Economic Growth in a Politically Fragmented World*

Byeongju Jeong

CERGE-EI
POB 882, Politickych veznu 7
111 21 Prague 1
Czech Republic

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ABSTRACT

I explore the effects of economic and political integration on economic growth in a model of vintage human capital and sequential intergenerational bargains. Adoption of a new technology raises not only the productivity but also the bargaining position of the future generations, creating a bias for the current generations to preserve the current technology. Economic integration (i.e., the sharing of frontier technology among countries) promotes growth if there is a diversity in human capital distribution or a coordination failure across countries. On the other hand, political integration (i.e., the merging of countries into a single bargain) promotes stagnation as it eliminates the diversity and coordination failures.

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1. Introduction

A number of authors suggested political fragmentation as an important determinant of the modern economic growth in Europe. This is based on the observation that a country alone may go through a spurt of growth but invariably loses the growth momentum and that European countries took turns in maintaining the growth momentum, thus avoiding the stagnation of Europe as a whole for centuries. A plausible inference is that a single country could determine its growth path (i.e, political autonomy) but the growth path was constrained by the diffusion of technology across countries (i.e., economic integration). In other words, the European growth may have to do with the growth imperative that each country faces under economic integration and political fragmentation.

The economic integration defined in terms of diffusion of technology was probably at a continental scale a couple of hundred years ago.[1] In comparison with the European growth then, an economically and politically integrated China stagnated after remarkable spurts of growth in preceding centuries. In Section 2, I review theses along this line in the literature. Since the European growth, the economic integration has expanded to a global scale and every country faces the growth imperative in an economically integrated and politically fragmented world. Growth-oriented reforms in Asia, Latin America, and former-communist countries in recent decades provide the examples.

In Sections 3 to 7, I explore the above notion of the growth imperative in a model environment where human capital is transmitted from the old to the young generation, but the young generation can opt out for a new technology. The choice of technology adoption is made in an intergenerational bargain in each period. The adoption of new technology raises tomorrow's productivity at today's cost of not fully utilizing the existing human capital. Further, it allows tomorrow's young generation to adopt an even better

[1] In terms of the model to be presented, the diffusion of technology is the opportunity for a country to adopt the frontier technology within one generation. This is not to deny the global diffusion of technology in the long run.

technology, strengthening the bargaining position of tomorrow's young generation relative to that of tomorrow's old generation or, equivalently, today's young generation. This creates a bias against adopting a new technology, leading to a stagnation if the average human capital of the old generation is close enough to the new technology.

I examine the variations of the environment along two dimensions. First, I consider economic integration, that is, the sharing of the frontier technology among two countries initially separated from each other. A condition for perpetual growth in economic integration is the diversity of human capital distribution among countries: In each period, the world economy grows as long as the country furthest from the frontier technology has the incentive to adopt the frontier technology. When the diversity is limited, the world economy may still grow due to a coordination failure among countries: Each country adopts the frontier technology since, from the perspective of an individual country, stagnation is advantageous only when the other country stagnates as well. Second, I consider political integration, that is, the merging of countries into a political union with a single intergenerational bargain. The political union corrects the coordination failure and aligns the incentives of individual countries for stagnation by means of side payments that are implicit in bargaining. Thus, political integration can lead to a stagnation.

The growth mechanics of the model builds on the vintage human capital model of Chari and Hopenhayn (1991). The assumed embodiment of technology in human capital delivers a conceptually simple and tractable trade-off between the future productivity gain and fully utilizing the existing human capital. In this aspect, the model is close to Krusell and Rios-Rull (1996) in which a segment of population with human capital in existing technologies may outvote the others and block the adoption of a new technology, thereby creating rents for themselves. An alternative political mechanism explored in the literature is the lobbying model in which various interest groups bid to influence policy. In Bridgman, Livshits, and MacGee (2007), old skilled workers lobby against technology adoption in their industries and a government decides on technology adoption across industries weighing the aggregate output and the bribes from industry lobbies. Related works outside the vintage human capital framework include Bellettini and Ottaviano (2005) in which the young and the old bid for and against the upgrading of the aggregate technology to a regulator who maximizes the collection of bids, facing the trade-off between the current gain from the maturing of the old technology and the future gain from switching to a new technology. Acemoglu, Aghion, and Zilibotti (2006) presents a model that combines elements of technology adoption and innovation and shows that old firms with financial resources but a limited innovative capacity may bribe the government to maintain their monopoly rents retarding innovation.

In this paper, the society-wide decision process is modeled as a bargain between the old and the young generations. The bargain delivers efficient adoption and matching behavior *for the current generations within a country* and assigns a payoff to each generation. The aggregate path of the economy is independent of individual activities and payoffs as long as they aggregate to the bargaining outcome. I consider efficient bargaining as a means of abstracting from the variations in the details of economic and political institutions that add up to deliver efficient adoption and matching behavior for the current generations. This abstraction is in contrast with the previous studies, some of which are mentioned above, that focused on the conflict among the current generations under a particular institutional arrangement. In comparing Europe and China over a period lasting centuries, any sharp institutional assumptions seem unwarranted. Rather, the logic of growth and stagnation is assumed to rely on the sequential structure of intergenerational bargaining under various international economic and political environments: Bargaining is generally not efficient in the sense of maximizing the sum of the discounted utilities of all generations of all countries due to the segmentation of bargaining between the current and the future generations and between the current generations across countries.

Although the model is about the effects of economic and political integration on economic growth and not its causes, in Section 7, I characterize the welfare effects of economic and political integration and the incentives to form or break-up a political union under simplifying assumptions. The incentives for integration and fragmentation are due to the growth dynamics under the bargaining structure. In comparison, the models of integration and fragmentation such as Alesina and Spolaore (1997) and Bolton and Roland (1997) focused on the incentives due to the provision of public goods and re-distribution. These models are in part motivated by events such as the formation of the European Union and the break-up of the Soviet Union in an economically integrated world. This paper has little to say about such a regional integration and fragmentation since an economically integrated world needs a worldwide political union, a possibility in the future, to hold back the advancement of technology.

2. Review of Literature on Political Fragmentation and Growth

Mokyr (1990) frames the effect of political fragmentation on the modern economic growth in Europe in terms of what he calls Cardwell's Law. "Cardwell (1972, p.210) has pointed out that "no nations has been (technologically) very creative for more than an historically short period." As stated, Cardwell's Law is no more than an empirical regularity, and a crude one at that. The definition of a "nation," let alone that of a "short" period, is ambiguous. What is more, Cardwell gives us no inkling as to the possible economic and social reasons his law might be true. Yet if we accept it as an empirical regularity, Europe's advantage becomes clear. Europe always consisted of many nations. The Technological center of gravity of Europe moved over the centuries, residing at various times in Italy, southern Germany, the Netherlands, France, England, and again in Germany. Political fragmentation did not inhibit the flow of information from technological leaders to followers in Europe, and so it came to pass that the technology used in Europe always eventually settled on the best-practice technique in use regardless of where it had been invented." (p. 207, Mokyr 1990)

We can break down Mokyr's reasoning in the above excerpt into components: (1) The technological creativity of a country is temporary (i.e., Cardwell's Law); (2) If there are many countries, there is a good chance that at least one country is technologically creative at any moment despite Cardwell's Law; and (3) For the countries as a whole to grow for a sustained period, the new technology must fairly quickly spread from the currently creative countries to the countries that will be creative in the future. "Europe's advantage" lies in there being many countries across which the new technology spread quickly.

Many authors have commented on some sort of political control over the technological innovation in a country while observing the uncontrollable diffusion of technology across countries in Europe. Jones (1981) writes, "the culture, science, technology and commercial practice of the Italian city-states, Antwerp, Amsterdam, and London were passed from each to the next, and were diffused across agrarian and backward economies that showed little sign of attaining the same level on their own. Galileo's trial silenced Italian scientists, but the 'scientific revolution' continued in Protestant lands. Books might be smashed by mobs, entrepreneurs banished and investors expropriated by governments, but Europe as a whole did not experience technological regression." (p. 123-4, Jones 1981)

Diamond (2005) discusses the often-used example of the sudden political decision to stop the overseas voyages in China in the early 15th century following a few decades of well funded expeditions, in contrast to the sustained and uncontrolled European exploration starting with Columbus soon afterwards:[2] "In fact, precisely because Europe was fragmented, Columbus succeeded on his fifth try in persuading one of Europe's hundreds

[2] Diamond goes on and discuss the ecological conditions that were conducive to political fragmentation in Europe in comparison with China: "Hence the real problem in understanding China's loss of political and technological preeminence to Europe is to understand China's chronic unity and Europe's chronic disunity. The answer is again suggested by maps. Europe has a highly indented coastline, with five large peninsulas that approach islands in their isolation, and all of which evolved independent languages, ethnic groups, and governments: Greece, Italy, Iberia, Denmark, and Norway/Sweden. China's coastline is much smoother, and only the nearby Korean Peninsula attained separate importance." (p. 413-4, Diamond 2005)

of princes to sponsor him. Once Spain had thus launched the European colonization of America, other European states saw the wealth flowing into Spain, and six more joined in colonizing America. The story was the same with Europe's cannon, electric lighting, printing, small firearms, and innumerable other innovations: each was at first neglected or opposed in some parts of Europe for idiosyncratic reasons, but once adopted in one area, it eventually spread to the rest of Europe." (p. 413, Diamond 2005)

Rosenberg and Birdzell (1986) invoke the notion of countries in competition for trade as a factor contributing to the European growth: "Unlike China and the ancient empires, the Europe of the late medieval city-states and the early monarchies came to the age of discovery without a central authority strong enough to check the determination of its merchants to gain access to profitable trading opportunities, even though some satrap or other had forbidden such access or claimed it as a private preserve. The central authorities which eventually emerged did not take the form of a single monolithic empire, but of a group of nation-states which continued, among themselves, the early city-state competition for trade." (p. 60, Rosenberg and Birdzell 1986)

The "competition for trade" can be considered as a channel of the uncontrolled diffusion of technology across countries. In the same vein, North (2005) writes, "It was the dynamic consequences of the competition among fragmented political bodies that resulted in an especially creative environment. Europe was politically fragmented; but it had both a common belief structure derived from Christendom and information and transportation connections that resulted in scientific, technological, and artistic developments in one part spreading rapidly throughout Europe." (p.138, North 2005)

Olson (1982), writing about the European growth and the subsequent slow-down after the World War 2, provides a reason for the slow-down of the technological creativity in a country, namely, the accumulation of special interest groups, each of which organizes to maintain the rents from the existing technology. Olson was skeptical of the efficiency of the comprehensive bargain among all groups in a country while acknowledging that an encompassing group (i.e., a group that represents a substantial portion of the population) has some incentive to reduce inefficient redistribution and to make the society prosperous. The models of Krusell and Rios-Rull (1996), Bridgman, Livshits, and MacGee (2007), Bellettini and Ottaviano (2005), and Acemoglu, Aghion, and Zilibotti (2006) discussed earlier, embody Olson's thesis to varying degrees and feature conflicts in the current generations that may lead to an inefficient outcome.

As mentioned, in this paper any conflicts between the current generations in a country are resolved efficiently through bargaining. Nonetheless, the slow-down can occur due to the changing incentive for stagnation given the sequential structure of intergenerational bargaining as a growth over one or more periods pulls the distribution of human capital toward the frontier technology. Thus the source of inefficiency is the conflicts between the current and the future generations, mediated by the evolution of the distribution of human capital. Further, conflicting interests between the current generations across countries may mitigate or exacerbate the inefficiency. Thus the focus is on how international environments such as economic integration and political fragmentation affect the nature of these conflicts and the growth paths of countries.

3. Preview of Main Results

Imagine that there are one old person and one young person. The old person's human capital is h. The old and the young can work together, in which case the joint output is h and the young person inherits the old person's human capital in the next period. Alternatively, the old person can work alone and produce ϕh, where $\phi < 1$, while the young person adopts a new technology in which case the young person produces no output in the current period but obtains human capital λh, where $\lambda > 1$, in the next period. The old and the young bargain over whether to work together and how to divide the output in case they work together, where the outside option of the old person is to work alone and that of the young person is to adopt the new technology. The situation repeats in the next period with the young person having become old and a new young person having been born.

Depending on the parameter values, the economy may stagnate with the young and the old working together in each period, or follow a perpetual growth path with the old working alone and the young adopting the new technology in each period. In particular, the economy stagnates if ϕ is low enough, holding the other parameter values. Either country on the row headed by Autarky on Table 1 illustrates the stagnation equilibrium. As will be discussed below, Table 1 also illustrates the equilibrium paths of a two country world under alternative international economic and political environments and under variations in the diversity of human capital across the two countries.

Table 1: Evolution of Human Capital under (No) Diversity across Countries

	No Diversity:	Diversity:
Autarky: Country 1	$h \to h \to h$	$\lambda h \to \lambda h \to \lambda h$
Autarky: Country 2	$h \to h \to h$	$h \to h \to h$
	No Diversity:	Diversity:
Economic Integration: Country 1	$h \to h \to h$	$\lambda h \to \lambda h \to \lambda^3 h$
Economic Integration: Country 2	$h \to h \to h$	$h \to \lambda^2 h \to \lambda^2 h$
Political Union of Countries 1 and 2	$(h, h) \to (h, h) \to (h, h)$	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$

Now suppose that there are two countries, each populated by one old person and one young person. From the perspective of each country, the only difference from the autarky described above is that the two countries share the new technology: If the young person adopts the new technology, he obtains human capital $\lambda \max\{h_1, h_2\}$, where h_i is the human capital of the old person in country i. Suppose that $h_1 = h_2$: The old people have the same human capital across the countries. It is an equilibrium for both countries to stagnate, replicating the equilibrium of the autarky. The cell headed by Economic Integration and No Diversity on Table 1 illustrates the stagnation equilibrium.

Now suppose that $h_1 = \lambda h_2$: The human capital of the old person in country 1 is one-level above that of the old person in country 2. It is an equilibrium for country 1 to stagnate but for country 2 to adopt the new technology if ϕ is neither too high nor too low: The country with the current technology further away from the new technology has a greater incentive to adopt the new technology. Then, the human capital of country 1 will be h_1 and that of country 2 will be λh_1 in the next period so that country 1 will adopt the new technology and country 2 will stagnate. The world economy will grow perpetually, each country adopting a new technology in every two periods. The cell headed by Economic Integration and Diversity on Table 1 illustrates the growth equilibrium. Comparing the cells headed by Autarky/Economic Integration and Diversity/No Diversity, observe that economic integration (i.e., the transition of the two countries in autarky to countries sharing the new technology) can turn a stagnating country to a growing one if there is a diversity in human capital across the two countries.

Now suppose that the two countries with $h_1 = \lambda h_2$ merge into a political union, effectively becoming a single country. Then, there will be two old people, one with h_1 and the other with h_2, in the union. Bargaining is potentially more complicated with multiple and heterogeneous old people in the union but it is plausible for the union to stagnate depending on the parameter values since, intuitively, the incentive for country 2 to adopt the new technology and that for country 1 to stagnate would be averaged in the union. The cell headed by Political Union and Diversity on Table 1 illustrates the stagnation equilibrium. Comparing the cells headed by Economic Integration/Political Union and Diversity on Table 1, observe that political integration (i.e., the transition of the two countries under economic integration to one country with a single bargain) can turn a growing world economy to a stagnating one due to the elimination of the diversity in the incentive for growth across the two countries.

Table 2 illustrates the equilibrium paths of a two country world under alternative international economic and political environments and under variations in coordination across the two countries. Suppose that there are two countries, each populated by two old people and two young people. Further, in each country the human capital of one old person is one-level above that of the other old person: $h_1 = \lambda h_2$ where h_1 and h_2 are the two human capital levels. Each country is then just like the political union considered above and may stagnate in autarky depending on the parameter values. The row headed by Autarky on Table 2 illustrates the stagnation equilibrium. If the two countries integrate economically, it is plausible that the integrated world economy stagnates since there is no diversity across the two countries, unlike the case of economic integration considered earlier. The cell headed by Economic Integration and No Coordination Failure on Table 2 illustrates the stagnation equilibrium.

