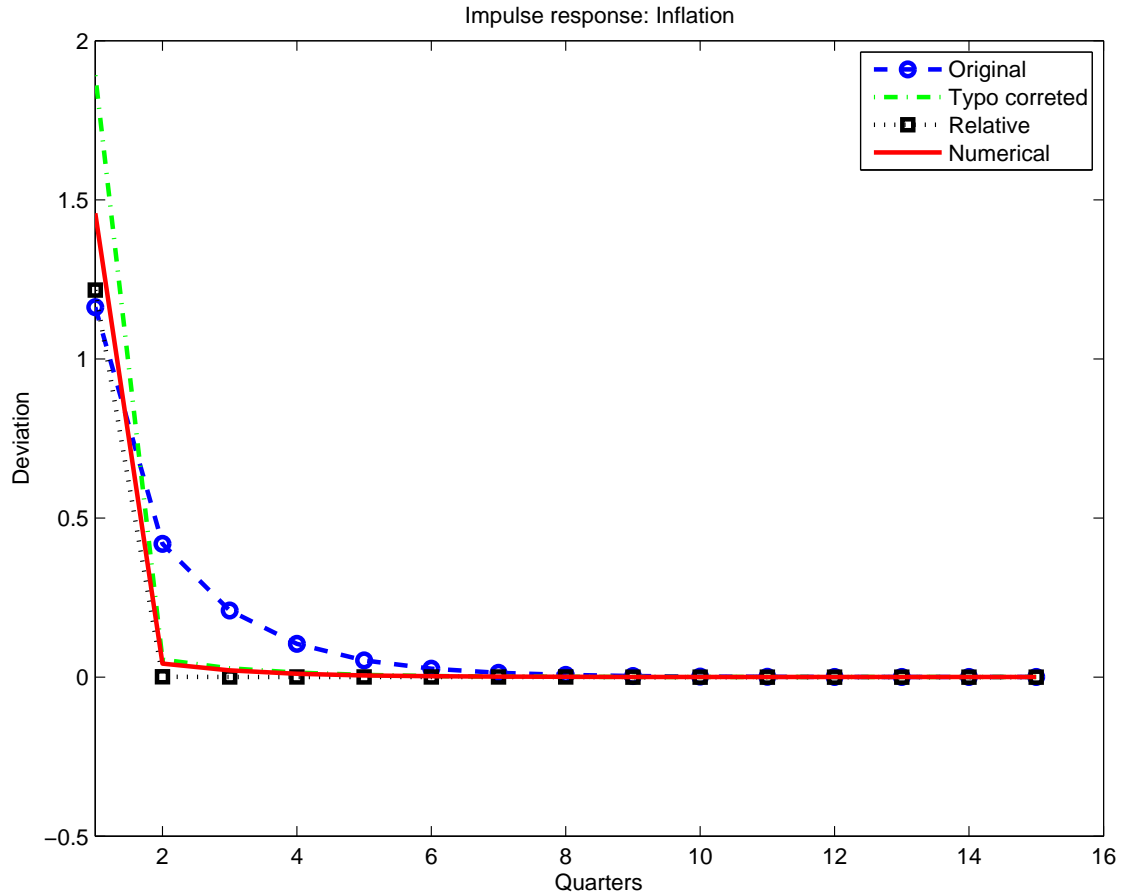

Figure 1: Inflation response to money growth shock

The increase in the money supply can have two effects. First, it can reduce the real interest rate while more liquidity leads excess supply of money, therefore pressures for lower price of money occur which is equivalent to lowering the interest rate, this is called the "liquidity effect". Second, the foreseen higher future inflation leads via the Fischer equation to increase in the nominal interest rate, this is called Fisher effect.

Therefore to generate a decrease of the nominal interest rate in response to

a increase in a supply of money, it is required that the liquidity effect outweighs the Fisher effect. However, in in the presented model money does not influence real variables such as the real interest rate. It only influences inflation, therefore increases in the money supply leads to higher inflation and so the nominal interest rate rises as there is only a Fisher effect.
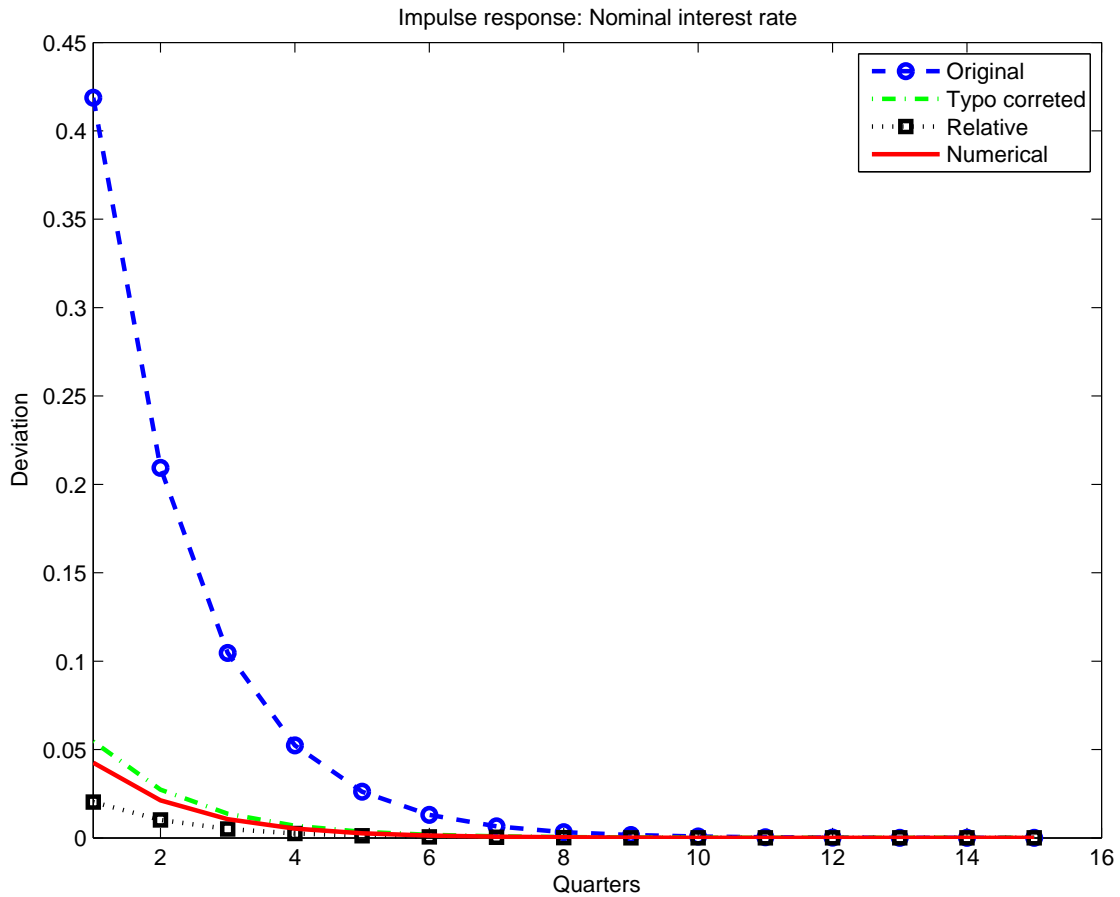

Figure 2: Nominal interest rate response to money growth shock

The results presented in the Figure (2) show that the response of nominal interest rate is significantly muted in comparison with the original model. Therefore, it signals that the reports reported in Walsh (2003) overvalue the Fischer effect. In the following chapter, we show how to generate some liquidity effect (lower nominal interest rates for higher money supply) by introduction of explicit timing constraints.