Table 2: Evolution of Human Capital under (No) Coordination Failure across Countries

Autarky: Country 1	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$
Autarky: Country 2	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$
	No Coordination Failure:	Coordination Failure:
Eco. Integration: C1	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$	$(\lambda h, h) \to (\lambda h, \lambda^2 h) \to (\lambda^3 h, \lambda^2 h)$
Eco. Integration: C2	$(\lambda h, h) \to (\lambda h, h) \to (\lambda h, h)$	$(\lambda h, h) \to (\lambda h, \lambda^2 h) \to (\lambda^3 h, \lambda^2 h)$
Political Union of Countries 1 and 2	$(\lambda h, \lambda h, h, h) \to (\lambda h, \lambda h, h, h) \to (\lambda h, \lambda h, h, h)$	$(\lambda h, \lambda h, h, h) \to (\lambda h, \lambda h, h, h) \to (\lambda h, \lambda h, h, h)$

However, the perpetual growth can also be an equilibrium, which can be reasoned as follows. Suppose that country 1 adopts the new technology while country 2 stagnates: In country 1, one young person works with the old person with h_1 and the other young person adopts the new technology while in country 2 both of the young people work with the old people. Technology adoption by country 1 benefits tomorrow's young generation in country 2 since they will be able to adopt a technology that is one-level more advanced than otherwise. This raises the outside option of tomorrow's young generation in bargaining with tomorrow's old generation, and thereby lowers the utility of tomorrow's old generation or, equivalently, today's young generation in country 2. Note that if country 2 also adopts the new technology, technology adoption by country 1 does not affect the outside option of tomorrow's young generation and the utility of today's young generation in country

2. Thus technology adoption by country 1 makes stagnation less advantageous to today's young generation in country 2. Country 2 may adopt the new technology since country 1 adopts it, and vice-versa. The perpetual growth path, whereby in each country one young person works with one old person and the other young person adopts the new technology in every period, can be sustained due to the coordination failure between the two countries. The cell headed by Economic Integration and Coordination Failure on Table 2 illustrates the growth equilibrium. Comparing the cells headed by Autarky/Economic Integration and Coordination Failure/No Coordination Failure on Table 2, observe that economic integration can turn a stagnating country to a growing one if there is a coordination failure between the two countries.

Now suppose that the two countries, each populated by two old people and two young people as described above, merge into a political union. Then, there will be two old people with h_1 and two old people with h_2, where $h_1 = \lambda h_2$, in the union. The union is identical to a country where one old person has h_1 and another old person has h_2 (i.e., either country on the row headed by Autarky on Table 2), except that the population is doubled. Suppose that the growth path is independent of the population size. Then, the union will stagnate given that a country where one old person has h_1 and another old person has h_2 stagnates. The row headed by Political Union on Table 2 illustrates the stagnation equilibrium. On the other hand, the perpetual growth path cannot be sustained since the union eliminates the coordination failure. Comparing the cells headed by Economic Integration/Political Union and Coordination Failure on Table 2, observe that political integration can turn a growing world economy to a stagnating one due to the elimination of the coordination failure between the two countries.

4. The Model Economy: A Single Country

There are two overlapping generations in each period. There are many people in each generation whose number is normalized to one. Human capital embodies technology with a higher level of h associated with a superior technology. Technologies are ordered such that the ratio of the human capital levels associated with any two adjacent technologies is fixed at $\lambda > 1$. Let $\{h_s\}_{s \geq 0}$ denote the ordered set of human capital levels, where h_0 is associated with the frontier technology available for adoption; h_s, $s \geq 1$, is associated with a previously adopted technology; and $h_s/h_{s+1} = \lambda$ for all s. Note that h_s, $s \geq 0$, is a variable over time: If the frontier technology advances to the next level, h_s rises by λ for all s and the technology previously indexed by s is now indexed by $s+1$.

Let n_s denote the fraction of old people with human capital h_s. I have $\sum_{s \geq 1} n_s = 1$. An old person can work alone or work with a young person; A young person is endowed with no human capital and can work with an old person or adopt a technology alone. A team of a young person and an old person with human capital h_s produces h_s units of output. Within a team, the young person inherits the human capital of the old person in the next period. An old person with human capital h_s alone produces ϕh_s units of output. A young person alone can only adopt a technology: He produces no output but obtains human capital h_s, where $s \geq 0$ is his choice, in the next period.[3] Assume that $\phi < 1$, which implies that team production has an advantage over individual production, holding technology.

The frontier technology advances to the next level in the next period if at least one young person adopts the current frontier technology. This is as in Krusell and Rios-Rull (1996). Since there are an infinite number of young people, the country as a whole can

[3] The previous version of this paper assumed that the young person alone always obtains human capital associated with the frontier technology h_0. This would be a natural assumption if one were to interpret technology adoption as a development of a new technology whose nature prevents predetermining the productivity of a new technology to be lower than one-level above the previously developed technology. That the young person is allowed to adopt a technology below the frontier one, h_s with $s \leq 1$, as assumed here, lowers the cutoff-value $\tilde{x}$ for *immediate* stagnation, discussed in paragraphs following Proposition 2, as it allows the young people to adopt the best technology in use h_1 instead of being matched with old people far from the frontier technology. However, it does not alter the long-run equilibrium paths as $\tilde{x}$ only affects the timing of stagnation. A similar property holds in the two-country world in Section 5 (see footnote 9).

trivially advance the frontier technology by having one young person adopt the current frontier technology. In this sense, there are (virtually) no costs of innovating technology while there are individual costs of adopting technology in terms of foregone output. With this understanding, I will treat the society-wide *advancement* of technology as a separate decision from the individual *adoption* of a technology. Let τ denote the indicator of technological advancement: $\tau = 1$ in case of advancement; $\tau = 0$ otherwise. Let $\tilde{n}_s$, $s \geq 0$, denote the fraction of young people who will have human capital h_s when old. Let $n \equiv (n_1, n_2, n_3, \ldots)$ and $\tilde{n} \equiv (\tilde{n}_0, \tilde{n}_1, \tilde{n}_2, \ldots)$. The feasibility requires: $\tilde{n}_s \leq n_s$ for all $s \geq 1$, and $\tilde{n} = (0, n)$ if $\tau = 0$.[4]

I assume that the growth dynamics and the division of output across generations is independent of the value of h_0. Then, I can normalize h_0 to one. The aggregate output is:

$$Y(n, \tilde{n}) = \sum_{s=1}^{\infty} \left(\frac{1}{\lambda^s} \cdot \min\{n_s, \tilde{n}_s\} + \frac{\phi}{\lambda^s} \cdot \max\{0, n_s - \tilde{n}_s\} \right). \tag{1}$$

Let $V_o(n)$ denote the aggregate utility of the old generation, and $V_y(n)$ the aggregate utility of the young generation. Let $W(n, \tilde{n}, \tau)$ denote the aggregate utility across the young and the old generations conditional on τ and $\tilde{n}$:

$$W(n, \tilde{n}, \tau) = Y(n, \tilde{n}) + \beta(\tau \cdot \lambda V_o(\tilde{n}) + (1 - \tau) \cdot V_o(\tilde{n}_{-0})), \tag{2}$$

where β is the discount rate and $\tilde{n}_{-0} \equiv (\tilde{n}_1, \tilde{n}_2, \ldots)$. Note that the aggregate utility includes the discounted utility of the young generation's utility when it becomes old.

The advancement of technology, the individual adoption and matching behavior, and the utilities of generations are determined in a sequential, intergenerational bargains. Let

[4] It is feasible to have $\tilde{n} = (0, n)$ and $\tau = 1$. Strictly speaking, this implies that the frontier technology can advance even if no young person adopts the current frontier technology. As mentioned above, the country as a whole can trivially advance the frontier technology by having one young person adopt the current frontier technology. In other words, the intergenerational bargaining outcome and the other equilibrium properties, as will be discussed, are not substantively altered by modeling the technology adoption by one young person. I abstract from this modeling detail for a technical convenience.

$\bar{V}_o(n)$ denote the reservation utility of the old generation; and $\bar{V}_y$ the reservation utility of the young generation:

$$\bar{V}_o(n) = \sum_{s=1}^{\infty} \frac{\phi n_s}{\lambda^s} \tag{3}$$

and

$$\bar{V}_y = \beta \cdot \max\{\max_{\tilde{n}}\{\lambda V_o(\tilde{n})\}, \max_{\tilde{n}_0=0}\{V_o(\tilde{n}_{-0})\}\} = \beta\lambda \max_{\tilde{n}}\{V_o(\tilde{n})\}. \tag{4}$$

The reservation utilities are obtained when there are no matches across the generations: Each old person works alone and each young person adopts a technology. The second equality in (4) follows from the fact that the young generation can always raise its utility by the factor $\lambda > 1$ on any human capital distribution with $\tilde{n}_0 = 0$, i.e., for any $\tilde{n}$ with $\tilde{n}_0 = 0$, $V_o(\tilde{m}) = \lambda V(\tilde{n}_{-0})$ if we set $\tilde{m}_s = \tilde{n}_{s+1}$ for all $s \geq 0$. Note that neither generation can force any matches; They can only bargain for a mutually beneficial set of matches. This is in contrast to a voting model in which a generation can potentially impose a set of matches against the preference of the other generation.

Let $q(n)$ denote the technology advancement rule, that is, a function from $\{n\}$ to $\{\tau\}$. Let $g(n) \equiv (g_0(n), g_1(n), \ldots)$ denote the matching behavior, that is, a function from $\{n\}$ to $\{\tilde{n}\}$. Given n, the values of these functions and the utilities of generations solve the Nash bargaining problem: For all n,

$$(g(n), q(n), V_o(n), V_y(n)) = \arg\max_{\tilde{n},\tau,v_o,v_y} \left\{(v_o - \bar{V}_o(n))^{\mu}(v_y - \bar{V}_y)^{1-\mu}\right\}, \tag{5}$$

where the maximization is subject to $v_o + v_y \leq W(n, \tilde{n}, \tau)$. Note that the bargain takes place in each period given that the equilibrium utility functions and the associated policy functions will hold in the future. Equation (5) ensures that the bargaining outcome coincides with the equilibrium utilities and policies in all states of the economy.

Note also that the bargain is across the generations; the individual utilities do not need to be specified. One interpretation is the collective bargain between the generations. Given the ex-ante homogeneity among young people, I could assume that the young generation's aggregate lifetime utility $V_y(n)$ is divided equally among the young people, which may involve an unequal division of output in each of the two periods of the lifetime. In Appendix 1, I show how the bargaining outcome can be implemented in a decentralized bargain with a tax and subsidy policy. An equilibrium is the value functions, W, V_o, and V_y, and the policy functions, g and q, which solve (2) and (5) given (1), (3), and (4).

Given the transferability of utility across the generations, (5) is equivalent to maximizing the aggregate utility and dividing the surplus by μ and $1-\mu$ shares across the generations. In this sense, any conflicts between the current generations are resolved efficiently. This is in contrast to a voting model in which the voting outcome may not maximize the aggregate utility. Efficient bargaining allows me to abstract from the details of institution, in particular, from the separation of economic and political processes as is typical in the literature. Given the sequential structure of bargaining, however, conflicts between the current generations and the future generations are in general not resolved efficiently, and may lead to the stagnation of the economy in the current model.

Given that the surplus is divided by μ and $1-\mu$ across the generations, I can rewrite the value functions as:

$$\hat{W}(n) = (1-q(n)) \cdot \max_{\tilde{n}_0=0}\{\hat{W}_1(n,\tilde{n})\} + q(n) \cdot \max_{\tilde{n}}\{\hat{W}_2(n,\tilde{n})\}, \tag{6}$$

where

$$\hat{W}_1(n,\tilde{n}) = Y(n,\tilde{n}) + \beta\hat{V}_o(\tilde{n}_{-0}); \tag{7}$$

$$\hat{W}_2(n,\tilde{n}) = Y(n,\tilde{n}) + \beta\lambda\hat{V}_o(\tilde{n}); \tag{8}$$

$$\hat{V}_o(n) = \mu \cdot (\hat{W}(n) - \bar{V}_y) + (1-\mu) \cdot \bar{V}_o(n). \tag{9}$$

I assume that $\beta\lambda\mu < 1$ in order to ensure the existence of value functions that solve (6) to (9) given (1), (3), and (4). Further, I assume that $\mu > 0$ in order to avoid a degenerate solution to (6) to (9). (See Step [1] in Appendix 2.) The following proposition summarizes some of the equilibrium properties.

Proposition 1. There is a unique set of continuous and bounded value functions, $\hat{W}$, $\hat{W}_1$, $\hat{W}_2$, and $\hat{V}_o$ that solve (6) to (9). Further, the following equilibrium properties hold:

(a) The value functions are increasing, i.e., $\hat{W}(n) \geq \hat{W}(m)$ for any n and m with $\sum_{s\leq u} n_s \geq \sum_{s\leq u} m_s$ for all $u \geq 1$. This implies that the outside option of the young generation is the all-out-technology-adoption, i.e., $\bar{V}_y = \beta\lambda\hat{V}_o(\bar{n})$ where $\bar{n} \equiv (1, 0, 0, \ldots)$.

(b) The equilibrium features either a perpetual growth or a perpetual stagnation possibly after some periods of continued growth, i.e., for any n, if $q(n) = 0$, $q^t(n) = 0$ for all $t \geq 2$, where $q^t(n)$ denotes the technology advancement decision in period t given n when $t = 1$.

(c) The country stagnates perpetually possibly after some periods of growth if ϕ is low enough and n is concentrated enough to the high human capital levels, i.e., there are cutoff values of ϕ, $\{\bar{\phi}(\bar{S})\}_{\bar{S}\geq 1}$ with $\bar{\phi}(\bar{S})$ decreasing in $\bar{S}$, so that $q^t(n) = 0$ for some t if $\phi < \bar{\phi}(\bar{S})$ and $\sum_{s=1}^{\bar{S}} n_s = 1$.

Proof: See Appendix 2. The existence of the unique set of continuous and bounded value functions is Result 1. Property (a) combines Lemmas 1 and 2. Property (b) is Result 2. Property (c) is from Result 3.

In order to gain intuition for the logic of growth and stagnation in equilibrium, first consider a social planner who maximizes the sum of discounted utilities of all generations. Note that there are no costs of advancing the frontier technology at the aggregate level although there are costs of adopting the frontier technology at the individual level (see footnote 4). Then, the perpetual stagnation can be improved on by advancing the frontier technology continuously and discarding the old technologies (i.e., having some young people adopt the frontier technology instead of matching with the old people whose human capital is far from the frontier technology) eventually. As alluded to above, therefore, any stagnation chosen by the current generations in equilibrium represents dynamic inefficiency.

Let the aggregate utility gain from holding the frontier technology in equilibrium be called the stagnation premium:

$$SP(n) \equiv \max_{\tilde{n}_0=0}\{\hat{W}_1(n,\tilde{n})\} - \max_{\tilde{n}}\{\hat{W}_2(n,\tilde{n})\}. \tag{10}$$

To see that the stagnation premium can be positive, suppose that the old generation's human capital is concentrated on the best technology in use (i.e., $n_1 = 1$). Then, depending on the parameter values, it can be optimal for no young person to adopt the frontier technology even if the frontier technology advances, since today's cost of output loss relative to tomorrow's productivity gain, that accrues to today's young generation when it become the old generation, may be too high (i.e., $g_0(\bar{n}) = 0$ under the constraint that $\tau = 1$). Then, the young generation will not benefit from the higher productivity associated with the technological advancement. On the other hand, today's technological advancement allows tomorrow's young generation to adopt an even better technology, raising their outside option and reducing the size of surplus to be divided. Since a fixed μ share of the surplus accrues to the old generation, this lowers the utility of tomorrow's old generation or, equivalently, today's young generation. Therefore, in contrast to the social planner discussed above, the current generations will strictly prefer not to advance the technology today if no young person will adopt the frontier technology (i.e., if $g_0(\bar{n}) = 0$ under the constraint that $\tau = 1$, $q(\bar{n}) = 0$ without the constraint). Expanding on the above reasoning, we can see that if the old generation's human capital is sufficiently skewed to the best technology in use h_1, the current generations may prefer not to advance the technology at all.