Finally, to complete our comparison with the original model as in Walsh (2003), we show the correlations of variables with the output. The results of calculations are summarized in the correlation table. The results shown in Table (5) reveal

	Original	Typo correct.	Replication	Relative	Numerical
Capital	0.27	0.27	0.27	0.27	0.27
Money holdings	0.82	0.82	0.25	0.25	0.25
Output	1.00	1.00	1.00	1.00	1.00
Consumption	0.95	0.95	0.95	0.95	0.95
Employment	0.89	0.89	0.89	0.89	0.89
Nominal rate	0.06	0.06	0.05	0.05	0.05
Real rate	0.97	0.97	0.96	0.96	0.96
Inflation	-0.09	-0.09	-0.06	-0.06	-0.06
Investment	1.00	1.00	1.00	1.00	1.00

Table 5: Correlation with output

that the correction in the amplitude of inflation does still delivers high money and output correlation in comparison with rest of the models. The rest of the correlation parameters is consistent across the model derivation, linearization and solution techniques.

4 Real Wealth as a State

In the previous section, we solved the model by setting money balances and capital as the state variables. However, Walsh (2003) states in the section 2.7.1 that the state of the system can be summarized by the real financial wealth. To solve the model we have to redefine the optimization problem. Household solves the following problem:

$$V(a, k, s) = \max_{c, n, k', m', b'} U(c, m', 1 - n) + \beta E V(a', k', s'), \qquad (31)$$

where maximization is over (c, n, k', m', b') and subject to the modified budget constraint:

$$y + \frac{M}{P} + \frac{(1+i)B}{P} + \tau = c + k' - (1-\delta)k + \frac{M'}{P} + \frac{B'}{P}.$$

In here, the real financial wealth of household is $\omega' = \frac{M'}{P'} + \frac{(1+i')B'}{P'} + \tau'$, transfers τ are received at the beginning of the period, and s is a set of shocks that includes monetary and technology shock.

The modification follows by defining the real money balances as $m' \equiv M'/P$ and real bond holdings $b' \equiv B'/P$. Using these two definitions, we can state the real budget constraint as follows:

$$y' + (1-\delta)k + \omega = c' + k' + m' + b' \tag{32}$$

$$\omega' = \frac{m'}{1+\pi'} + \frac{(1+i')b'}{1+\pi'} + \tau'. \tag{33}$$

Differentiating with respect to c, n, k', m', and b' the first order necessary conditions can be stated as follows:

$$0 = U_c(c, m', 1-n) - \lambda$$

$$0 = U_n(c, m', 1-n) - \lambda f_n(k, n)$$

$$0 = \beta E V_k(a', k', s')' - \lambda$$

$$0 = \beta E \left[\frac{V_a'(a', k', s')}{1+\pi'} \right] - \lambda$$

$$0 = U_{m'}(c, m', 1-n) + \beta E \left[\frac{V_a'(a', k', s')(1+i')}{1+\pi'} \right] - \lambda \tag{34}$$

The envelope theorem gives following two equations:

$$V_\omega(\omega', k', s') = \lambda$$

$$V_k(\omega', k', s') = \lambda \left(f_k(k, n) + 1 - \delta \right).$$

Plugging these two equations into the system of equations (34), the necessary conditions are transformed to the following system:

$$
\begin{aligned}
\frac{U_n(c, m', 1 - n)}{U_c(c, m', 1 - n)} &= \lambda f_n(k, n) \\
U_c(c, m', 1 - n) &= \beta E \left[U_c'(c, m', 1 - n) \left(f_k'(k, n) + 1 - \delta \right) \right] \\
\frac{U_{m'}(c, m', 1 - n)}{U_c(c, m', 1 - n)} &= 1 - \beta E \left[\frac{U_c'(c, m', 1 - n)}{(1 + \pi') \, U_c(c, m', 1 - n)} \right] \\
U_c(c, m', 1 - n) &= \beta E \left[\frac{U_c'(c, m', 1 - n)(1 + i')}{1 + \pi'} \right]
\end{aligned}
$$

Further, the first order conditions are completed with the resource constraint:

$$k' = z f(k, n) + (1 - \delta)k - c,$$

the description of the evolution of the aggregate money supply in real terms:

$$m' = \frac{1 + u}{1 + \pi} m,$$

and the productivity and money growth shock processes:

$$
\begin{aligned}
z' &= \rho_z z + \varepsilon_z{}' \\
u' &= \gamma_u u + \gamma_z z + \varepsilon_u{}',
\end{aligned}
$$

where the same restriction for parameters ρ_z, γ_u and γ_z as in the previous apply.

In this model, we would like to be very specific about the timing of the variables. Therefore, to stress the timing we rewrite first order conditions in terms of variables as functions contingent on the state of the economy:

$$\frac{U_n\left[c(\omega,k,s),m'(\omega,k,s),1-n(\omega,k,s)\right]}{U_c\left[c(\omega,k,s),m'(\omega,k,s),1-n(\omega,k,s)\right]} = f_n\left(k,n\right)$$

$$U_c\left[c(\omega,k,s),m'(\omega,k,s),1-n(\omega,k,s)\right] =$$

$$\beta E[U_c\left[c(\omega',k',s'),m'(\omega',k',s'),1-n(\omega',k',s')\right] \times \left(f_k\left(k',n'\right)+1-\delta\right)]$$

$$\frac{U_{m'}\left[c(\omega,k,s),m'(\omega,k,s),n(\omega,k,s)\right]}{U_c\left[c(\omega,k,s),m'(\omega,k,s),1-n(\omega,k,s)\right]} =$$

$$1-\beta E\left[\frac{U_c\left[c(\omega',k',s'),m'(\omega',k',s'),1-n(\omega',k',s')\right]}{\left[1+\pi\left(\omega',k',s'\right)\right]U_c\left[c(\omega,k,s),m'(\omega,k,s),1-n(\omega,k,s)\right]}\right]$$

$$U_c\left[c(\omega,k,s),m'(\omega,k,s),1-n(\omega,k,s)\right] =$$

$$\beta E\left[\frac{U_c\left[c(\omega',k',s'),m'(\omega',k',s'),1-n(\omega',k',s')\right]\left[1+i'(\omega,k,s)\right]}{1+\pi\left(\omega',k',s'\right)}\right].$$

$$(35)$$

5   Alternative timing: CWID versus CIA

In this section, we analyze the implications of different timing of money balances relevant for the utility function in these models.

The most common setup of money timing used in monetary model is the one labeled by Carlstrom and Fuerst (2001) with label "cash-when-I'm-done" (CWID). This timing assumes that money that delivers utility are those at the end of the period. This means money that held after receiving transfers, selling bonds, and purchasing consumption.

In this model of timing there are segregated good and financial markets. However, households still consist of a worker-shopper pair. The typical period can be described by the following steps. At first, household collects information on shocks and the prices. Then asset markets are opened, household receives payoff from its portfolio chosen in the previous period and receives transfers/tax from the monetary authority. Households decides on a new portfolio and asset market closes. Further, good market opens. Worker part of household goes to the market

to sell their own product; shopping part goes to the market to buy consumption good with money. After, the good market is closed household consumes and enjoy money held.

Nevertheless, as Carlstrom and Fuerst (2001) point out, it is very difficult to justify CWID timing on logical grounds. As the CWID supposes that consumption expenditures are restricted, not by the money held as the consumer enters markets, but instead by the money held after consumers leaves the goods market, and after even more transactions in the asset market. Following Carlstrom and Fuerst (2001) in the next Section, we will assume a more natural specification of preferences, where money services are derived from the money holding before the transactions happens rather than after them. This set up is consistent with the so-called cash-in-advance timing.

5.1   CIA Timing of Used Money Balances

With CWID timing current income is included as part of current money balances. As Carlstrom and Fuerst (2001) state this violates Clower's (1967) dictum that "money buys goods, and goods buy money but goods do not buy goods".

An obvious candidate for an alternative timing that is immune to this critique is "cash-in-advance"(CIA) timing. Under this timing money accumulated in previous periods are used in the current period transactions [e.g. Clower (1967), Lucas (1980), and Svensson (1985)].

The budget constraint is unchanged and is as follows:

$$c + k' + m' + b' = zf(k,n) + (1 - \delta)k + \frac{m}{1 + \pi} + \frac{(1 + i)b}{1 + \pi} + \tau, \qquad (36)$$

but the relevant money balances are those after the trades in asset market are

settled. Therefore, we define a new variable d as follows

$$d = \frac{m}{1+\pi} - b' + \frac{(1+i)b}{1+\pi}, \tag{37}$$

where d are the money holdings that deliver the utility to household. As in the previous section, the money balances, capital and bond holdings are chosen to be the state variable. With these state variables, the household maximization problem can be stated as follows:

$$V(m, b, k, s) = \max_{c,d,n,k',m',b'} U(c, d, 1-n) + \beta E V(m', b', k', s') \tag{38}$$

subject to the budget constraint (36) and the equation for money balances (37).

As in the solution of the standard money in utility model, the first order necessary conditions with respect to control variables c, d, n, k', b' and m' can be stated as follows:

$$
\begin{aligned}
U_c - \lambda &= 0 \\
U_d - \mu &= 0 \\
U_x - \lambda z f_n &= 0 \\
\beta E V'_k - \lambda &= 0 \\
\beta E V'_b - \lambda - \mu &= 0 \\
\beta E V'_m - \lambda &= 0.
\end{aligned}
$$

The envelope theorem for the state variables k, m and b gives following three equations:

$$
\begin{aligned}
V_k &= \lambda \left(z f_k + 1 - \delta \right) \\
V_m &= (\lambda + \mu) \frac{1}{1+\pi}
\end{aligned}
$$

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$$V_b = (\lambda + \mu) \frac{1+i}{1+\pi}. \tag{39}$$

By plugging the equations (39) into the equations (39), the first order conditions can be rewritten as follows:

$$\frac{U_x}{U_c} = z f_n$$

$$U_c = \beta E \left[U_c' \left(f_k' + 1 - \delta \right) \right]$$

$$U_c + U_d = \beta E \left[\frac{\left(U_c' + U_d' \right) \left(1 + i' \right)}{1 + \pi'} \right]$$

$$U_c = \beta E \left[\frac{U_c' + U_d'}{1 + \pi'} \right]. \tag{40}$$

The model is closed with the following resource constraint:

$$k' = z f(k, n) + (1 - \delta)k - c,$$

the evolution of the aggregate money supply in real terms as given by the equation (6), and two shock processes given as in the standard model by equations (7) and (4).

Again, we would like to be more specific about timing of variables in the equations (40), therefore we express the first order conditions using the variables explicitly declared as functions contingent on the state of the economy:

$$\frac{U_x \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right]}{U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right]} = z f_n \left[k, n(\Xi) \right]$$

$$U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right] = \beta E \{ U_c \left[c(\Xi'), d(\Xi'), 1 - n(\Xi') \right]$$
$$\times \left(z' f_k \left[k'(\Xi), n(\Xi') \right] + 1 - \delta \right) \}$$

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$$\beta E \left\{ \frac{\left[U_c\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right] + U_{\hat{m}}\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right]\right]\left[1 + i'(\Xi)\right]}{\left[U_c\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right] + U_{\hat{m}}\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]\right]\left[1 + \pi(\Xi')\right]} \right\}$$
$$= 1 \quad (41)$$

$$\beta E \left\{ \frac{U_c\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right] + U_{\hat{m}}\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right]}{U_c\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]\left[1 + \pi(\Xi')\right]} \right\}$$
$$= 1 \quad (42)$$

where as in the derivation of the standard model Ξ is the consolidated the vector of states, i.e. $\Xi \equiv (m, b, k, s)$. In here, it can be seen that due to CIA constraint, households must carry on money one period in advance to deliver utility, as the intertemporal constraint changes in comparison with the Euler equation (15).