The above discussion brings out two forces that determine the growth path of the economy. One is tomorrow's productivity gain, that accrues to today's young generation when it becomes old, from individual technology adoption by today's young generation relative to today's cost of output loss from not fully utilitizing the individual human capital of today's old generation. Since tomorrow's productivity gain relative to today's cost of output loss is maximized when adopting the frontier technology, this force, call it *productivity effect*, incentivizes the current generations to advance the frontier technology unless, as discussed above, no young person would adopt the frontier technology in any case. The other force is the negative effect of advancing the frontier technology on tomorrow's utility of today's young generation, which operates through tomorrow's bargain. The latter force, call it *bargaining effect*, incentivizes the current generations not to advance the frontier technology, leading to a stagnation if it is stronger than the productivity effect. The following proposition summarizes the conditions for growth and stagnation.

Proposition 2. There are cut-off values, $\bar{\phi}$, $\tilde{\phi}$, $\bar{\mu}$, and $\bar{x}$, which depend on (the other) model parameter values, so that:

(a) If $\phi > \bar{\phi}$, the country grows perpetually regardless of n.

(b) If $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$, the country grows perpetually if $n_1 \in (\bar{x}, 1-\bar{x})$, and stagnates perpetually possibly after, at most, two periods of growth if $n_1 \in [0, \bar{x}] \cup [1-\bar{x}, 1]$.

(c) If $\phi < \tilde{\phi}$ or if $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu > \bar{\mu}$, the country stagates perpetually possibly after some periods of growth if $n_1 + n_2 = 1$.

Proof: See Appendix 2. Part (a) is Result 8. Part (b) combines Results 5 and 6. Part (c) combines Results 3, 4, and 7. In order to simplify notation, I have set: $\bar{\phi} \equiv \bar{\phi}(1)$, $\bar{\mu} \equiv 1/(\beta + \beta\lambda)$, $\bar{x} \equiv 1 - \check{n}$ if $\tilde{\phi} < \phi \leq \hat{\phi}(2)$, $\bar{x} \equiv \hat{n}$ if $\hat{\phi}(2) \leq \phi < \bar{\phi}(1)$.

If $\phi > \bar{\phi}$, the current generations always choose to advance the frontier technology. Intuitively, a higher value of ϕ implies a lower opportunity cost of adopting a technology in terms of the lost current output. It also implies a larger gain from adopting a technology since it raises the share of the productivity gain that accrues to today's young generation, by raising its outside option tomorrow, when it becomes the old generation. Thus the productivity effect is stronger with a higher value of ϕ. If $\phi \geq \bar{\phi}$, the productivity effect dominates the bargaining effect regardless of the human capital distribution n.

If $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$, where $\tilde{\phi} > \bar{\phi}(2)$, the economy may stagnate depending on n. Call this equilibrium Conditional Growth Equilibrium (CGE) in anticipation of comparing with the world equilibria in Section 5. As discussed earlier, a human capital distribution that is more skewed to the best technology in use h_1 imply, on average, a higher opportunity cost of adopting a technology, so the bargaining effect is more likely to dominate the productivity effect. There are four possible growth paths:

(1) Immediate and perpetual stagnation: There is a cutoff value $\tilde{x} \in [1-\bar{x}, 1]$, that possibly depends on n_2, such that the economy stagnates immediately and perpetually if $n_1 \in [\tilde{x}, 1]$. In the first period of stagnation, some old people, whose human capital is below the best technology in use h_1, may not be matched with the young people, allowing some young people to adopt h_1.

(2) Perpetual stagnation after one period of growth: If $n_1 \in [0, \bar{x}]$, at least $1 - n_1$ fraction of the old people, whose human capital is below the best technology in use h_1, are not matched with the young people, allowing at least $1 - n_1$ fraction of the young people to adopt the frontier technology h_0. This leads to tomorrow's $n_1 \in [\tilde{x}, 1]$.

(3) Perpetual stagnation after two periods of growth: If $n_1 \in [\bar{x}, \tilde{x})$, $1 - n_1$ fraction of the old people, whose human capital is below the best technology in use h_1, are not matched with the young people, allowing $1 - n_1$ fraction of the young people to adopt the frontier technology h_0. This leads to tomorrow's $n_1 \in [0, \bar{x}]$ and to $n_1 \in [\tilde{x}, 1]$ the day after tomorrow.

(4) Perpetual growth: If $n_1 \in (\bar{x}, 1 - \bar{x})$, $1 - n_1$ fraction of the old people, whose human capital is below the best technology in use h_1, are not matched with the young people, allowing at least $1 - n_1$ fraction of the young people to adopt the frontier technology h_0. This leads to tomorrow's $n_1 \in (\bar{x}, 1 - \bar{x})$. By induction, the economy grows perpetually.

If $\phi < \tilde{\phi}$ or if $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu > \bar{\mu}$, the country eventually enters and stays in the stagnation zone of the human capital distribution if $n_1 + n_2 = 1$ initially. The intuition for the threshold $\tilde{\phi}$ is the converse of the threshold $\bar{\phi}$. As for $\bar{\mu}$, a higher value of μ raises the (old generation's share of) stagnation premium, raising the incentive of current generations to stagnate or move into the stagnation zone of the human capital distribution.[5] I have only a partial characterization of the equilibrium when $n_1 + n_2 < 1$, (c) in Proposition 1. Thus I cannot rule out the possibility of perpetual growth when $n_1 + n_2 < 1$. Throughout the analysis in this paper, the focus is on the equilibrium path starting from the initial distribution with $n_1 + n_2 = 1$. The current model is about countries near the frontier technology and not about countries catching up from far behind: A country can upgrade its technologies to the frontier technology in one generation.

5. The Model Economy: Two Countries

There are two countries, indexed by $i = A, B$, each of which faces the same environment as in Section 4, except that the frontier technology is shared across countries.[6] Let

[5] To elaborate, the relevant general equilibrium effects are not easy to disentangle, but can be roughly summarized as follows. A higher value of μ raises the old generation's share of the surplus. This raises the aggregate utility of the current generations, that includes the discounted utility of today's young generation when it becomes the old generation, which in turn raises the size of the surplus and the old generation's utility. Due to this compounding of the surplus, the aggregate utility under stagnation can rise substantially. The same mechanism operates on the growth path too. However, the size of the surplus is smaller on the growth path since the human capital on average stays further away from the frontier. Thus a higher value of μ raises the stagnation premium, the gap in the surplus between stagnation and growth. As mentioned earlier, a higher value of ϕ reduces the stagnation premium in part by raising the old generation's utility on the growth path by more than that on the stagnation path. The difference from μ is that a higher value of ϕ raises the old generation's utility via the outside option of the old generation and not via the surplus. The effect of a higher value of ϕ on the outside option is larger on the growth path than under stagnation since a growth scales up the outside option. The effects of the other parameters, λ and β, are less clear than those of ϕ and μ, with a mixture of growth-enhancing and stagnation-promoting effects as their values change.

[6] The previous version of this paper assumed that there are arbitrarily many countries. The main results of the two-country world, assumed here, can be generalized to a many-country world at a significant cost of notation and extended proofs. An insight that can be gained from modeling the many-country world is that the perpetual growth is more likely when there are more countries. Loosely speaking, with

n, $\tilde{n}$, and τ denote the vectors of human capital distributions and technology advancement decisions across countries: $\tau \equiv (\tau^A, \tau^B)$; $n \equiv (n^A, n^B)$; $n^i \equiv (n^i_1, n^i_2, n^i_3, \ldots)$; $\tilde{n} \equiv (\tilde{n}^A, \tilde{n}^B)$; and $\tilde{n}^i \equiv (\tilde{n}^i_0, \tilde{n}^i_1, \tilde{n}^i_2, \ldots)$. The feasibility requires: $\tilde{n}^i_s \leq n^i_s$ for all $s \geq 1$; and $\tilde{n}^i = (0, n)$ if $\tau^i = 0$.[7] Let $q^i(n)$ and $g^i(n)$ denote the policy functions of country i. The value functions are:

$$\check{W}^i(n) = (1 - \max_j\{q^j(n)\}) \cdot \max_{\tilde{n}^i_0=0}\{\check{W}^i_1(n, \tilde{n}^i)\} + \max_j\{q^j(n)\} \cdot \max_{\tilde{n}^i}\{\check{W}^i_2(n, \tilde{n}^i)\}, \quad (6)'$$

where

$$\check{W}^i_1(n, \tilde{n}^i) = Y(n^i, \tilde{n}^i) + \beta \check{V}^i_o(G^{-i}_{-0}(n, \tilde{n}^i)); \quad (7)'$$

$$\check{W}^i_2(n, \tilde{n}^i) = Y(n^i, \tilde{n}^i) + \beta\lambda \check{V}^i_o(G^{-i}(n, \tilde{n}^i)); \quad (8)'$$

$$\check{V}^i_o(n) = \mu \cdot (\check{W}^i(n) - \bar{V}^i_y(n)) + (1-\mu) \cdot \bar{V}^i_o(n^i); \quad (9)'$$

$$\bar{V}^i_o(n^i) = \sum_{s=0}^{\infty} \frac{\phi n^i_s}{\lambda^s}; \quad (3)'$$

$$\bar{V}^i_y(n) = \beta \cdot \max\{\max_{\tilde{n}^i}\{\lambda \check{V}^i_o(G^{-i}(n, \tilde{n}^i))\}, (1 - q^j(n)) \max_{\tilde{n}^i_0=0}\{\check{V}^i_o(G^{-i}_{-0}(n, \tilde{n}^i))\}\}; \quad (4)'$$

$\tilde{n}^i_{-0} \equiv (\tilde{n}^i_1, \tilde{n}^i_2, \ldots)$; $g^i_{-0}(n) \equiv (g^i_1(n), g^i_2(n), \ldots)$; $G^{-A}(n, \tilde{n}^A) \equiv (\tilde{n}^A, g^B(n))$; $G^{-B}(n, \tilde{n}^B) \equiv (g^A(n), \tilde{n}^B)$; $G^{-A}_{-0}(n, \tilde{n}^A) \equiv (\tilde{n}^A_{-0}, g^B_{-0}(n))$; $G^{-B}_{-0}(n, \tilde{n}^B) \equiv (g^A_{-0}(n), \tilde{n}^B_{-0})$; and $i \neq j$ in (4)′. An equilibrium is the value functions, $\{\check{W}^i\}$ and $\{\check{V}^i_o\}$, and the policy functions, $\{g^i(n)\}$ and $\{q^i(n)\}$, that solve (6)′ to (9)′ given (1), (3)′, and (4)′. The stagnation premium is:

$$SP^i(n) = \begin{cases} 0 & \text{if } q^j(n) = 1; \\ \max_{\tilde{n}^i_0=0}\{\check{W}^i_1(n, \tilde{n}^i)\} - \max_{\tilde{n}^i}\{\check{W}^i_2(n, \tilde{n}^i)\} & \text{if } q^j(n) = 0, \end{cases} \quad (10)'$$

where $i \neq j$. Note that the stagnation premium of a country is zero if the other country advances the frontier technology: It takes only one country to advance the frontier technology. Thus, there can be a negative effect of technological advancement by one country

more countries, the conditions of diversity (i.e., the technological advancement being in the interests of at least one country at all times) and coordination failures (i.e., not all countries expecting every country to withhold technological advancement), to be discussed below, are more easily satisfied.

[7] It is feasible to have $\tilde{n}^i = (0, n^i)$ and $\tau^i = 1$. See footnote 4.

on the current generations in the other country. On the other hand, technological advancement benefits the future generations in the world by raising the future productivity and raising the outside option of tomorrow's young generation as in Section 4.

If $\phi > \bar{\phi}$, case (a) in Proposition 2, the only world equilibrium is the perpetual growth as in autarky. (See Results 9 and 10 in Appendix 3.) If $\phi < \tilde{\phi}$ or if $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu > \bar{\mu}$, case (c) in Proposition 2, there is no simple characterization of the world equilibrium. In particular, the symmetric equilibrium (i.e., equilibria in which switching country names do not change the equilibrium path and the utilities of either country) may not exist depending on the parameter values. Below I characterize the equilibrium when $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$, case (b) in Proposition 2 that supports Conditional Growth Equilibrium (CGE).

Unlike the equilibrium of a country in autarky, the equilibrium of the two-country world is not unique. This is perhaps not surprising given that each country makes decisions taking as given the decisions of the other country as well as those of the future generations in the country. Multiple equilibria can result, in particular, due to the coordination failure among countries as will be discussed below. My strategy is, on the one hand, to look for an equilibrium that is heuristically similar to that in autarky. Since there are (trivially) no coordination failures in the equilibrium in autarky, it is plausible that a similar equilibrium in the multiple-country world represents a reasonable benchmark biased toward no coordination failure (see footnote 9 for some details). On the other hand, I consider the equilibrium with a complete coordination failure, stated in the following proposition.

Proposition 3. If $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$, there exists an equilibrium in which the world economy grows perpetually regardless of n.

Proof: See Result 11 in Appendix 3.

In this equilibrium, the old people with the best technology in use h_1 are matched with the young people and the remaining young people adopt the frontier technology h_0 regardless of n. Consequently, the world economy grows perpetually. Call this equilibrium Unconditional Growth World Equilibrium (UGWE). From the perspective of a single country, the frontier technology advances exogenously regardless of its own technology advancement decision. Therefore, there is no stagnation premium and both countries advance the frontier technology, sustaining the growth path. Given the lack of stagnation premium, the outside option of the young generation is the all-out-technology adoption as in autarky:

$$\bar{V}_y^i(n) = \max_{\tilde{n}^i}\{\lambda \check{V}_o^i(G^{-i}(n, \tilde{n}^i))\} = \lambda \check{V}_o^i(G^{-i}(n, \bar{n})). \tag{4$''$}$$

This equilibrium hinges on the complete coordination failure: The current generations across countries are unable to coordinate their decisions whenever they prefer the world economy to stagnate.[8] Thus I have the following remark.

Remark. The coordination failure among countries promotes the growth of the world economy.

The following proposition characterizes an equilibrium that is heuristically similar to Conditional Growth Equilibrium (CGE), (b) in Proposition 2. Generally, I restrict the

[8] The extreme case of the coordination failure is when $n^1 = n^2 = \bar{n}$. I have $g^1(n) = g^2(n) = (0, \bar{n})$while $q^1(n) = q^2(n) = 1$. By choosing not to advance the frontier technology, the current generations in either country are not worse off regardless of the technology advancement decision of the other country, and would be better off if the other country chose not to advance the frontier technology either. Recall that I assumed away any costs of advancing the frontier technology at the aggregate level (see footnotes 4 and 7). If there is a non-negligible cost of advancing the frontier technology at the aggregate level, the perpetual growth path may not be sustainable for some n. On the other hand, there are costs of stagnation from which I have abstracted, too. For example, suppose that not all of the young people are able to adopt the frontier technology. Then, the world economy would never be at or near the state of $n^1 = n^2 = \bar{n}$, and the current generations in either country would be always better off by having some young people adopt the frontier technology as long as the other country advances the frontier technology. All these additional modeling details would not add substance to the logic of stagnation in the equilibrium as modeled here.

equilibrium to be symmetric across countries: Shuffling country names does not change the equilibrium.

Proposition 4. If $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$, there exists an equilibrium with following properties. There are cutoff values, $\hat{\phi}$, $\bar{y}$, and $\hat{y}$, which depend on (the other) model parameter values, so that:

(a) If $n_1^j \in [0, 1-\bar{y})$ for some j and $n_1^j \in (\bar{y}, 1]$ for some j, the world economy grows perpetually.

(b) If $n_1^j \in [0, \bar{y}]$ for both j or $n_1^j \in [1-\bar{y}, 1]$ for both j, the world economy stagnates perpetually after, at most, two periods of growth, except when the condition for (c) applies.