Equation (41) can be further expressed as

$$\beta E \left\{ \frac{\left[U_c\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right] + U_{\hat{m}}\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right]\right]}{U_c\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]\left[1 + \pi(\Xi')\right]} \right\} \left[1 + i'(\Xi)\right]$$
$$= 1 + \frac{U_{\hat{m}}\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]}{U_c\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]}$$

and so if we divide it by (42) we get

$$\frac{U_{\hat{m}}\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]}{U_c\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]} = i'(\Xi)$$

which is the money demand function depending on the interest rate, $i'(\Xi)$, prevailing between the current and next period. The detailed derivation of the model using the functional forms for the production function and the utility function are provided in the Appendix (B).

5.2 Simplified CIA Timing of Used Money Balances

In this section, we analyze the effect of another kind of simplifications which often appears in the literature. It is the situation when the trade in nominal bonds is explicitly introduced in the model economy, however, the trade structure - the

trades on the asset and goods markets - is neglected. While this neglect has no effect on the results in the model with CWID timing it has an important effect in the model with CIA timing.[4]

As in the previous section, household solves the following maximization problem

$$V(m, b, k, s) = \max_{c,d,n,m',b',k'} U(c, d, 1 - n) + \beta EV(m', b', k', s')$$

subject to the standard budget constraint as given by equation (36). However, in this model the money that deliver utility are given by the following equation:

$$d = \frac{m}{1 + \pi}.$$

Similarly, as in the previous cases the first order condition are derived with respect to control variables c, d, n, k', m', and b' as follows:

$$
\begin{aligned}
U_c - \lambda &= 0 \\
U_d - \mu &= 0 \\
U_x - \lambda z f_n &= 0 \\
\beta EV_k' - \lambda &= 0 \\
\beta EV_m' - \lambda &= 0 \\
\beta EV_b' - \lambda &= 0
\end{aligned}
$$

and the envelope theorem implies following equations:

$$V_k = \lambda \left(z f_k + 1 - \delta \right)$$

---

[4]The same problem it creates in the models with money introduced via the cash-in-advance model.

$$V_m = \lambda \frac{1}{1+\pi} + \mu \frac{1}{1+\pi}$$
$$V_b = \lambda \frac{1+i}{1+\pi}.$$

Using these equations the first order conditions can rewritten as follows:

$$\frac{U_x}{U_c} = z f_n$$
$$U_c = \beta E \left[U_c' \left(f_k' + 1 - \delta \right) \right]$$
$$U_c = \beta E \left[\frac{U_c'(1+i')}{1+\pi'} \right]$$
$$U_c = \beta E \left[\frac{U_c' + U_d'}{1+\pi'} \right]$$

As in the previous section, the model is closed with the following resource constraint, the evolution of the aggregate money supply (6), and two shock processes given as in the standard model by equations (7) and (4). We also rewrite the necessary conditions as functions contingent on the state of the economy Ξ :

$$\frac{U_x \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right]}{U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right]} = z f_n \left[k, n\left(\Xi \right) \right]$$
$$U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right] = \beta E \{ U_c \left[c(\Xi'), d(\Xi'), 1 - n(\Xi') \right]$$
$$\times \left(z' f_k \left[k'\left(\Xi \right), n\left(\Xi' \right) \right] + 1 - \delta \right) \}$$

$$\beta E \left\{ \frac{U_c \left[c(\Xi'), d(\Xi'), 1 - n(\Xi') \right] \left[1 + i'(\Xi) \right]}{U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right] \left[1 + \pi\left(\Xi' \right) \right]} \right\} = 1 \tag{43}$$
$$\beta E \left\{ \frac{U_c \left[c(\Xi'), d(\Xi'), 1 - n(\Xi') \right]}{U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right] \left[1 + \pi\left(\Xi' \right) \right]} \right\} = 1 +$$
$$+ \frac{U_d \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right]}{U_c \left[c(\Xi), d(\Xi), 1 - n(\Xi) \right]} \tag{44}$$

where by Ξ we noted the vector of states, i.e. $\Xi \equiv (m, b, k, s)$. Equation (43) can

be further expressed as follows:

$$\beta E \left\{ \frac{U_c\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right]}{U_c\left[c(\Xi), d(\Xi), 1 - n(\Xi)\right]\left[1 + \pi\left(\Xi'\right)\right]} \right\} \left[1 + i'(\Xi)\right] = 1.$$

The equation (45) can be further modified by use of equation (44), and following equations is derived:

$$\frac{E\left\{\frac{U_c[c(\Xi'), d(\Xi'), 1 - n(\Xi')]}{1 + \pi(\Xi')}\left[1 + \frac{U_d[c(\Xi'), d(\Xi'), 1 - n(\Xi')]}{U_c[c(\Xi'), m(\Xi'), 1 - n(\Xi')]}\right]\right\}}{E\left\{\frac{U_c[c(\Xi'), d(\Xi'), 1 - n(\Xi')]}{1 + \pi(\Xi')}\right\}} = 1 + i'(\Xi).$$

Dividing by the $E\left\{\frac{U_c[c(\Xi'), d(\Xi'), 1 - n(\Xi')]}{1 + \pi(\Xi')}\right\}$, we get the money demand function in the following form:

$$E\left\{\frac{U_d\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right]}{U_c\left[c(\Xi'), d(\Xi'), 1 - n(\Xi')\right]}\right\} = i'(\Xi), \tag{45}$$

for more detailed derivation of the equation (45) see the details in Appendix E. Further, for a detailed solution of the model of the simplified cash-in-advance timing see the details in Appendix (D). The equation (45) relates the expected next period marginal utility from money holding to marginal utility of consumption. So, the MRS between money and consumption depends on the current period nominal interest rate, $i'(\Xi)$.

5.3 Simulation results

As for the standard money in utility model presented in the previous section, we solve model of cash-in-advance timing of utility and its simplified version by log-linearization. The detailed log-linearization of these two models is presented in Appendices C and F, respectively.

Also, as in the section on standard money in utility model, we focus our atten-

tion to standard deviations of the series generated by models and impulse response functions.

In here, we again presents results for the benchmark model but with the different parametrization. While the original parametrization delivers non-intuitive impulse response of the model. Therefore, we set $b = 2$. In the following tables, for the CIA and SCIA, we report the statistics of the money balances d that are relevant to utility function.

	Benchmark	Re-calibration	CIA	SCIA
Capital	0.270	0.270	0.270	0.270
Money balances	0.736	0.758	0.567	0.567
Output	1.088	1.087	1.088	1.088
Consumption	0.235	0.235	0.235	0.235
Employment	0.315	0.314	0.316	0.316
Nominal rate	0.044	0.035	1.078	0.539
Real rate	0.034	0.034	0.034	0.034
Inflation	1.407	1.426	0.695	0.695
Investment	4.038	4.034	4.040	4.040

Table 6: Standard deviations

Table (6) present the results of standard deviations calculations. In this table, we also present the comparison of benchmark model with the re-calibrated model to asses the changes implied by change of the utility function parameter b. The re-calibrated model delivers slightly higher values for money balances and inflation as the elasticity of money demand is increased. At the same time, nominal exchange rate volatility decreases while smaller changes in interest rate are needed to adjust for change in money demand.

The CIA and SCIA timing present a restriction on the money holdings, therefore the observed volatility of money holdings is much lower then in the benchmark model. Due to this restriction, interest rate volatility is increasing to compensate for the restriction on money holdings flexibility.

In the Figures (3) and (4), we present impulse response function of the benchmark (blue dashed line); re-calibrated (green dash-dotted line); CIA timing (black
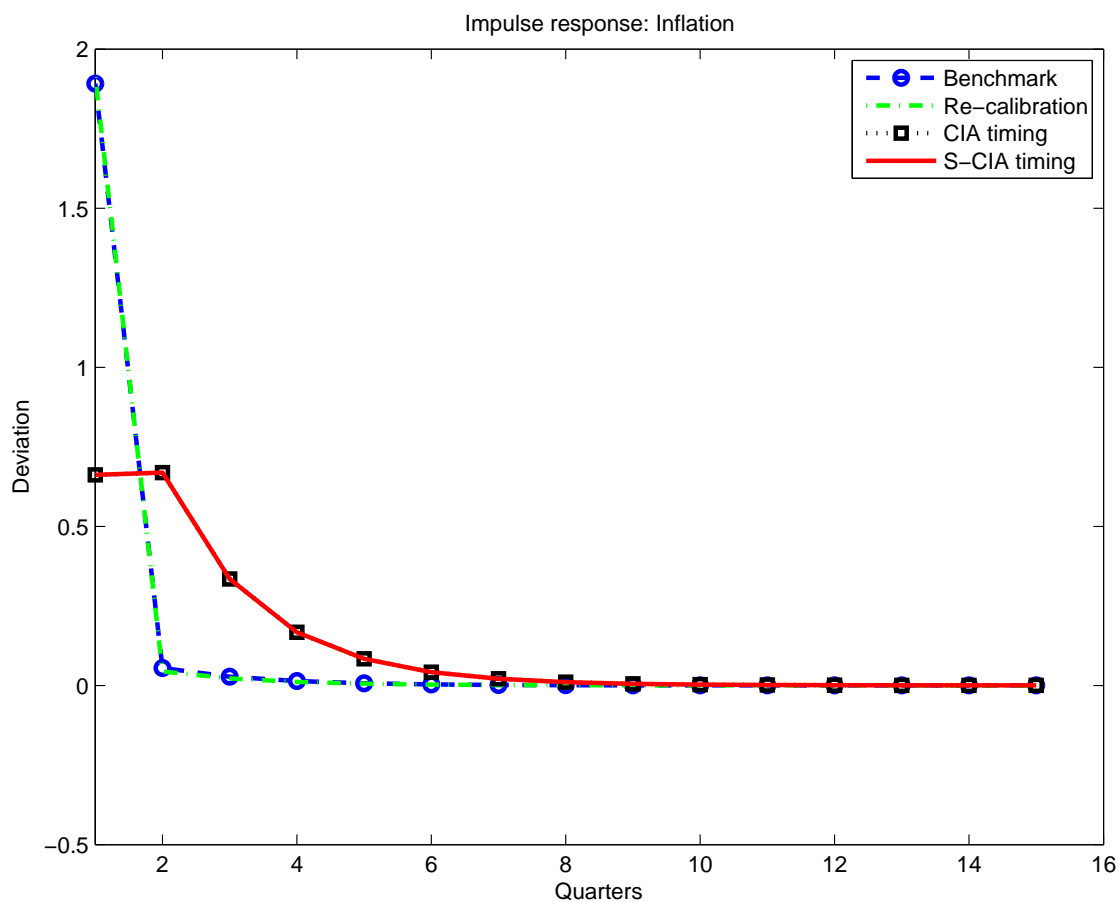
Figure 3: Inflation response to money growth shock

dashed line); and SCIA timing (red solid line) model. Figure (3) shows that the re-calibration does not significantly change the response of inflation. The introduction of CIA timing changes inflation response significantly.

	Benchmark	Re-calibration	CIA	SCIA
Capital	0.27	0.27	0.27	0.27
Money balances	0.27	0.27	0.37	0.37
Output	1.00	1.00	1.00	1.00
Consumption	0.95	0.95	0.95	0.95
Employment	0.89	0.89	0.89	0.89
Nominal rate	0.03	0.03	0.02	0.03
Real rate	0.97	0.97	0.97	0.97
Inflation	-0.06	-0.06	-0.11	-0.11
Investment	1.00	1.00	1.00	1.00