(c) If $\hat{\phi} \leq \phi \leq \bar{\phi}$, $n_1^j \in [\max\{\bar{y}, 1-\bar{y}\}, 1]$ for all j, and $n_1^j \in [\max\{\bar{y}, 1-\bar{y}\}, \hat{y}]$ for some j, the world economy grows perpetually.

Proof: See Results 12 and 13 in Appendix 3. In order to simplify notation, I have set: $\hat{\phi} \equiv \hat{\phi}(2)$, $\bar{y} \equiv 1 - \check{m}$ if $\tilde{\phi} < \phi \leq \hat{\phi}$, $\bar{y} \equiv \hat{m}$ if $\hat{\phi} \leq \phi < \bar{\phi}$, and $\hat{y} \equiv 1 - \hat{\rho}\hat{m}$.

Call the above equilibrium Conditional Growth World Equilibrium (CGWE). There are several points of comparison with Conditional Growth Equilibrium (CGE) and Unconditional Growth World Equilibrium (UGWE). The outside option of the young generation is the all-out-technology adoption as in CGE and UGWE: Equation (4)$''$ holds for CGWE as well. The advantage of adopting the frontier technology h_0 over adopting a technology in use, h_s with $s \leq 1$, overcomes any strategic disadvantage there may be if there are no gains from joint production with the old generation. The cutoff value $\bar{y}$ segments the range of n_1^j in terms of growth paths and is analogous to $\bar{x}$ in CGE with some differences stated below.[9]

[9] The world economy stagnates immediately if $n_1^j \geq \tilde{y}$ for both j. The value $\tilde{y}$ is not unique. A high value implies that the countries may grow even though they both wish to stagnate. A low value implies that the countries may stagnate even though they both wish to grow. Thus, there can be coordination failures of both types. However, $\tilde{y}$ only affects the timing of stagnation in CGWE as $\tilde{x}$ does in CGE, except

First, observe the asymmetry between conditions for a growth and those for stagnation in the two-country world: The perpetual stagnation, case (b), requires the same conditions across countries while the perpetual growth, case (a), requires different conditions across countries. In particular, the world economy grows perpetually regardless of the cutoff value of $\bar{y}$ if $n^A = 0$ and $n^B = 1$, the maximum diversity across countries. Intuitively, holding the frontier technology requires both countries to cooperate; the frontier technology advances as long as it is advantageous to, at least, one country. Diversity raises the chance of the technological advancement to be in the interests of at least one country at all times. Thus I have the following remark.

Remark. The diversity of countries in human capital distribution promotes the growth of the world economy.

Second, we have $\bar{y} > \bar{x}$ so that $(\bar{y}, 1 - \bar{y}) \subset (\bar{x}, 1 - \bar{x})$. This implies that, if the two countries have the same human capital distribution, $n^A = n^B$, the zone of perpetual growth is smaller in the two-country world economy than in autarky. This reflects a general equilibrium effect which raises the advantage of stagnation in the two-country world in comparison to the single-country world. Intuition can be gained in steps. In the stagnating single-country world, the young generation's outside option of all-out-technology-adoption always leads to a stagnation after one period of growth, so the outside option carries the stagnation premium in the following period. This premium raises the outside option of the young generation and thereby lowers the old generation's utility. In the stagnating two-country world, the young generation's outside option in one country triggers the perpetual growth of the world economy. The old generation's utility in a country in the stagnating

when the 'lock-out' effect, case (c) in Proposition 4, applies: a higher value of $\tilde{y}$ permits a greater chance of a temporary growth that may lead to a perpetual growth due to the strategic behavior of a country, as discussed toward the end of this section. The range of n_1^j as stated in (c), Proposition 4, holds if $\tilde{y} = 1$ when $\hat{\phi} \leq \phi < \bar{\phi}$, maximizing the 'lock-out' effect. I chose the maximum value of $\tilde{y}$ to simplify the proof when $\hat{\phi} \leq \phi < \bar{\phi}$.

two-country world is higher than that in the stagnating single-country world due to the absence of the stagnation premium attached to the young generation's outside option.[10] Since the aggregate utility of current generations includes the discounted value of the current young generation's utility when old, the two-country world has a higher incentive to move into, or stay in, the stagnation zone. The above discussion can be summarized as a 'lock-in' effect. Thus I have the following remark.

Remark. A 'lock-in' effect may lead to a perpetual stagnation of the world economy.

Third, case (c) in Proposition 4 reflects the strategic move by one country: One of the countries strategically chooses to have a sufficient fraction of young people to adopt the frontier technology so as to put the world economy on a perpetual growth path and obtain a greater utility when the current young generation becomes old, leaving some of the currently old people with the best technology in use h_1 unmatched.[11] This can be described as a 'lock-out' effect. Thus I have the following remark.

[10] Further, in the stagnating two-country world, the outside option exercised by the young generation of a country leaves the other country fall behind the frontier technology so that it makes the all-out-technology-adoption in the following period. In the country whose young generation (call it YG 1) exercised the outside option, the outside option exercised by the young generation (call it YG 2) in the following period would lead to a world where both countries have made the all-out-technology-adoption. The world economy would then stagnate in subsequent periods. Since the old generation's utility is higher in the stagnating two-country world than in the stagnating single-country world, YG 2 has a better outside option and the utility of YG 1 is lower in the period following the initial outside option than if they were in the single-country world. This further lowers the outside option of the young generation (YG 1) and raises the old generation's utility in the stagnating two-country world in comparison to the single-country world.

[11] To elaborate, without the strategic move, the two countries would stagnate after two periods of growth, similar to the growth path (3) under Conditional Growth Equilibrium (CGE) in autarky. In particular, $1 - n_1$ fraction of the young people would adopt the frontier technology today, while all young people would adopt the frontier technology tomorrow in order to maximize the stagnation premium the day after tomorrow. Tomorrow's all-out-technology-adoption implies that the utility of tomorrow's old generation or, equivalently, today's young generation is only the outside option. By having additional young people adopt the frontier technology today, a country can enduce tomorrow's human capital distribution to be in the perpetual growth zone, similar to the growth path (4) under Conditional Growth Equilibrium (CGE), thereby creating surplus to be bargained over tomorrow.

Remark. A 'lock-out' effect may lead to a perpetual growth of the world economy.

6. The Comparison of World Economies

I compare three versions of the world economy. The first world economy is that each of the two countries is in autarky as described in Section 4. The second world economy is that the two countries share the frontier technology but are separated in the intergenerational bargain as described in Section 5. The third world economy is a political union with a single frontier technology and a single bargain, equivalent to a single country in autarky in Section 4.

I consider economic integration as the transition from the first to the second world economy. From the perspective of a country, economic integration is the exogenously given opportunity to adopt the frontier technology in the world rather than the endogenous act of adopting the frontier technology in the world, although the opportunity may lead to the act in equilibrium. From the perspective of the world economy, economic integration is the expansion of the opportunity to adopt the frontier technology in the world. In other words, the opportunity to adopt the frontier technology in the world defines the boundary of an economically integrated world.[12]

I consider political integration as the transition from the second to the third world economy. Thus, political integration is efficient bargaining among the current generations in an economically integrated world. From the perspective of a country, political integration may or may not be a voluntary event. It may be a consequence of a conquest by a foreign country or, more fundamentally, of ecological conditions conducive to conquests (see footnote 2). On the other hand, the current generations of the world may choose

[12] Economic integration modeled in terms of technology adoption is admittedly stylized and abstracts from various channels such as trade and investment. Similarly, political integration modeled in terms of efficient bargaining, as discussed below, is stylized and abstracts from various channels such as taxes and subsidies.

political integration as a means of stalling the growth of the world economy and extracting the stagnation premium, as will be shown in Section 7.

I assume that the transition is an unexpected or a small-probability event so that the evaluation of the transition can be done in terms of the value functions in Sections 4 and 5 in approximation. Comparing the equilibria in the two sections, I have the following variations of transition. If $\phi > \bar{\phi}$, a country grows unconditionally both in autarky and in the economically integrated world. If $\phi < \tilde{\phi}$ or if $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu > \bar{\mu}$, country i, for $i = A, B$, stagnates eventually in autarky if $n_1^i + n_2^i = 1$ while, if an equilibrium exists, there may be a perpetual growth in the economically integrated world depending on parameter values and the distribution of human capital across countries. A consequence is that economic integration will *not* turn a perpetual growth to an eventual stagnation if $n_1^j + n_2^j = 1$ for both j.

Below I present the case of $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$ in detail, comparing Conditional Growth Equilibrium (CGE), Unconditional Growth World Equilibrium (UGWE), and Conditional Growth World Equilibrium (CGWE). First, consider economic transition assuming an uncoordinated world economy upon economic integration. I have the following summary of the stagnation-to-growth transition.

Corollary 1. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$. Under the transition from CGE to UGWE, both countries stagnate eventually in autarky but grow perpetually in the economically integrated world if $n_1^j \in [0, \bar{x}] \cup [1 - \bar{x}, 1]$ for both j.

Corollary 1 combines the stagnation condition in (b) in Proposition 2 and Proposition 3, and reflects the coordination failure present in UGWE.

Now, consider economic transition assuming a (fairly) coordinated world economy upon economic integration. I have the following summary of the stagnation-to-growth and the growth-to-stagnation transitions.

Corollary 2. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$. Under the transition from CGE to CGWE,

(a) Both countries stagnate eventually in autarky but grow perpetually in the economically integrated world if $n_1^i \in [0, \min\{\bar{x}, 1-\bar{y}\}]$ and $n_1^j \in [\max\{1-\bar{x}, \bar{y}\}, 1]$ for $i \neq j$.

(b) Both countries grow perpetually in autarky but stagnate eventually in the economically integrated world if $n_1^j \in [\bar{x}, \min\{1-\bar{x}, \bar{y}\}]$ for both j or $n_1^j \in [\max\{\bar{x}, 1-\bar{y}\}, 1-\bar{x}]$ for both j, except when the condition for (c) in Proposition 4 (i.e, $\hat{\phi} \leq \phi < \bar{\phi}$, $n_1^j \in [\max\{\bar{y}, 1-\bar{y}\}, 1]$ for all j, and $n_1^j \in [\max\{\bar{y}, 1-\bar{y}\}, \hat{y}]$ for some j) applies.

(c) Both countries stagnate eventually in autarky but grow perpetually in the economically integrated world if $\hat{\phi} \leq \phi \leq \bar{\phi}$, $n_1^j \in [\max\{1-\bar{x}, \bar{y}\}, 1]$ for all j, and $n_1^j \in [\max\{1-\bar{x}, \bar{y}\}, \hat{y}]$ for some j.

Case (a) combines the stagnation condition in (b) in Proposition 2 and the growth condition in (a) in Proposition 4, and reflects the effect of diversity. Case (b) combines the growth condition in (b) in Proposition 2 and the stagnation condition in (b) in Proposition 4, and reflects the 'lock-in' effect. Case (c) combines the stagnation condition in (b) in Proposition 2 and the growth condition in (c) in Proposition 4, and reflects the 'lock-out' effect.[13]

Now consider political integration. The distribution of human capital in the political union is:

$$n^u = \eta^A n^A + \eta^B n^B,$$

where η^i is country i's share of the world population: $\eta^A + \eta^B = 1$. First, consider political transition assuming an uncoordinated world economy under political fragmentation. I have

[13] Corollaries 1 and 2 exhaust the growth-to-stagnation or the stagnation-to-growth transitions in which both countries either stagnate eventually or grow perpetually in autarky, but does not address the transitions from the world economy in which one country in autarky stagnates eventually while the other country in autarky grows perpetually. These hybrid transitions can be easily classified but their interpretation is problematic: The technological advancement of one country on the perpetual growth path would eventually diffuse to the other country in stagnation, which is not modeled in this paper.

the following summary of the growth-to-stagnation transition.

Corollary 3. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$. Under the transition from UGWE to CGE, the world economy grows perpetually in political fragmentation but stagnates eventually in political union if $n_1^u \in [0, \bar{x}] \cup [1-\bar{x}, 1]$.

Corollary 3 is essentially equivalent to Corollary 1 except that the change is in the opposite direction, and reflects the elimination of coordination failures upon political integration.

Now, consider political transition assuming a (fairly) coordinated world economy under political fragmentation. I have the following summary of the stagnation-to-growth and the growth-to-stagnation transitions.

Corollary 4. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$ and $\mu \leq \bar{\mu}$. Under the transition from CGWE to CGE,

(a) The world economy grows perpetually in political fragmentation but stagnates eventually in political union if $n_1^u \in [0, \bar{x}] \cup [1-\bar{x}, 1]$, $n_1^i \in [0, 1-\bar{y}]$, and $n_1^j \in [\bar{y}, 1]$ for $i \neq j$.

(b) The world economy stagnates eventually in political fragmentation but grows perpetually in political union if $n_1^u \in [\bar{x}, 1-\bar{x}]$ and $n_1^j \in [0, \bar{y}]$ for both j, or if $n_1^u \in [\bar{x}, 1-\bar{x}]$ and $n_1^j \in [1-\bar{y}, 1]$ for both j, except when the condition for (c) in Proposition 4 (i.e, $\hat{\phi} \leq \phi < \bar{\phi}$, $n_1^j \in [\max\{\bar{y}, 1-\bar{y}\}, 1]$ for all j, and $n_1^j \in [\max\{\bar{y}, 1-\bar{y}\}, \hat{y}]$ for some j) applies.

(c) The world economy grows perpetually in political fragmentation but stagnates eventually in political union if $\hat{\phi} \leq \phi \leq \bar{\phi}$, $n_1^j \in [\max\{1-\bar{x}, \bar{y}\}, 1]$ for all j, and $n_1^j \in [\max\{1-\bar{x}, \bar{y}\}, \hat{y}]$ for some j.

Corollary 4 is essentially equivalent to Corollary 2 except that the change is in the opposite direction. Case (a) reflects the elimination of diversity upon political integration. Case (b) reflects the elimination of the 'lock-in' effect. Case (c) reflects the elimination of the 'lock-out' effect.

Corollaries 1 to 4 formalize the growth enhancing effect of the diversity and coordination failures, applicable when the world economy is economically integrated but politically fragmented. They also show that when the diversity and coordination failures are limited, growth paths are affected by general equilibrium effects like 'lock-in' and 'lock-out' effects.

7. Welfare Effects

Now consider the effect of economic integration on the aggregate utility of the country in transition. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and Unconditional Growth World Equilibrium (UGWE) holds after economic integration. In order to motivate the choice of UGWE as the equilibrium, note the fundamental asymmetry between the advancement and the stagnation of the frontier technology when there are two countries: It takes only one country to advance the technology while it takes both countries to hold the technology. Loosely speaking outside the current model, random factors can disrupt stagnation more easily than growth in a multi-country world. Further, focus on the equilibrium path starting from the initial distribution with $n_1^j + n_2^j = 1$ for both j. This is to streamline presentation. Substantively, the current model is about countries near the frontier technology and not about countries catching up from far behind, as mentioned earlier.

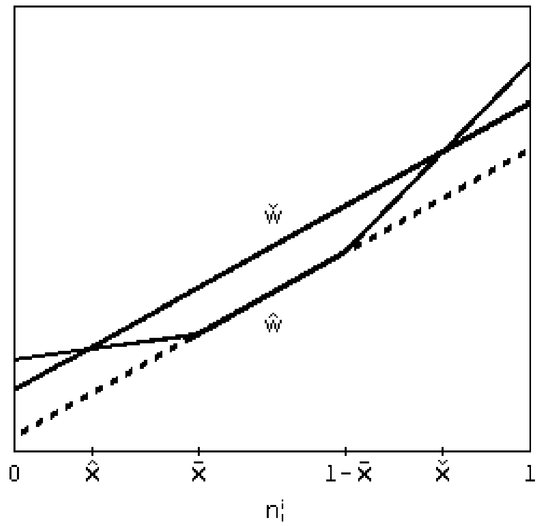
Under UGWE, the aggregate utility of a country is not affected by the human capital distribution of the other country: $\check{W}^i(n)$ is independent of n^j for $i \neq j$. Thus, the utility changes of the country can be characterized in terms of the human capital distribution of the country in transition only.