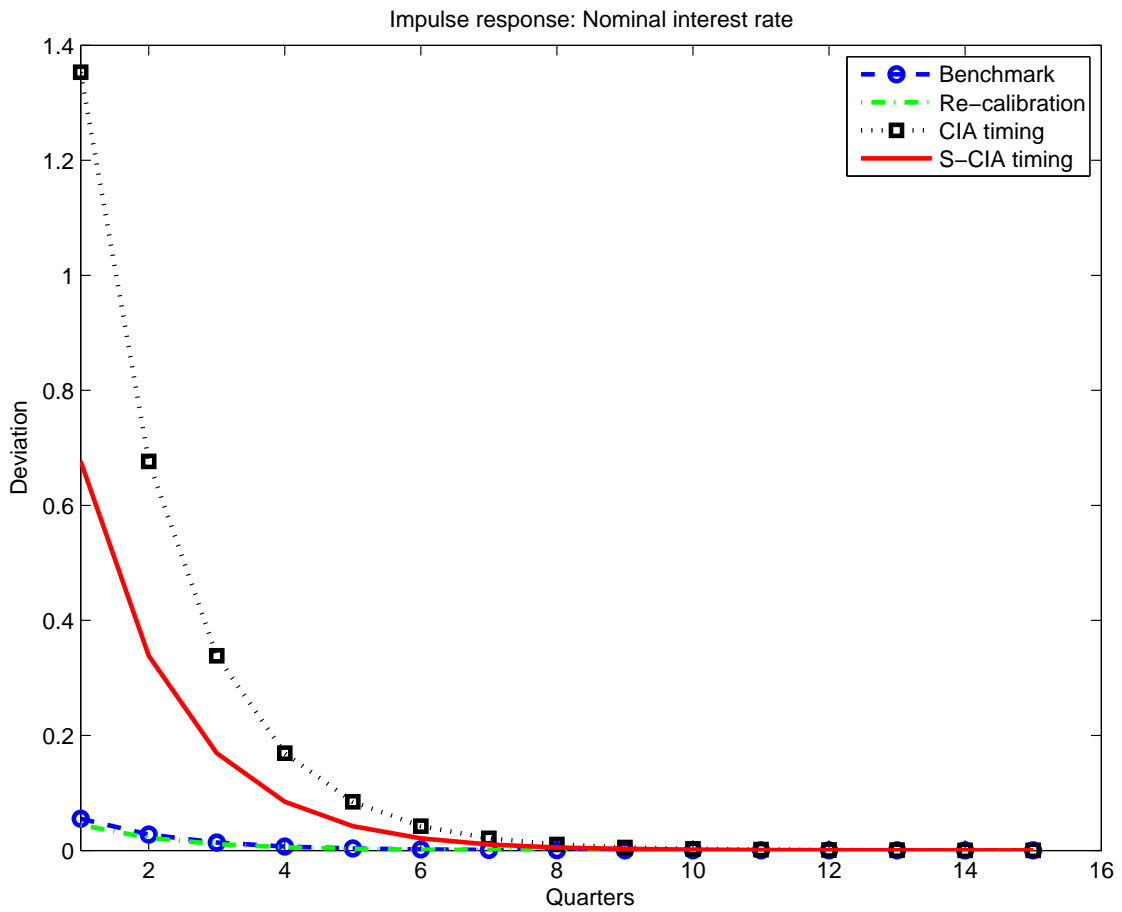Table 7: Correlation with output

Figure 4: Nominal interest rate response to money growth shock

Table (7) present the comparison of variables correlation of variables with output in the examined models. As it can be seen, the change of inverse of the interest elasticity of real money demand does not significantly affects the results. However, the introduction of cash-in-advance restriction induces significant changes. In money in the utility models nominal interest rate does not show a significant relation with output. This is because increases in the nominal interest rate are driven by the Fisher effect that dominates the liquidity effect. The introduction of the CIA timing and the SCIA modification drives the change in the nominal interest rate-output correlation.

As the Table (7) shows the most significant change in correlation occurs for inflation-output. Figures (3) and (4) shows under the CIA timing, households rise their expectations of inflation, the nominal interest rate rises accordingly. The response of nominal interest rate is much larger than in the benchmark model, thus the liquidity effect is suppressed relatively to Fischer effect. This is also confirmed by the more negative value of nominal interest rate-output correlation then under the benchmark model. However, under the SCIA modification as Figure (4) shows lower deviation of nominal interest rate than under the CIA timing. This indicates that the importance of the liquidity effect is increased in SCIA timing.

	Re-calibration	CIA	SCIA
Capital	-0.02	-0.02	-0.02
Money balances	0.15	0.24	0.24
Output	0.71	0.71	0.71
Consumption	0.61	0.61	0.61
Employment	0.73	0.73	0.73
Nominal rate	0.04	0.01	0.03
Real rate	0.74	0.74	0.74
Inflation	-0.05	-0.09	-0.09
Investment	0.72	0.72	0.72

Table 8: Correlation with lagged output

Table (8) shows correlations of variables lagged by one period with the current period output. Comparing the CIA and SCIA timing modification with the standard money in the utility model (Re-calibration) reveals that a inflation more correlated with output than it is in the re-calibrated model.

6 Conclusion

The model introduced by Sidrauski (1967b) introduces demand for money based on the utility delivered by holding the money as form of asset. The goal of this note was to present the effects of choice of the state variables when solving the money in utility model. We also provide a detailed derivation of the handbook model by Walsh (2003).

We also show, how to modify the first order conditions of the money in utility in order to describe the effects of the choice of the real wealth of household as the state variable.

Inspired by the lack of explicit timing specifications in the recent literature for the money that deliver utility to household, we show how the various timing assumptions affect the size of the liquidity and Fischer effect. We show that the most significant change occurs when the cash held before the shopping is used to deliver utility. Our results show that cash-in-advance constraints lead to decrease in volatilities of money holdings and inflation.

Further, by setting-up and solving various versions of the money in utility model, we performed analysis of implications of various assumptions. Also, we demonstrate the effects of approximations methods choice. We also explained the deviations from the reference literature.

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A CWID-timing model: Log-linearization

In this appendix, we present solution of the model by log-linearization, where all variables are expressed as the relative deviations from the steady state. Rearranging the first order conditions and laws of motions, we get the following set of thirteen equations:

$$x' = [ac'^{1-b} + (1-a)m'^{1-b}], \tag{46}$$

$$\lambda' = ac'^{-b}x'^{\frac{b-\Phi}{1-b}}, \tag{47}$$

$$\frac{i'}{1+i'} = \frac{1-a}{a}\left(\frac{m'}{c'}\right)^{-b}, \tag{48}$$

$$\Psi(1-n')^{-\eta} = \lambda'(1-\alpha)\frac{y'}{n'}, \tag{49}$$

$$\lambda' = \beta E'[\lambda_{t+1}(1+r')], \tag{50}$$

$$E'[\lambda'\frac{1+i}{1+\pi'}] = E'[\lambda_{t+1}(1+r')], \tag{51}$$

$$r' = exp(z')\alpha k^{\alpha-1}n'^{1-\alpha} - \delta, \tag{52}$$

$$y' = c' + k' - (1-\delta)k, \tag{53}$$

$$y' = exp(z')k^{\alpha}n'^{1-\alpha}, \tag{54}$$

$$z' = \rho_z z + \varepsilon_z', \tag{55}$$

$$m' = \frac{1+\theta'}{1+\pi'}m_{t-1}, \tag{56}$$

$$\theta' = \bar{\theta} + u', \tag{57}$$

$$u' = \gamma_u u + \gamma_z z + \varepsilon_u', \tag{58}$$

To solve the model, the log-linearization of the model around its the steady state is needed. Model will be formulated in the logarithmic deviations around the steady state. To do this, a new variable is introduced in the the following form:

$$\tilde{\chi}' = log(\chi') - log(\bar{\chi}),$$

where χ is the steady state value of the variable $\chi = \{c, m, n, k, x, \lambda, i, \pi, r, y, zm, \theta, u\}$. For equation (46), this transformation implies:

$$\tilde{x}' = \frac{a\bar{c}^{1-b}(1-b)}{\bar{x}}\tilde{c}' + \frac{(1-a)\bar{m}^{1-b}(1-b)}{\bar{x}}\tilde{m}',$$

where $\bar{x} = a\bar{c}^{1-b} + (1-a)\bar{m}^{1-b}$. The $\tilde{x}'$ can be expressed in form that uses the steady state money–consumption ratio, by dividing $\bar{c}^{1-b}$:

$$\tilde{x}' = \frac{a}{a + (1-a)(\frac{\bar{m}}{\bar{c}})^{1-b}}(1-b)\tilde{c}' + \frac{(1-a)(\frac{\bar{m}}{\bar{c}})^{1-b}}{a + (1-a)(\frac{\bar{m}}{\bar{c}})^{1-b}}(1-b)\tilde{m}'.$$