Proposition 5. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and $n_1^i + n_2^i = 1$ for some country i. There are cutoff values $\hat{x}$ and $\check{x}$, which depend on model parameter values and satisfy $\hat{x} < \bar{x}$ and $1 - \bar{x} < \check{x}$, so that under the transition from CGE to UGWE, economic integration raises the aggregate utility of country i if $n_1^i \in (\hat{x}, \check{x})$, and lowers the aggregate utility of country i if $n_1^i \in [0, \hat{x}) \cup (\check{x}, 1]$.

Proof: See Appendix 4.

The utility changes are due to the elimination of stagnation premium and can be understood as the combination of two effects, illustrated by the figure below.

Figure: Aggregate Utility of Country i



Under UGWE, $\check{W}^i(n)$ is linear in n_1^i. Under CGE, $\hat{W}(n^i)$ consists of three segments, each of which is linear in n_1^i. The gaps between the first segment ($n_1^i \in [0, \bar{x}]$) and the dotted line and between the third segment ($n_1^i \in [1-\bar{x}, 1]$) and the dotted line represent the (anticipated) stagnation premium. One effect of economic integration is to eliminate the (anticipated) stagnation premium. The other effect is similar to the 'lock-in' effect associated with Conditional Growth World Equilibrium (CGWE) in Section 5: The lack of stagnation premium lowers the outside option of the young generation and thereby raises the old generation's utility. Since the aggregate utility of current generations includes the discounted value of the current young generation's utility when old, economic integration raises the aggregate utility for all n_1^i. The second effect leaves segments of n_1^i, $(\hat{x}, \bar{x}) \cup (1-\bar{x}, \check{x})$, where the aggregate utility rises upon economic integration despite the (anticipated) loss of stagnation premium. These properties of the utility changes imply:

Corollary 5. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and $n_1^i + n_2^i = 1$ for some country i. Under the transition from CGE to UGWE, country i would have stagnated perpetually after at most one period of growth in autarky if economic integration lowers the aggregate utility of the country.

The above proposition highlights that economic integration is an exogenous and possibly involuntary event to a country: Economic integration eliminates the option to hold the frontier technology and to enjoy the stagnation premium. Thus Corollary 5 can be interpreted as the following remark.

Remark. If economic integration is involuntary, it is due to the loss of stagnation premium upon economic integration.

Now consider a once-and-for-all formation of political union. As above, assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and that Unconditional Growth World Equilibrium (UGWE) holds in political fragmentation, and focus on the equilibrium path starting from the initial distribution with $n_1^j + n_2^j = 1$ for both j. Since $\check{W}^i(n)$ is linear in n^i and independent of n^j for $i \neq j$, with a slight abuse of notation, I can write:

$$\sum_i \eta^i \check{W}(n^i) = \check{W}\left(\sum_i \eta^i n^i\right) = \check{W}\left(n^u\right) \tag{11}$$

for all $\{\eta^i\}$ and $\{n^i\}$. Thus, the utility comparison of a politically fragmented world and a political union is analogous to the utility comparison of a country in autarky and a country in economic integration in Proposition 5 and Corollary 5.

Proposition 6. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and $n_1^j + n_2^j = 1$ for both j. Under the transition from UGWE to CGE, political integration raises the worldwide aggregate utility if $n_1^u \in [0, \hat{x}) \cup (\check{x}, 1]$, and lowers the worldwide aggregate utility if $n_1^u \in (\hat{x}, \check{x})$.

Corollary 6. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and $n_1^j + n_2^j = 1$ for both j. Under the transition from UGWE to CGE, the political union, once formed, stagnates perpetually after at most one period of growth if political integration raises the worldwide aggregate utility.

Reversing the reasoning in the previous remark, I have the following interpretation of Corollary 6.

Remark. If political integration is voluntary, it is due to the gain of stagnation premium upon political integration.

Now consider the choice of political integration, building on the above characterization of a once-and-for-all formation of political union. Suppose that the opportunity to form or break up a political union is a small probability event in every period so that the utility comparison can be done as if the formation or the break-up is permanent in approximation. Assume that the decision of forming or breaking up a political union is made through a cross-country bargain and that the cross-country bargain is efficient: The bargaining outcome maximizes the worldwide aggregate utility of the current generations.

Proposition 7. Assume that $\tilde{\phi} \leq \phi \leq \bar{\phi}$, $\mu \leq \bar{\mu}$, and $n_1^j + n_2^j = 1$ for both j. Assume that the formation and the break-up are the transitions between UGWE and CGE in approximation. The political union will form eventually if $n_1^u \in [0, \max\{\hat{x}, 1-\check{x}\}) \cup (\min\{\check{x}, 1-\hat{x}\}, 1]$. Once formed, the political union will stagnate perpetually after, at most, one period of growth, and it will never break-up. The world economy will always remain politically fragmented and grow perpetually if $n^u \in [\max\{\hat{x}, 1-\check{x}\}, \min\{\check{x}, 1-\hat{x}\}]$.

Proof: See Appendix 4.

That the formation of the political union leads to a perpetual stagnation underlines its nature: The political union is a means of eliminating the coordination failure and the diversity in the growth-incentive across countries through side payments that are implicit in the cross-country bargain. The outcome is (a path to) the stagnation of the world economy if it is advantageous to the current generations. Since the logic of stagnation applies to all subsequent generations in sequence, stagnation is maintained perpetually.

8. Conclusion

I have modeled economic integration as the sharing of the frontier technology among countries: The opportunity to adopt the frontier technology defines the boundary of an economically integrated world. I have modeled political integration as efficient bargaining among the current generations in an economically integrated world. Since a political union is an economy in autarky, political integration is essentially a reverse of economic integration. The implicit asymmetry is that economic integration is exogenous to a country while political integration is, to an extent, a collective choice by all countries.

I highlight two reasons why the economically integrated but politically fragmented world is more likely to grow than an economy in autarky or a political union. First, when there are diverse incentives to grow among countries, the world economy grows if it is in the interests of at least one country. Second, even when the stagnation is in the interests of all countries, the world economy may still grow due to the coordination failure. When the diversity and the coordination failure are limited, growth paths are affected by general equilibrium effects (e.g., 'lock-in' and 'lock-out' effects) of the multiple-country world on the advantage of stagnation. However, these general equilibrium effects are perhaps more model-specific than the effects of diversity and the coordination failure.

The model is a formalization of what may be termed 'the political fragmentation hypothesis' in understanding European growth in contrast to Chinese stagnation prior to its adoption of European technology. As mentioned earlier, economic integration defined as the sharing of the frontier technology was probably at the continental scale prior to the European growth validating the China-Europe comparison in terms of political integration and fragmentation.[14] Since then, economic integration has expanded to a global scale so that nothing short of a global political union can withhold the advancement of technology.

[14] In considering China as an economically integrated area, I have ignored the periphery countries like Japan, Korea, and Vietnam. In terms of the model, it is arguable that China had such a dominant influence on the periphery countries in the centuries preceding European growth that the diversity was limited, and the coordination was easier to achieve in East Asia than in Europe. No single European country held such a dominant position.

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Appendix 1: Implementation of the Bargaining Outcome (For Online Posting)

The economy-wide bargaining outcome (i.e, $g(n)$, $V_y(n)$, and $V_o(n)$ in equation 5) can be implemented in a decentralized bargain with a tax and subsidy policy. (The technology advancement decision $q(n)$ is maintained to be simply the decision of the economy-wide intergenerational bargain without a need for implementation in a decentralized setting. See footnote 4.) Suppose that initially all young and old people are randomly matched and the pair-wise bargains take place. The outside option of an old person is working alone. The outside option of a young person is to adopt a technology, i.e., produce no output and obtain human capital $h_s = 1/\lambda^s$, where $s \geq 0$ is his choice, in the next period. Let $\upsilon_o(h_s; n)$ denote the individual utility of an old person with h_s, and $\upsilon_y(h_s; n)$ that of a young person initially matched with an old person with h_s. Consider any values of $\{\upsilon_o(h_s; n)\}$ and $\{\upsilon_y(h_s; n)\}$ that add up to the equilibrium aggregate utilities: $\sum_{s\geq 1} n_s \cdot \upsilon_o(h_s; n) = V_o(n)$ and $\sum_{s\geq 1} n_s \cdot \upsilon_y(h_s; n) = V_y(n)$. Note that the equality of utilities across all young people (i.e., $\upsilon_y(h_s; n) = V_y(n)$ for all s) is a special case.

Suppose that the following tax/subsidy policy is imposed in order to implement υ_o and υ_y. Each young person is taxed $\kappa_y(n)$ and each old person with h_s is taxed $\kappa_o(h_s; n)$, regardless of the pair-wise bargaining outcome. For each s, $g_s(n)$ fraction of pairs are (randomly) chosen and given the subsidy of $\varsigma(h_s; n)$ conditional on the pair working with each other; Each old person of the remaining $n_s - g_s(n)$ fraction of pairs is given a subsidy of $\tilde{\varsigma}_o(h_s; n)$ conditional on working alone; Each young person of the remaining $n_s - g_s(n)$ fraction of pairs is given a subsidy of $\tilde{\varsigma}_y(h_s; n)$ conditional on adopting a technology associated with human capital h_0 if $q(n) = 1$, and h_1 if $q(n) = 0$. That all lone young people adopt the best available technology constrained by q is an equilibrium property since W is increasing (see Property (a) in Proposition 1). The equal subsidy across all young people adopting a technology (i.e., $\tilde{\varsigma}_y(h_s; n) = \tilde{\varsigma}_y(h_u; n)$ for all s and u) is a special case that delivers the equal utility across all young people (i.e., $\upsilon_y(h_s; n) = V_y(n)$ for all s) mentioned above and is perhaps more plausible than the other cases. The budget constraint is:

$$\kappa_y(n) + \sum_{s\geq 1} \kappa_o(h_s; n) = \sum_{s\geq 1} (g_s(n)\varsigma(h_s; n) + (n_s - g_s(n))(\tilde{\varsigma}_y(h_s; n) + \tilde{\varsigma}_o(h_s; n))). \quad \text{(A1)}$$

The pair-wise bargain is assumed to solve the Nash bargaining problem. For the $g_s(n)$ fraction of pairs chosen for a joint production, the sum of the utilities of the young and the old must be greater or equal to the sum of the reservation utilities:

$$\begin{aligned} h_s + \varsigma(h_s; n) - \kappa_o(h_s; n) - \kappa_y(n) + \beta(q(n) \cdot \lambda \upsilon_o(h_{s+1}; g(n)) + (1 - q(n)) \cdot \upsilon_o(h_s; g_{-0}(n))) \\ \geq \phi h_s - \kappa_o(h_s; n) - \kappa_y(n) + \bar{\upsilon}_y(h_s; n) \end{aligned} \quad \text{(A2)}$$

where $g_{-0}(n) \equiv (g_1(n), g_2(n), \ldots)$; $\bar{\upsilon}_y(h_s; n) = \beta \max_{s'} \{I_{q(n)=0} \cdot I_{s'>0} \cdot \lim_{\epsilon\to 0} \upsilon_o(h_{s'}; g_{-0}(n) + \epsilon(\chi_{s'} - \chi_s)) + (1 - I_{q(n)=0} \cdot I_{s'>0}) \cdot \lim_{\epsilon\to 0} \lambda\upsilon_o(h_{s'+1}; g(n) + \epsilon(\chi_{s'} - \chi_s))\}$; χ_s is a vector $(\ldots, 0, 1, 0, \ldots)$ with the $s + 1'$th element set to one. Taking the limit is to deal with the possibility of a discontinuity of υ_o when evaluated at $(h_{s'}; g_{-0}(n))$ or $(h_{s'+1}; g(n))$. The utility of the old person is his reservation utility plus μ share of the joint surplus:

$$\begin{aligned}\upsilon_o(h_s; n) = \phi h_s - \kappa_o(h_s; n) + \mu\big[&h_s(1-\phi) + \varsigma(h_s; n) - \bar{\upsilon}_y(h_s; n) \\ &+ \beta(q(n) \cdot \lambda\upsilon_o(h_{s+1}; g(n)) + (1 - q(n)) \cdot \upsilon_o(h_s; g_{-0}(n)))\big].\end{aligned} \tag{A3}$$

Similarly, the utility of the young person is his reservation utility plus $1 - \mu$ share of the joint surplue:

$$\begin{aligned}\upsilon_y(h_s; n) = -\kappa_y(n) + \bar{\upsilon}_y(h_s; n) + (1-\mu)\big[&h_s(1-\phi) + \varsigma(h_s; n) - \bar{\upsilon}_y(h_s; n) \\ &+ \beta(q(n) \cdot \lambda\upsilon_o(h_{s+1}; g(n)) + (1 - q(n)) \cdot \upsilon_o(h_s; g_{-0}(n)))\big].\end{aligned} \tag{A4}$$

For the $n_s - g_s(n)$ fraction of pairs chosen for a lone production, the utilities of the old and the young are:

$$\upsilon_o(h_s; n) = \phi h_s + \tilde{\varsigma}_o(h_s; n) - \kappa_o(h_s; n) \tag{A5}$$

and

$$\upsilon_y(h_s; n) = \tilde{\varsigma}_y(h_s; n) - \kappa_y(n) + \beta(q(n) \cdot \lambda\upsilon_o(h_1; g(n)) + (1 - q(n)) \cdot \upsilon_o(h_1; g_{-0}(n))). \tag{A6}$$

The sum of the reservation utilities must be greater than or equal to the sum of the utilities from a joint production:

$$\begin{aligned}&\phi h_s - \kappa_o(h_s; n) - \kappa_y(n) + \tilde{\varsigma}_o(h_s; n) + \tilde{\varsigma}_y(h_s; n) \\ &+ \beta(q(n) \cdot \lambda\upsilon_o(h_1; g(n)) + (1 - q(n)) \cdot \upsilon_o(h_1; g_{-0}(n))) \\ &\geq h_s - \kappa_o(h_s; n) - \kappa_y(n) \\ &+ \beta \lim_{\epsilon\to 0}(q(n) \cdot \lambda\upsilon_o(h_s; g(n) + \epsilon(\chi_s - \chi_1)) + (1 - q(n)) \cdot \upsilon_o(h_s; g_{-0}(n) + \epsilon(\chi_s - \chi_1)).\end{aligned} \tag{A7}$$

To see that there are values of $\kappa_y(n)$, $\{\kappa_o(h_s; n)\}$, $\{\varsigma(h_s; n)\}$, $\{\tilde{\varsigma}_y(h_s; n)\}$, and $\{\tilde{\varsigma}_o(h_s; n)\}$ that satisfy (A1) to (A7), fix the value of $\kappa_y(n)$ arbitrarily. Given the value of $\kappa_y(n)$, the values of $\{\varsigma(h_s; n)\}$ are given by (A4). Then, the values of $\{\kappa_o(h_s; n)\}$ are given by (A3). Then, the values of $\{\tilde{\varsigma}_o(h_s; n)\}$ are given by (A5). If $g_s(n) = 0$, the values of both $\kappa_o(h_s; n)$ and $\tilde{\varsigma}_o(h_s; n)$ are given by (A5), and are not unique. The values of $\{\tilde{\varsigma}_y(h_s; n)\}$ are given by (A6). Observe that, by setting the value of $\kappa_y(n)$ high enough, we can raise the values of $\{\varsigma(h_s; n)\}$, $\{\tilde{\varsigma}_o(h_s; n)\}$, and $\{\tilde{\varsigma}_y(h_s; n)\}$ without a bound. Therefore, (A2) and

(A7) are satisfied if the value of $\kappa_y(n)$ is set high enough. Finally, consulting (2) and using the expressions of υ_o and υ_y from (A3) to (A6), we have:

$$\begin{aligned}
V_o(n) + V_y(n) &= \sum_{s\geq 1} g_s(n)(h_s + \varsigma(h_s; n) - \kappa_o(h_s; n) - \kappa_y(n)) \\
&\qquad\qquad + \beta(q(n)\cdot\lambda\upsilon_o(h_{s+1}; g(n)) + (1-q(n))\cdot\upsilon_o(h_s; g_{-0}(n))) \\
&+ \sum_{s\geq 1}(n_s - g_s(n))(\phi h_s + \tilde{\varsigma}_o(h_s; n) + \tilde{\varsigma}_y(h_s; n) - \kappa_o(h_s; n) - \kappa_y(n) \\
&\qquad\qquad\qquad + \beta(q(n)\cdot\lambda\upsilon_o(h_1; g(n)) + (1-q(n))\cdot\upsilon_o(h_1; g_{-0}(n))) \\
&= V_o(n) + V_y(n) \\
&+ \sum_{s\geq 1} g_s(n)(\varsigma(h_s; n) - \kappa_o(h_s; n) - \kappa_y(n)) \\
&+ \sum_{s\geq 1}(n_s - g_s(n))(\tilde{\varsigma}_y(h_s; n) + \tilde{\varsigma}_o(h_s; n) - \kappa_o(h_s; n) - \kappa_y(n)).
\end{aligned}$$

Then, (A1) is satisfied.

The economy-wide bargaining outcome in a country in the two-country world (i.e, $g^i(n)$, $V_y^i(n)$, and $V_o^i(n)$ in Section 5) can be implemented in a decentralized bargain with a tax and subsidy policy in the same way as above.

Appendix 2: Proofs of Propositions in Section 4 (For Online Posting)

[1] I can rewrite the equilibrium conditions in a way that is convenient for the analysis. From (6) to (9), I have

$$\hat{W}(n) = (1-q(n))\cdot\max_{\tilde{n}_0=0}\{\hat{W}_1(n,\tilde{n})\} + q(n)\cdot\max_{\tilde{n}}\{\hat{W}_2(n,\tilde{n})\}; \tag{B1}$$

$$\hat{W}_1(n,\tilde{n}) = Y(n,\tilde{n}) + \beta\mu(\hat{W}(\tilde{n}_{-0}) - \bar{V}_y) + \beta(1-\mu)\cdot\bar{Y}(\tilde{n}_{-0}); \tag{B2}$$

$$\hat{W}_2(n,\tilde{n}) = Y(n,\tilde{n}) + \beta\lambda\mu(\hat{W}(\tilde{n}) - \bar{V}_y) + \beta\lambda(1-\mu)\cdot\bar{Y}(\tilde{n}); \tag{B3}$$

where $\bar{Y}(n) \equiv \sum_{s=1}^{\infty}\phi n_s/\lambda^s$. If $\mu = 0$, (B1) to (B3) reduce to a static problem whose solution is $\tilde{n} = (\sum_{s=\bar{s}+1}^{\infty} n_s, n_1, n_2, \ldots, n_{\bar{s}}, \ldots)$ where $\bar{s} = \arg\max\{s : \phi \leq 1/(1-\beta+\beta\lambda^s)\}$. This degenerate solution also holds for each country in the two-country world in Section 5, regardless of the human capital distribution of the other country. With this result in mind, assume that $\mu > 0$ from now on. Further, assume that $\beta\lambda\mu < 1$ in order to ensure a well-defined problem in (B1) to (B3).

[2] Suppose that there exists a bounded $\hat{W}$ that solves (B1) to (B3), and that $\hat{W}(n) < \hat{W}(m)$ for some n and m with $\sum_{s\leq u} n_s \geq \sum_{s\leq u} m_s$ for all $u \geq 1$. If $q(m) = 0$, There is a γ with $\gamma_0 = 0$, $\sum_{s\leq u} \gamma_s \geq \sum_{s\leq u} g_s(m)$ for all $u \geq 1$, and $Y(n,\gamma) \geq Y(m, g(m))$. Then, $\hat{W}(n) - \hat{W}(m) \geq \hat{W}_1(n,\gamma) - \hat{W}_1(n, g(m)) \geq \beta\mu(\hat{W}(\gamma_{-0}) - \hat{W}(g_{-0}(m)))$ and $\hat{W}(\gamma_{-0}) < \hat{W}(g_{-0}(m))$. If $q(m) = 1$, There is a γ with $\sum_{s\leq u} \gamma_s \geq \sum_{s\leq u} g_s(m)$ for all $u \geq 0$, and $Y(n,\gamma) \geq Y(m, g(m))$. Then, $\hat{W}(n) - \hat{W}(m) \geq \hat{W}_2(n,\gamma) - \hat{W}_2(n, g(m)) \geq \beta\lambda\mu(\hat{W}(\gamma) - \hat{W}(g(m)))$ and $\hat{W}(\gamma) < \hat{W}(g(m))$. Let $\tilde{\gamma} \equiv \gamma_{-0}$ and $\tilde{g} \equiv g_{-0}(m)$ if $q(m) = 0$, and $\tilde{\gamma} \equiv \gamma$ and $\tilde{g} \equiv g(m)$ if $q(m) = 1$. By induction, there is a sequence of $(\tilde{\gamma}^t, \tilde{g}^t)_{t\geq 1}$ so that $\hat{W}(n) - \hat{W}(m) \geq \beta\lambda\mu(\hat{W}(\tilde{\gamma}) - \hat{W}(\tilde{g})) \geq (\beta\lambda\mu)^2(\hat{W}(\tilde{\gamma}^2) - \hat{W}(\tilde{g}^2)) \geq \ldots \geq \lim_{t\to\infty}(\beta\lambda\mu)^t(\hat{W}(\tilde{\gamma}^t) - \hat{W}(\tilde{g}^t)) = 0$ since $\hat{W}$ is bounded. This is a contradiction. Therefore, $\hat{W}$ is "increasing" in n if it is bounded and solves (B1) to (B3):

Lemma 1. If a value function $\hat{W}$ is bounded and solves (B1) to (B3), $\hat{W}(n) \geq \hat{W}(m)$ for any n and m with $\sum_{s\leq u} n_s \geq \sum_{s\leq u} m_s$ for all $u \geq 1$.

Given that $\hat{W}$ is increasing, observe in (9) that the outside option of the young generation is the all-out-technology-adoption in which every young person adopts the frontier technology:

Lemma 2. $\arg\max_{\tilde{n}}\{\hat{V}_o(\tilde{n})\} = \bar{n} \equiv (1, 0, \ldots)$.

Given that $\hat{W}$ is increasing, in (B2) and (B3), we have the following properties of g.

Lemma 3. If $q(n) = 0$, $g_s(n) \leq n_s$ for all $s \geq 2$; $g_s(n) = n_s$ if $g_u(n) > 0$ for any s and u with $2 \leq s < u$; and $g_1(n) - n_1 = \sum_{s\geq 2}(n_s - g_s(n))$. Similarly, if $q(n) = 1$, $g_s(n) \leq n_s$ for all $s \geq 1$; $g_s(n) = n_s$ if $g_u(n) > 0$ for any s and u with $1 \leq s < u$; and $g_0(n) = \sum_{s\geq 1}(n_s - g_s(n))$.

[3] Suppose that there exists a bounded $\hat{W}$ that solves (B1) to (B3), and that $g_s(n) > 0$ for some n and $s \geq 2$. If $q(n) = 0$, I have

$$\begin{aligned}
\hat{W}(n) = \hat{W}_1(n, g(n)) & \\
\geq\ & Y(n, (0, g_1(n) + g_s(n), g_2(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots)) \\
& + \beta(1-\mu)\cdot\bar{Y}(g_1(n) + g_s(n), g_2(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots) \\
& + \beta\mu(\hat{W}(g_1(n) + g_s(n), g_2(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots) - \beta\lambda\hat{V}_o(\bar{n})) \\
\geq\ & \hat{W}(n) + g_s(n)\cdot(-(1-\phi)/\lambda^s + \beta(1-\mu)\cdot\phi(1/\lambda - 1/\lambda^s)) \\
& + \beta\mu(\hat{W}(g_1(n) + g_s(n), g_2(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots) - \hat{W}(g_{-0}(n))).
\end{aligned}$$

Since $\hat{W}(n)$ is increasing in n (consult Lemma 1), $(1-\phi)/\lambda^s - \beta(1-\mu)\cdot\phi(1-1/\lambda^s) \geq 0$. This inequality does not hold if s is large enough. If $q(n) = 1$, I have

$$\begin{aligned}
\hat{W}(n) &= \hat{W}_2(n, g(n)) \\
&\geq\ Y(n, (g_0(n) + g_s(n), g_1(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots)) \\
&\quad + \beta\lambda(1-\mu)\cdot Y((g_0(n) + g_s(n), g_1(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots), \bar{n}) \\
&\quad + \beta\lambda\mu(\hat{W}(g_0(n) + g_s(n), g_1(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots) - \beta\lambda\hat{V}_o(\bar{n})) \\
&\geq\ \hat{W}(n) + g_s(n)\cdot(-(1-\phi)/\lambda^s + \beta\lambda(1-\mu)\cdot\phi(1/\lambda - 1/\lambda^{s+1})) \\
&\quad + \beta\lambda\mu(\hat{W}(g_0(n) + g_s(n), g_1(n), \ldots, g_{s-1}(n), 0, g_{s+1}(n), \ldots) - \hat{W}(g(n))).
\end{aligned}$$

Then, $(1-\phi)/\lambda^s - \beta\lambda(1-\mu)\cdot\phi(1/\lambda - 1/\lambda^{s+1}) \geq 0$. This inequality does not hold if s is large enough. Therefore, $g_s(n)$ will have the zero value for any n if s is large enough:

Lemma 4. If a value function $\hat{W}$ is bounded and solves (B1) to (B3), there exists an S so that for all n and $s \geq S$, $g_s(n) = 0$, where g is the policy function associated with $\hat{W}$.

Set $n \equiv (n_1, \ldots, n_S)$, where S satisfies the above condition from now on. Thus we are limiting n to have a finite number of elements. Lemma 4 ensures that this is not a binding constraint on the equilibrium path.

[4] Let Z denote the set of continuous and bounded functions on the domain of $\{n : \sum_{s=1}^{S} n_s = 1\}$. Let T denote the operator that maps a $\hat{W}$ in Z to another $\hat{W}$ in Z according to (B1) to (B3) given $\hat{V}_o(\bar{n})$:

$$T(\hat{w}|\bar{v}_o)(n) = \max\{\max_{\tilde{n}}\{\bar{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}, \max_{\tilde{n}}\{\tilde{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}\};$$

$$\bar{T}(\hat{w})(n,\tilde{n}|\bar{v}_o) = Y(n,\tilde{n}) + \beta\mu(\hat{w}(\tilde{n}_{-0}) - \beta\lambda\bar{v}_o) + \beta(1-\mu)\cdot\bar{Y}(\tilde{n}_{-0})\};$$

$$\tilde{T}(\hat{w})(n,\tilde{n}|\bar{v}_o) = Y(n,\tilde{n}) + \beta\lambda\mu(\hat{w}(\tilde{n}) - \beta\lambda\bar{v}_o) + \beta\lambda(1-\mu)\cdot\bar{Y}(\tilde{n})\}.$$

If $\hat{w}$ is continuous in n, $\max_{\tilde{n}}\{\bar{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}$ is continuous in n according to the Theorem of the Maximum. Similarly, $\max_{\tilde{n}}\{\tilde{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}$ is continuous in n. For any $\hat{w}$, $\hat{\omega}$, and n, let $g|\hat{w} \equiv \arg\max_{\tilde{n}}\{T(\hat{w}|\bar{v}_o)(n)\}$ and $g|\hat{\omega} \equiv \arg\max_{\tilde{n}}\{T(\hat{\omega}|\bar{v}_o)(n)\}$. I have $T(\hat{w}|\bar{v}_o)(n) - T(\hat{\omega}|\bar{v}_o)(n) \leq \beta\mu(\hat{w}(g_{-0}|\hat{w}) - \hat{\omega}(g_{-0}|\hat{w}))$ if $\max_{\tilde{n}}\{\bar{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\} \geq \max_{\tilde{n}}\{\tilde{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}$; and $T(\hat{w}|\bar{v}_o)(n) - T(\hat{\omega}|\bar{v}_o)(n) \leq \beta\lambda\mu(\hat{w}(g|\hat{w}) - \hat{\omega}(g|\hat{w}))$ if $\max_{\tilde{n}}\{\bar{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\} \leq \max_{\tilde{n}}\{\tilde{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}$. Similarly, $T(\hat{\omega}|\bar{v}_o)(n) - T(\hat{w}|\bar{v}_o)(n) \leq \beta\mu(\hat{\omega}(g_{-0}|\hat{\omega}) - \hat{w}(g_{-0}|\hat{\omega}))$ if $\max_{\tilde{n}}\{\bar{T}(\hat{\omega})(n,\tilde{n}|\bar{v}_o)\} \geq \max_{\tilde{n}}\{\tilde{T}(\hat{\omega})(n,\tilde{n}|\bar{v}_o)\}$; and $T(\hat{\omega}|\bar{v}_o)(n) - T(\hat{w}|\bar{v}_o)(n) \leq \beta\lambda\mu(\hat{\omega}(g|\hat{\omega}) - \hat{w}(g|\hat{\omega}))$ if $\max_{\tilde{n}}\{\bar{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\} \leq \max_{\tilde{n}}\{\tilde{T}(\hat{w})(n,\tilde{n}|\bar{v}_o)\}$. Then, $\sup_n |T(\hat{w}|\bar{v}_o)(n) -$

$T(\hat{\omega}|\bar{v}_o)(n)| \leq \beta\lambda\mu \sup_n |\hat{w}(n) - \hat{\omega}(n)|$. According to the Contraction Mapping Theorem, there is a unique fixed point of T in Z:

Lemma 5. Mapping T is a contraction, and there is a unique continuous and bounded $\hat{W} \in Z$ that solves $T(\hat{w}|\bar{v}_o) = \hat{w}$. Further, $\hat{W} = \lim_{t\to\infty} T^t(\hat{w}|\bar{v}_o)$ for any $\hat{w} \in Z$, where $T^{t+1}(\hat{w}|\bar{v}_o) \equiv T(T^t(\hat{w}|\bar{v}_o)|\bar{v}_o)$.

[5] For any $\hat{w}$, $\bar{v}_o$, and $\bar{\nu}_o$ with $\bar{v}_o < \bar{\nu}_o$, $0 \leq T(\hat{w}|\bar{v}_o)(\bar{n}) - T(\hat{w}|\bar{\nu}_o)(\bar{n}) \leq (\bar{\nu}_o - \bar{v}_o) \cdot \max\{\beta^2\lambda\mu, \beta^2\lambda^2\mu\} = \beta^2\lambda^2\mu$ and

$$\begin{aligned}
0 &\leq T^{t+1}(\hat{w}|\bar{V}_o)(\bar{n}) - T^{t+1}(\hat{w}|\bar{\nu}_o)(\bar{n}) \\
&\leq \max\{(\bar{\nu}_o - \bar{v}_o) \cdot \beta^2\lambda\mu + \beta\mu(T^t(\hat{w}|\bar{V}_o)(\bar{n}) - T^t(\hat{w}|\bar{\nu}_o)(\bar{n})), \\
&\qquad (\bar{\nu}_o - \bar{v}_o) \cdot \beta^2\lambda^2\mu + \beta\lambda\mu(T^t(\hat{w}|\bar{V}_o)(\bar{n}) - T^t(\hat{w}|\bar{\nu}_o)(\bar{n}))\} \\
&= (\bar{\nu}_o - \bar{v}_o) \cdot \beta^2\lambda^2\mu + \beta\lambda\mu(T^t(\hat{w}|\bar{V}_o)(\bar{n}) - T^t(\hat{w}|\bar{\nu}_o)(\bar{n})) \\
&\leq (\bar{\nu}_o - \bar{v}_o) \cdot \beta^2\lambda^2\mu \cdot (1 - \beta^t\lambda^t\mu^t)/(1 - \beta\lambda\mu).
\end{aligned}$$

Then, $0 \leq T^\infty(\hat{w}_2|\bar{V}_o)(\bar{n}) - T^\infty(\hat{w}_2|\bar{\nu}_o)(\bar{n}) \leq (\bar{\nu}_o - \bar{v}_o)(\beta^2\lambda^2\mu)/(1 - \beta\lambda\mu)$. This implies that $\hat{W}(\bar{n})$ is continuous and non-increasing in $\bar{v}_o$. Further, from (B1) and (B3), $\hat{W}(\bar{n}) \geq \bar{Y}(\bar{n}) \cdot (1 + \beta\lambda) = \phi(1 + \beta\lambda)/\lambda$ when $\bar{v}_o = \bar{Y}(\bar{n}) = \phi/\lambda$. On the other hand, in (9), $\hat{W}(\bar{n})$ is continuous and increasing in $\bar{v}_o$ without an upper bound and $\hat{W}(\bar{n}) = \phi(1 + \beta\lambda)/\lambda$ when $\bar{v}_o = \phi/\lambda$. Therefore, there is a unique $\bar{v}_o \geq \phi/\lambda$ that, along with the associated $\hat{W}$, solves (B1),(B2),(B3), and (9).