For the equation (47) it follows:

$$\tilde{\lambda}' = -b\tilde{c}' + \frac{b - \Phi}{1 - b}\tilde{x}'.$$

The equation (48) is known as the money demand equation, by use of log-linearization following equation is derived

$$\frac{1}{1 + \bar{i}}\tilde{i}' = -b\tilde{m}' + b\tilde{c}'.$$

For labor supply equation (49), by use of log-linearization following equation is derived

$$\frac{1 + \bar{n}(\eta - 1)}{1 - \bar{n}}\tilde{n}' = \tilde{\lambda}' + \tilde{y}'.$$

Log-linearization of the Langrange multiplier–real rate relation (50) and Fischer equation (51) gives:

$$\tilde{\lambda}' = \tilde{\lambda}_{t+1} + \frac{\bar{r}}{1 + \bar{r}}\tilde{r}',$$
$$\bar{i}\tilde{i} = \frac{1}{\bar{r} + \bar{\pi} + \bar{r}\bar{\pi}}(\bar{\pi}\tilde{\pi}' + \bar{r}\tilde{r}' + \bar{\pi}\bar{r}(\tilde{\pi}' + \tilde{r}')).$$

For the rate of return on capital given by equation (52) it follows:

$$\frac{\bar{r}}{\bar{r} + \delta}\tilde{r}' = \tilde{z}' + (\alpha - 1)\tilde{k} + (1 - \alpha)\tilde{n}'.$$

The log-linearization of the budget constraint and production function given by equations (53) and (54), gives:

$$\tilde{y}' = \frac{\bar{c}}{\bar{y}}\tilde{c}' + \frac{\bar{k}}{\bar{y}}\tilde{k}' - \frac{\bar{k}}{\bar{y}}(1 - \delta)k, \tag{59}$$
$$\tilde{y}' = \tilde{z}' + \alpha\tilde{k} + (1 - \alpha)\tilde{n}'.$$

While the equations (55) and (58) for technological process shock and monetary growth shock are already in linear form of deviations from the steady, it is straightforward to rewrite them in log-linear form:

$$\tilde{z}' = \rho_z\tilde{z} + \varepsilon_z',$$
$$\tilde{m}' = \frac{\bar{\theta}}{1 + \bar{\theta}}\tilde{\theta}' - \frac{\bar{\theta}}{1 + \bar{\theta}}\tilde{\pi}' + \tilde{m},$$
$$\tilde{\theta}' = \frac{1}{\bar{\theta}}\tilde{u}',$$
$$\tilde{u}' = \gamma_u\tilde{u} + \gamma_z\tilde{z} + \varepsilon_u'.$$

B CIA timing model: derivation of FOCs

$$
\begin{aligned}
U_x(\Xi) &= U_c(\Xi)e^z f_n(\Xi) \\
y(\Xi) &= e^z f(\Xi) \\
y(\Xi) &= c(\Xi) + k'(\Xi) - (1-\delta)k \\
r^e(\Xi) &= E\left\{ e^{z'} f_k(k, n(\Xi')) - \delta \right\} \\
E\left\{ \frac{U_c(\Xi')}{U_c(\Xi)} r(\Xi') \right\} &= 1 \\
E\left\{ \frac{U_c(\Xi')}{U_c(\Xi)} r(\Xi') \right\} &= E\left\{ \frac{[U_c(\Xi') + U_d(\Xi')]}{[U_c(\Xi) + U_d(\Xi)]} \frac{1}{1 + \pi(\Xi')} \right\} [1 + i'(\Xi)] \\
\frac{U_d(\Xi)}{U_c(\Xi)} &= i'(\Xi) \\
d(\Xi) &= \frac{m}{1 + \pi(\Xi)} \\
m'(\Xi) &= m \frac{1 + e^u}{1 + \pi(\Xi)}
\end{aligned}
$$

Derivation of the Fisher equation

$$
\begin{aligned}
U_c(\Xi) + U_d(\Xi) &= \beta E\left\{ \frac{U_c(\Xi') + U_d(\Xi')}{1 + \pi(\Xi')} \right\} [1 + i'(\Xi)] \\
U_c(\Xi)\left[1 + \frac{U_d(\Xi)}{U_c(\Xi)} \right] &= \beta E\left\{ \frac{U_c(\Xi')\left[1 + \frac{U_d(\Xi')}{U_c(\Xi')} \right]}{1 + \pi(\Xi')} \right\} [1 + i'(\Xi)] \\
U_c(\Xi)\left[1 + \frac{U_d(\Xi)}{U_c(\Xi)} \right] &= U_c(\Xi)[1 + i'(\Xi)] \\
U_c(\Xi)[1 + i'(\Xi)] &= \beta E\left\{ \frac{U_c(\Xi')[1 + i'(\Xi')]}{1 + \pi(\Xi')} \right\} [1 + i'(\Xi)] \\
U_c(\Xi) &= \beta E\left\{ \frac{U_c(\Xi')[1 + i'(\Xi')]}{1 + \pi(\Xi')} \right\} \\
E\left\{ \frac{U_c(\Xi')}{U_c(\Xi)} r(\Xi') \right\} &= E\left\{ \frac{U_c(\Xi')}{U_c(\Xi)} \frac{1 + i'(\Xi')}{1 + \pi(\Xi')} \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
U_x(\Xi) &= \Psi\left[1 - n(\Xi) \right]^{-\eta} \\
U_c(\Xi) &= a\left[X(\Xi) \right]^{\frac{b-\phi}{1-\phi}} \left[c(\Xi) \right]^{-b} \\
X(\Xi) &= \left[c(\Xi) \right]^{1-b} \left\{ a + (1-a)\left[\frac{d(\Xi)}{c(\Xi)} \right]^{1-b} \right\}
\end{aligned}
$$

$$\frac{U_d(\Xi)}{U_c(\Xi)} = \left(\frac{1-a}{a}\right)\left[\frac{d(\Xi)}{c(\Xi)}\right]^{-b}$$

$$e^z f_n(\Xi) = (1-\alpha)\frac{y(\Xi)}{n(\Xi)}$$

$$e^z f_k(\Xi) = \alpha\frac{y(\Xi)}{k}.$$

C CIA-timing model: Log-linearization

$$\tilde{\lambda}_t = \tilde{U}_{c,t} = \Omega_1\tilde{c}_t + \Omega_2\tilde{d}_t$$