Result 1. There is a unique set of $\hat{W}$ and $\hat{V}_o$ in Z that solve (B1) to (B3) and (9).

In Steps [14], [15], and [17], I derive the policy functions, g and q, and find that they are unique except possibly when the parameters are on the borderline between zones (e.g., $\phi = \hat{\phi}(2)$): Holding n, the values of the policy functions can jump as the value of ϕ crosses the borderline so that there could be two valid policy functions on the borderline. Since there is little substance in the multiple policy functions on the borderline, the equilibrium is essentially unique under the restriction of continuous value functions. This does not rule out the possibility of an equilibrium with discontinuous value functions.

[6] Let $q^1(n) \equiv q(n)$; $g^1(n) = g_{-0}(n)$ if $q(n) = 0$; and $g^1(n) = g(n)$ if $q(n) = 1$. For $t \geq 2$, let $q^t(n) \equiv q(g^{t-1}(n))$; $g^t(n) = g_{-0}(g^{t-1}(n))$ if $q(g^{t-1}(n)) = 0$; and $g^t(n) = g(g^{t-1}(n))$

if $q(g^{t-1}(n)) = 1$. Let $\bar{W}(n)$ denote the value function conditional on $q^t(n) = 0$ and $g^t(n) = (0, n)$ for all t:

$$\begin{aligned}\bar{W}(n) &= Y(n,(0,n)) + \beta\mu(\bar{W}(n) - \beta\lambda\hat{V}_o(\bar{n})) + \beta(1-\mu)\cdot\bar{Y}(n) \\ &= (Y(n,(0,n)) - \beta^2\lambda\mu\hat{V}_o(\bar{n}) + \beta(1-\mu)\cdot\bar{Y}(n))/(1-\beta\mu).\end{aligned} \tag{B4}$$

We have: $\hat{W}(n) \geq \hat{W}_1(n,(0,n))) = Y(n,(0,n)) + \beta\mu(\hat{W}(n) - \beta\lambda\hat{V}_o(\bar{n})) + \beta(1-\mu)\cdot\bar{Y}(n)$ so that $\hat{W}(n) \geq (Y(n,(0,n)) - \beta^2\lambda\mu\hat{V}_o(\bar{n}) + \beta(1-\mu)\cdot\bar{Y}(n))/(1-\beta\mu) = \bar{W}(n)$.

Lemma 6. For any n, $\hat{W}(n) \geq \bar{W}(n)$.

[7] Given Lemma 6, we have:

$$\hat{W}(n) \geq \hat{W}(n|0) \equiv \max_{\tilde{n}_0=0}\{\hat{W}(n,\tilde{n}|0)\}, \tag{B5}$$

where

$$\hat{W}(n,\tilde{n}|0) \equiv Y(n,\tilde{n}) + \beta\mu(\bar{W}(\tilde{n}|0) - \beta\lambda\hat{V}_o(\bar{n})) + \beta(1-\mu)\cdot\bar{Y}(\tilde{n}_{-0}). \tag{B6}$$

We have: $d\bar{W}(n)/dn_s \equiv \lim_{\epsilon\to 0}(\bar{W}(n_1+\epsilon,\ldots,n_s-\epsilon,\ldots) - \bar{W}(n)/\epsilon = (1/\lambda - 1/\lambda^s)(1 + \beta\phi(1-\mu))/(1-\beta\mu)$ and $d\hat{W}(n,\tilde{n}|0)/d\tilde{n}_s \equiv \lim_{\epsilon\to 0}(\hat{W}(n,(\tilde{n}_1+\epsilon,\ldots,\tilde{n}_s-\epsilon,\ldots)|0) - \hat{W}(n,\tilde{n}|0))/\epsilon = -(1-\phi)/\lambda^s + \beta\phi(1-\mu)(1/\lambda - 1/\lambda^s) + \beta\mu\cdot d\bar{W}(\tilde{n}_{-0})/dn_s$ while $\tilde{n}_1 \geq n_1$ and $\tilde{n}_s \leq n_s$. Then, $d\hat{W}(n,\tilde{n}|0)/d\tilde{n}_s = -(1-\phi)/\lambda^s + (1/\lambda - 1/\lambda^s)(\beta\phi(1-\mu) + \beta\mu)/(1-\beta\mu) \geq 0$ iff $\phi \geq \hat{\phi}(s) \equiv (1-\beta\lambda^{s-1}\mu)/(1-\beta+\beta\lambda^{s-1}(1-\mu))$. Then, a solution to the maximization problem in (B5) and (B6) is: $g(n|0) \equiv (0, n_1 + \sum_{s\geq\hat{S}} n_s, n_2, \ldots, n_{\hat{S}-1})$ where $\hat{S} = \arg\min_s\{\phi \geq \hat{\phi}(s)\}$. Substituting the expression of $g(n|0)$ in (B5) and (B6), we have: $\hat{W}(n|0) = c(0|0) + \sum_{s=1}^{\infty} c(s|0)n_s$, where

$$\begin{aligned}c(0|0) &= -\beta^2\lambda\mu\cdot\hat{V}_o(\bar{n})/(1-\beta\mu); \\ c(s|0) &= (1+\beta\phi(1-\mu))/(\lambda^s(1-\beta\mu)) &&\text{for } s < \hat{S}; \\ c(s|0) &= \phi/\lambda^s + \beta\phi(1-\mu)/\lambda + \beta\mu\cdot c(1|0) &&\text{for } s \geq \hat{S}.\end{aligned} \tag{B7}$$

[8] Given Lemma 6, we have:

$$\hat{W}(n) \geq \hat{W}(n|1) \equiv \max_{\tilde{n}}\{\hat{W}(n,\tilde{n}|1)\}, \tag{B8}$$

where

$$\hat{W}(n,\tilde{n}|1) \equiv Y(n,\tilde{n}) + \beta\lambda\mu(\bar{W}(\tilde{n}|0) - \beta\lambda\hat{V}_o(\bar{n})) + \beta\lambda(1-\mu)\cdot\bar{Y}(\tilde{n}). \tag{B9}$$

We have: $d\hat{W}(n,\tilde{n}|1)/d\tilde{n}_s \equiv \lim_{\epsilon\to 0}(\hat{W}(n,(\tilde{n}_0+\epsilon,\ldots,\tilde{n}_s-\epsilon,\ldots)|1) - \hat{W}(n,\tilde{n}|1))/\epsilon = -(1-\phi)/\lambda^s + \beta\lambda\phi(1-\mu)(1/\lambda - 1/\lambda^{s+1}) + \beta\lambda\mu\cdot d\bar{W}(\tilde{n})/d\tilde{n}_s = -(1-\phi)/\lambda^s + (1-1/\lambda^s)(\beta\phi(1-\mu)+\beta\mu)/(1-\beta\mu) \geq 0$ iff $\phi \geq \hat{\phi}(s+1) = (1-\beta\lambda^s\mu)/(1-\beta+\beta\lambda^s(1-\mu))$, or equivalently, $d\hat{W}_1(n,\tilde{n})/d\tilde{n}_{s-1} \geq 0$ iff $\phi \geq \hat{\phi}(s)$. Then, a solution to the maximization problem in (B8) and (B9) is: $g(n|1) \equiv (\sum_{s\geq \hat{S}-1} n_s, n_1, \ldots, n_{\hat{S}-2}, 0, \ldots)$ where $\hat{S} = \arg\min_s\{\phi \geq \hat{\phi}(s)\}$. Substituting the expression of $g(n|0)$ in (B8) and (B9), we have: $\hat{W}(n|1) = c(0|1) + \sum_{s=1}^{\infty} c(s|1)n_s$, where

$$
\begin{aligned}
&c(0|1) = -\beta^2\lambda^2\mu\cdot\hat{V}_o(\bar{n})/(1-\beta\mu);\\
&c(s|1) = (1+\beta\phi(1-\mu))/(\lambda^s(1-\beta\mu)) \qquad \text{for } s < \hat{S}-1;\\
&c(s|1) = \phi/\lambda^s + \beta\phi(1-\mu) + \beta\lambda\mu\cdot c(1|0) \quad \text{for } s \geq \hat{S}-1.
\end{aligned} \tag{B10}
$$

[9] Let $\hat{W}(n|\infty)$ denote the value function conditional on the perpetual growth:

$$\hat{W}(n|\infty) \equiv \max_{\tilde{n}}\{\hat{W}_2(n,\tilde{n}|\infty)\} \tag{B11}$$

where

$$\hat{W}_2(n,\tilde{n}|\infty) = Y(n,\tilde{n}) + \beta\lambda\mu(\hat{W}(n|\infty) - \beta\lambda\hat{V}_o(\bar{n})) + \beta\lambda(1-\mu)\cdot\bar{Y}(\tilde{n}). \tag{B12}$$

Conjecture that $\hat{W}(n|\infty)$ is linear and increasing in n, i.e., $\hat{W}(n|\infty) = c(0|\infty) + \sum_{s=1}^{\infty} c(s|\infty)n_s$ with $c_s > c_u$ for any $s < u$. Substituting this expression in (B12), we have: $d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_s \equiv \lim_{\epsilon\to 0}(\hat{W}_2(n,(\tilde{n}_0+\epsilon,\ldots,\tilde{n}_s-\epsilon,\ldots)|\infty) - \hat{W}_2(n,\tilde{n}|\infty))/\epsilon = -(1-\phi)/\lambda^s + \beta\phi(1-\mu)(1-1/\lambda^s) + \beta\lambda\mu(c(1|\infty) - c(s+1|\infty)) \geq 0$ iff $s \geq \bar{S}$ for some $\bar{S}$. Then, $g(n|\infty) = (\sum_{\bar{S}}^{\infty} n_s, n_1, \ldots, n_{\bar{S}-1}, 0, \ldots)$. Substituting the expressions of $\hat{W}(n|\infty)$ and $g(n|\infty)$ in (B11) and (B12), we have:

$$
\begin{aligned}
&c(0|\infty) = -\beta^2\lambda^2\mu\cdot\hat{V}_o(\bar{n})/(1-\beta\lambda\mu);\\
&c(s|\infty) = 1/\lambda^s + \beta\phi(1-\mu)/\lambda^s + \beta\lambda\mu\cdot c(s+1|\infty) \quad \text{for } s < \bar{S};\\
&c(s|\infty) = \phi/\lambda^s + \beta\phi(1-\mu) + \beta\lambda\mu\cdot c(1|\infty) \qquad \text{for } s \geq \bar{S}.
\end{aligned} \tag{B13}
$$

We can derive: $c(1|\infty) = (\sum_{s=1}^{\bar{S}-1}(\beta\mu)^{s-1}(1+\beta\phi(1-\mu))/\lambda + (\beta\mu)^{\bar{S}-1}\phi/\lambda + (\beta\lambda\mu)^{\bar{S}-1}\beta\phi(1-\mu))/(1-(\beta\lambda\mu)^{\bar{S}})$. We have: $\hat{W}(n|\infty)/d\tilde{n}_s \equiv \lim_{\epsilon\to 0}(\hat{W}(n_1+\epsilon,\ldots,n_s-\epsilon,\ldots)|\infty) - \hat{W}(n|\infty))/\epsilon = c(1|\infty) - c(s|\infty) = c(1|\infty)\cdot(1-\beta\lambda\mu) - \phi/\lambda^s - \beta\phi(1-\mu)$ for $s \geq \bar{S}$. We need: $d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_{\bar{S}} = -(1-\phi)/\lambda^{\bar{S}} + \beta\phi(1-\mu)(1-1/\lambda^{\bar{S}}) + \beta\lambda\mu\cdot\hat{W}(n|\infty)/d\tilde{n}_{\bar{S}+1} \geq 0$, which holds iff $\phi \geq \bar{\phi}(\bar{S})$, where $\bar{\phi}(s) \equiv (1/\lambda^s - (\beta\mu)^s - \beta\mu(1-\beta\lambda\mu)(1-(\beta\mu)^{s-1})/(1-\beta\mu))/((1-(\beta\lambda\mu)^s)(1/\lambda^s - \beta/\lambda^s + \beta(1-\mu)(1-\beta\lambda\mu)) + \beta\mu(1-\beta\lambda\mu)(\beta(1-\mu)(1-(\beta\mu)^{s-1})/(1-

$\beta\mu) + (\beta\mu)^{s-1} + \beta\lambda(1-\mu)(\beta\lambda\mu)^{s-1})$). We also need: if $\bar{S} \geq 2$, $d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_{\bar{S}-1} = -(1-\phi)/\lambda^{\bar{S}-1} + \beta\phi(1-\mu)(1-1/\lambda^{\bar{S}-1}) + \beta\lambda\mu \cdot \hat{W}(n|\infty)/d\tilde{n}_{\bar{S}} < 0$, which holds iff $\phi < \bar{\phi}(\bar{S}-1)$. We have: $c(s|\infty) - c(s+1|\infty) = \phi(1/\lambda^s - 1/\lambda^{s+1}) > 0$ for all $s \geq \bar{S}$; $c(\bar{S}-1|\infty) - c(\bar{S}|\infty) = 1/\lambda^{\bar{S}-1} + \beta\phi(1-\mu)/\lambda^{\bar{S}-1} + \beta\lambda\mu \cdot c(\bar{S}|\infty) - \phi/\lambda^{\bar{S}} - \beta\phi(1-\mu) - \beta\lambda\mu \cdot c(1|\infty) = d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_{\bar{S}-1} + \phi(1/\lambda^{\bar{S}-1} - 1/\lambda^{\bar{S}}) > 0$; and $c(s|\infty) - c(s+1|\infty) = (1/\lambda^s - 1/\lambda^{s+1})(1+\beta\phi(1-\mu)) + \beta\lambda\mu(c(s+1|\infty) - c(s+2|\infty)) > 0$ for all $s \in \{1, 2, \ldots, \bar{S}-2\}$, by induction, given $c(\bar{S}-1|\infty) - c(\bar{S}|\infty) > 0$. Then, $d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_s \geq 0$ for all $s \geq \bar{S}$; and $d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_s < 0$ for all $s < \bar{S}$. In summary, we have shown that:

Lemma 7. $\hat{W}(n|\infty)$ is linear and increasing with $d\hat{W}_2(n,\tilde{n}|\infty)/d\tilde{n}_s \geq 0$ iff $s \geq \bar{S}$, where $\bar{S} = 1$ if $\phi \geq \bar{\phi}(1)$; and $\bar{S}$ is given by $\bar{\phi}(\bar{S}) \leq \phi < \bar{\phi}(\bar{S}-1)$ otherwise.

[10] Suppose that, for some n, $q^t(n) = 1$ for all t, and that $\hat{W}(n) = \hat{W}(n|\infty) + \delta$ for some $\delta > 0$. We have: $\hat{W}(n) - \hat{W}(n|\infty) \leq \hat{W}_2(n, g(n)) - \hat{W}_2(n, g(n)|\infty) = \beta\lambda\mu(\hat{W}(g(n)) - \hat{W}(g(n)|\infty))$ so that $\hat{W}(g(n)) - \hat{W}(g(n)|\infty) \geq \delta/(\beta\lambda\mu)$. By induction, we have: $\hat{W}(g^t(n)) - \hat{W}(g^t(n)|\infty) \geq \delta/(\beta\lambda\mu)^t$ so that $\lim_{t\to\infty} \hat{W}(g^t(n)) = \infty$. This is a contradiction since $\hat{W}$ is bounded. Therefore, we have:

Lemma 8 For any n, $\hat{W}(n) \leq \hat{W}(n|\infty)$ if $q(g^t(n)) = 1$ for all t.