$$\left(1 + \eta\frac{\bar{n}}{1-\bar{n}}\right)\tilde{n}_t = \tilde{y}_t + \tilde{\lambda}_t$$

$$\tilde{y}_t = \alpha\tilde{k}_{t-1} + (1-\alpha)\tilde{n}_t + \tilde{z}_t$$

$$\left(\frac{\bar{y}}{\bar{k}}\right)\tilde{y}_t = \left(\frac{\bar{c}}{\bar{k}}\right)\tilde{c}_t + \tilde{k}_t - (1-\delta)\tilde{k}_{t-1}$$

$$\hat{r}_t = \alpha\left(\frac{\bar{y}}{\bar{k}}\right)\left[\tilde{y}_t - \tilde{k}_{t-1}\right]$$

$$E_t\left[\tilde{\lambda}_{t+1} - \tilde{\lambda}_t + \hat{r}_{t+1}\right] = 0$$

$$\bar{\Pi}E_t\hat{r}_{t+1} = E_t\hat{i}_{t+1} - \bar{R}E_t\hat{\pi}_{t+1}$$

$$\tilde{d}_t - \tilde{c}_t = -\frac{1}{b\bar{I}}\hat{i}_t$$

$$\tilde{d}_t = \tilde{m}_{t-1} - \frac{1}{\bar{\Pi}}\hat{\pi}_t$$

$$\tilde{m}_t = \tilde{m}_{t-1} + \frac{1}{1+\bar{\Xi}}\hat{u}_t - \frac{1}{\bar{\Pi}}\hat{\pi}_t$$

D Simplified CIA-timing model: Derivation of FOCs

$$U_x(\Xi) = U_c(\Xi)e^z f_n(\Xi)$$

$$y(\Xi) = e^z f(\Xi)$$

$$y(\Xi) = c(\Xi) + k'(\Xi) - (1-\delta)k$$

$$r^e(\Xi) = E\left\{e^{z'} f_k(k, n(\Xi')) - \delta\right\}$$

$$E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}r(\Xi')\right\} = 1$$

$$E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}r(\Xi')\right\} = E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}\frac{1}{1+\pi(\Xi')}\right\}[1 + i'(\Xi)]$$

$$E\left\{\frac{U_d(\Xi')}{U_c(\Xi')}\right\} = i'(\Xi)$$

$$m'\left(\Xi\right) = m\frac{1+e^u}{1+\pi\left(\Xi\right)}$$

$$d\left(\Xi\right) = \frac{m}{1+\pi\left(\Xi\right)}$$

Derivation of the Fisher equation

$$E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}r(\Xi')\right\} = E\left\{\frac{U_c(\Xi')}{U_c(\Xi)}\frac{1}{1+\pi\left(\Xi'\right)}\right\}\left[1+i'\left(\Xi\right)\right]$$

where

$$U_x(\Xi) = \Psi\left[1-n(\Xi)\right]^{-\eta}$$

$$U_c(\Xi) = a\left[X(\Xi)\right]^{\frac{b-\phi}{1-\phi}}\left[c(\Xi)\right]^{-b}$$

$$X(\Xi) = \left[c(\Xi)\right]^{1-b}\left\{a+(1-a)\left[\frac{d(\Xi)}{c(\Xi)}\right]^{1-b}\right\}$$

$$\frac{U_d(\Xi)}{U_c(\Xi)} = \left(\frac{1-a}{a}\right)\left[\frac{d(\Xi)}{c(\Xi)}\right]^{-b}$$

$$e^z f_n(\Xi) = (1-\alpha)\frac{y(\Xi)}{n(\Xi)}$$

$$e^z f_k(\Xi) = \alpha\frac{y(\Xi)}{k}.$$

E MRS between money and consumption in the simplified CIA-timing model

$$\frac{E\left\{\frac{U_c[c(\Xi'),m(\Xi'),1-n(\Xi')]}{1+\pi(\Xi')}\left[1+\frac{U_m[c(\Xi'),m(\Xi'),1-n(\Xi')]}{U_c[c(\Xi'),m(\Xi'),1-n(\Xi')]}\right]\right\}}{E\left\{\frac{U_c[c(\Xi'),m(\Xi'),1-n(\Xi')]}{1+\pi(\Xi')}\right\}}$$

$$= 1 + E\left\{\frac{U_m\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}{U_c\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}\right\}$$

$$+ Cov\left\{\frac{U_c\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}{1+\pi\left(\Xi'\right)}, \frac{U_m\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}{U_c\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}\right\}$$

$$= 1 + i'(\Xi)$$

neglecting the second-order terms we get

$$E\left\{\frac{U_m\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}{U_c\left[c(\Xi'),m(\Xi'),1-n(\Xi')\right]}\right\} = i'(\Xi).$$

F Simplified CIA timing model: Log-linearization

It is necessary to introduce an additional state variable i_t in order to solve the model by the Uhlig toolbox.

$$\tilde{\lambda}_t = \tilde{U}_{c,t} = \Omega_1 \tilde{c}_t + \Omega_2 \tilde{d}_t$$

$$\left(1 + \eta \frac{\bar{n}}{1 - \bar{n}}\right) \tilde{n}_t = \tilde{y}_t + \tilde{\lambda}_t$$

$$\tilde{y}_t = \alpha \tilde{k}_{t-1} + (1 - \alpha) \tilde{n}_t + \tilde{z}_t$$

$$\left(\frac{\bar{y}}{\bar{k}}\right) \tilde{y}_t = \left(\frac{\bar{c}}{\bar{k}}\right) \tilde{c}_t + \tilde{k}_t - (1 - \delta) \tilde{k}_{t-1}$$

$$\hat{r}_t = \alpha \left(\frac{\bar{y}}{\bar{k}}\right) \left[\tilde{y}_t - \tilde{k}_{t-1}\right]$$

$$E_t \left[\tilde{\lambda}_{t+1} - \tilde{\lambda}_t + \hat{r}_{t+1}\right] = 0$$

$$\bar{\Pi} E_t \hat{r}_{t+1} = \hat{i}_t - \bar{R} E_t \hat{\pi}_{t+1}$$

$$E_t \left\{\tilde{d}_{t+1} - \tilde{c}_{t+1}\right\} = -\frac{1}{b\bar{I}} \hat{i}_t$$

$$\tilde{m}_t = \tilde{m}_{t-1} + \frac{1}{1 + \Xi} \hat{u}_t - \frac{1}{\bar{\Pi}} \hat{\pi}_t$$

$$\tilde{d}_t = \tilde{m}_{t-1} - \frac{1}{\bar{\Pi}} \hat{\pi}_t$$