[11] Let $\tilde{g}(n|0) \equiv (0, n_1 + \sum_{s \geq \bar{S}+1} n_s, n_2, \ldots, n_{\bar{S}}, 0, \ldots)$. We have: $\hat{W}(n|0) \geq c(0|0) + \sum_{s=1}^{\infty} \tilde{c}(s|0) n_s$, where

$$\tilde{c}(s|0) = (1 + \beta\phi(1-\mu))/(\lambda^s(1-\beta\mu)) \qquad \text{for } s \leq \bar{S};$$
$$\tilde{c}(s|0) = \phi/\lambda^s + \beta\phi(1-\mu)/\lambda + \beta\mu \cdot \tilde{c}(1|0) \quad \text{for } s > \bar{S}.$$

We can show that, for all $\bar{S} \geq 2$, $\tilde{c}(\bar{S}|0) + c(0|0) > c(\bar{S}|\infty) + c(0|\infty)$ when $\phi = \bar{\phi}(\bar{S})$, in the following steps. From Lemma 6 and equations (9) and (B4), we have: $\hat{V}_o(\bar{n}) \geq (\mu + \phi(1-\mu))/(\lambda(1-\beta\mu+\beta\lambda\mu))$. This low bound of $\hat{V}_o(\bar{n})$ sets the upper bounds of $c(0|0)$ and $c(0|\infty)$. Denote these upper bounds by $c_u(0|0)$ and $c_u(0|\infty)$: $c_u(0|0) \equiv -\beta^2\lambda\mu/(1-\beta\mu) \cdot (\mu + \phi(1-\mu))/(\lambda(1-\beta\mu+\beta\lambda\mu))$ and $c_u(0|\infty) \equiv -\beta^2\lambda^2\mu/(1-\beta\lambda\mu) \cdot (\mu + \phi(1-\mu))/(\lambda(1-\beta\mu+\beta\lambda\mu))$. Now derive: $\tilde{c}(\bar{S}|0) = (1+\beta\bar{\phi}(\bar{S})(1-\mu))/(\lambda^{\bar{S}}(1-\beta\mu))$ and $c(\bar{S}|\infty) = \bar{\phi}(\bar{S})/\lambda^{\bar{S}} + (1-\bar{\phi}(\bar{S})(1-\beta))/(\lambda^{\bar{S}}(1-\beta\lambda\mu))$. Show that $\tilde{c}(\bar{S}|0) + c_u(0|0) = c(\bar{S}|\infty) + c_u(0|\infty)$ if $\bar{\phi}(\bar{S})$ is replaced by $\varphi(\bar{S}) \equiv (1-\beta\mu+\beta\lambda\mu-\beta\lambda^{\bar{S}}\mu)/((1-\beta)(1-\beta\mu+\beta\lambda\mu)+\beta\lambda^{\bar{S}}(1-\mu))$. Show that $\varphi(1) = \bar{\phi}(1)$ and that $\varphi(\bar{S}) < \bar{\phi}(\bar{S})$ for all $\bar{S} \geq 2$. Show that $\tilde{c}(\bar{S}|0) + c_u(0|0)$ declines more than $c(\bar{S}|\infty) + c_u(0|\infty)$ when $\bar{S} \geq 2$ and $\bar{\phi}(\bar{S})$ is replaced by $\varphi(\bar{S})$. Then, $\tilde{c}(\bar{S}|0) + c_u(0|0) > c(\bar{S}|\infty) + c_u(0|\infty)$ for all $\bar{S} \geq 2$. Observe that $c(0|0)$ declines less than

$c(0|\infty)$ as $\hat{V}_o(\bar{n})$ rises from the low bound. Then, $c_u(0|0) - c(0|0) \le c_u(0|\infty) - c(0|\infty)$. Therefore, we have: for all $\bar{S} \ge 2$, $\tilde{c}(\bar{S}|0) + c(0|0) > c(\bar{S}|\infty) + c(0|\infty)$ when $\phi = \bar{\phi}(\bar{S})$.

Observe that $\tilde{c}(\bar{S}|0)$, $c(\bar{S}|\infty)$, $c_u(0|0)$, and $c_u(0|\infty)$ are all continuous and linear in ϕ while $\phi \in [\bar{\phi}(\bar{S}+1), \bar{\phi}(\bar{S})]$. Then, repeating the above reasoning, we have: $\tilde{c}(1|0) + c(0|0) > c(1|\infty) + c(0|\infty)$ for all $\phi \in [\bar{\phi}(2), \bar{\phi}(1))$, and $\tilde{c}(\bar{S}|0) + c(0|0) > c(\bar{S}|\infty) + c(0|\infty)$ for all $\bar{S} \ge 2$ and all $\phi \in [\bar{\phi}(\bar{S}+1), \bar{\phi}(\bar{S})]$.

Now, we have: if $\bar{S} \ge 2$, $\tilde{c}(\bar{S}-1|0) - c(\bar{S}-1|\infty) = (1+\beta\phi(1-\mu))/(\lambda^{\bar{S}-1}(1-\beta\mu)) - (1+\beta\phi(1-\mu))/\lambda^{\bar{S}-1} - \beta\lambda\mu c(\bar{S}|\infty) > 1+\beta\phi(1-\mu))/(\lambda^{\bar{S}-1}(1-\beta\mu)) - 1/\lambda^{\bar{S}-1} - (1+\beta\phi(1-\mu))/\lambda^{\bar{S}-1} - \beta\lambda\mu(\tilde{c}(\bar{S}|0) + c(0|0) - c(0|\infty)) = -\beta\lambda\mu(c(0|0) - c(0|\infty))$ so that $\tilde{c}(\bar{S}-1|0) + c(0|0) - c(\bar{S}-1|\infty) - c(0|\infty) > (c(0|0) - c(0|\infty))(1-\beta\lambda\mu) > 0$. By induction, we have: $\tilde{c}(s|0) + c(0|0) > c(s|\infty) + c(0|\infty)$ for all $s \in \{1, 2, \dots, \bar{S}\}$ if $\bar{S} \ge 2$ and $\phi \in [\bar{\phi}(\bar{S}+1), \bar{\phi}(\bar{S})]$. Therefore, we have: $\hat{W}(n) \ge \hat{W}(n|0) \ge c(0|0) + \sum_{s=1}^{\infty} \tilde{c}(s|0) n_s > c(0|\infty) + \sum_{s=1}^{\infty} c(s|\infty) n_s = \hat{W}(n|\infty)$ if $\phi \in [\bar{\phi}(2), \bar{\phi}(1))$ and $n_1 = 1$, or if $\bar{S} \ge 2$, $\phi \in [\bar{\phi}(\bar{S}+1), \bar{\phi}(\bar{S})]$, and $\sum_{s=1}^{\bar{S}} n_s = 1$. Combined with Lemma 8, this implies that:

Lemma 9. If $\phi < \bar{\phi}(1)$ and $n_1 = 1$, or if $\bar{S} \ge 2$, $\phi \le \bar{\phi}(\bar{S})$, and $\sum_{s=1}^{\bar{S}} n_s = 1$, $q^t(n) = 0$ for some t.

In particular, Lemma 9 implies that $q^t(n) = 0$ for some t if $\phi \le \bar{\phi}(2)$ and $n_1 + n_2 = 1$.

[12] Suppose that $q(n) = 0$ and $q^2(n) = 1$ for some n. We have:

$$\begin{aligned}
\hat{W}(n) &= \hat{W}_1(n, g(n)) \\
&= Y(n, g(n)) + \beta\mu(\hat{W}(g^1(n)) - \beta\lambda\hat{V}_o(\bar{n})) + \beta(1-\mu)\cdot\bar{Y}(g^1(n)) \\
&\ge \hat{W}_2(n, g^2(n)) \\
&= Y(n, g^2(n)) + \beta\lambda\mu(\hat{W}(g^2(n)) - \beta\lambda\hat{V}_o(\bar{n})) + \beta\lambda(1-\mu)\cdot\bar{Y}(g^2(n))
\end{aligned}$$

and

$$\begin{aligned}
\hat{W}(g^1(n)) &= \hat{W}_2(g^1(n), g^2(n)) \\
&= Y(g^1(n), g^2(n)) + \beta\lambda\mu(\hat{W}(g^2(n)) - \beta\lambda\hat{V}_o(\bar{n})) + \beta\lambda(1-\mu)\cdot\bar{Y}(g^2(n)) \\
&\ge \hat{W}_1(g^1(n), g(n)) \\
&= Y(g^1(n), g(n)) + \beta\mu(\hat{W}(g^1(n)) - \beta\lambda\hat{V}_o(\bar{n})) + \beta(1-\mu)\cdot\bar{Y}(g^1(n)).
\end{aligned}$$

Then, $Y(n, g(n)) - Y(n, g^2(n)) \geq Y(g^1(n), g(n)) - Y(g^1(n), g^2(n))$. Consulting Lemma 3, we have:

$$\begin{aligned}
Y(n,g(n)) - Y(n,g^2(n)) &= \frac{n_1}{\lambda} + \sum_{s\geq 2} \frac{g_s(n) + \phi(n_s - g_s(n))}{\lambda^s} \\
&\quad - \frac{\min\{n_1, g_1^2(n)\} + \phi \max\{n_1 - g_1^2(n), 0\}}{\lambda} - \sum_{s\geq 2} \frac{g_s^2(n) + \phi(n_s - g_s^2(n)))}{\lambda^s} \\
&= \frac{(1-\phi)\cdot \max\{n_1 - g_1^2(n), 0\}}{\lambda} + \sum_{s\geq 2} \frac{(1-\phi)\cdot(g_s(n) - g_s^2(n))}{\lambda^s}
\end{aligned}$$

and

$$\begin{aligned}
Y(g^1(n), g(n)) - Y(g^1(n), g^2(n)) &= \sum_{s\geq 1} \frac{g_s(n)}{\lambda^s} - \sum_{s\geq 1} \frac{g_s^2(n) + \phi(g_s(n) - g_s^2(n))}{\lambda^s} \\
&= \sum_{s\geq 1} \frac{(1-\phi)\cdot(g_s(n) - g_s^2(n))}{\lambda^s}.
\end{aligned}$$

Since $g_1(n) \geq n_1$, $Y(n, g(n)) - Y(n, g^2(n)) \leq Y(g^1(n), g(n)) - Y(g^1(n), g^2(n))$ with the equality only when $g_s(n) = n_s$ for all $s \geq 1$. However, if $g_s(n) = n_s$ for all $s \geq 1$, $g^1(n) = n$ and $q^2(n) = q(g^1(n)) = q(n)$. This is a contradiction. Therefore, for any n, if $q(n) = 0$, $q^2(n) = 0$. By induction, we have the following lemma:

Result 2. For any n, if $q(n) = 0$, $q^t(n) = 0$ for all $t \geq 2$. Therefore, the country either grows perpetually or stagnates forever possibly after some periods of continued growth.

Lemma 9 and Result 2 imply:

Result 3. If $\phi < \bar{\phi}(1)$ and $n_1 = 1$, or if $\bar{S} \geq 2$, $\phi \leq \bar{\phi}(\bar{S})$, and $\sum_{s=1}^{\bar{S}} n_s = 1$, the country stagnates forever possibly after some periods of growth, i.e., there is $T \geq 1$ so that $q^t(n) = 1$ for all $t < T$ and $q^t(n) = 0$ for all $t \geq T$.

In particular, the economy stagnates forever possibly after some periods of growth if $\phi \leq \bar{\phi}(2)$ and $n_1 + n_2 = 1$.

[13] Consider $\phi \in [\bar{\phi}(2), \bar{\phi}(1)]$ and n with $n_1 + n_2 = 1$. We have $\hat{W}(n|\infty) \geq c_u(0|\infty) + n_1 \cdot c(1|\infty) + (1 - n_1|\infty) \cdot c(2|\infty)$. We have: $\hat{W}(n|0) \geq c_u(0|0) + n_1 \cdot \tilde{c}(1|0) + (1 - n_1) \cdot \tilde{c}(2|0)$. Let $c_u(0|1)$ denote upper bound of $c(0|1)$, corresponding to the low bound of $\hat{V}_o(\bar{n})$, as

discussed in Step [11]: $c_u(0|1) \equiv -\beta^2\lambda^2\mu/(1-\beta\mu)\cdot(\mu+\phi(1-\mu))/(\lambda(1-\beta\mu+\beta\lambda\mu))$. We have: $\hat{W}(n|0) \geq c_u(0|1) + \sum_{s=1}^{\infty}\tilde{c}(s|1)n_s$, where

$$\tilde{c}(s|1) = (1+\beta\phi(1-\mu))/(\lambda^s(1-\beta\mu)) \qquad \text{for } s=1;$$
$$\tilde{c}(s|1) = \phi/\lambda^s + \beta\phi(1-\mu) + \beta\lambda\mu\cdot c(1|0) \quad \text{for } s\geq 2.$$

We can show that: $c_u(0|0)+1/2\cdot\tilde{c}(1|0)+1/2\cdot\tilde{c}(2|0) = c_u(0|1)+1/2\cdot\tilde{c}(1|1)+1/2\cdot\tilde{c}(2|1) = c_u(0|\infty)+1/2\cdot c(1|\infty)+1/2\cdot c(2|\infty)$ when $\phi=\tilde{\phi}\equiv(1-\beta\mu-\beta\lambda\mu(1-\beta\lambda\mu)(\lambda-1))/((1-\beta)(1-\beta\mu+\beta\lambda\mu)+\beta\lambda^2(1-\mu)(1+\beta\mu-\beta\lambda\mu))$; $d(c_u(0|\infty)+1/2\cdot c(1|\infty)+1/2\cdot c(2|\infty))/d\phi > d(c_u(0|0)+1/2\cdot\tilde{c}(1|0)+1/2\cdot\tilde{c}(2|0))/d\phi$; $d(c_u(0|\infty)+1/2\cdot c(1|\infty)+1/2\cdot c(2|\infty))/d\phi > d(c_u(0|1)+1/2\cdot\tilde{c}(1|1)+1/2\cdot\tilde{c}(2|1))/d\phi$; $\tilde{c}(1|0)-\tilde{c}(2|0) > c(1|\infty)-c(2|\infty) > \tilde{c}(1|1)-\tilde{c}(2|1)$ if $\phi>\hat{\phi}(3)$; and $\bar{\phi}(1)>\tilde{\phi}>\bar{\phi}(2)>\hat{\phi}(3)$. These properties imply that $c_u(0|0)+n_1\cdot\tilde{c}(1|0)+(1-n_1)\cdot\tilde{c}(2|0) > c_u(0|\infty)+n_1\cdot c(1|\infty)+(1-n_1)\cdot c(2|\infty)$ for all n with $n_1=1-n_2\geq 1/2$ and $\phi\in[\bar{\phi}(2),\tilde{\phi})$; and $c_u(0|1)+n_1\cdot\tilde{c}(1|1)+(1-n_1)\cdot\tilde{c}(2|1) > c_u(0|\infty)+n_1\cdot c(1|\infty)+(1-n_1)\cdot c(2|\infty)$ for all n with $n_1=1-n_2\leq 1/2$ and $\phi\in[\bar{\phi}(2),\tilde{\phi})$. Since $c_u(0|0)-c(0|0)\leq c_u(0|1)-c(0|1)\leq c_u(0|\infty)-c(0|\infty)$, we have: $\hat{W}(n|0) > \hat{W}(n|\infty)$ for all n with n with $n_1+n_2=1$ and $\phi\in[\bar{\phi}(2),\tilde{\phi})$. Combined with Lemma 8 and Result 2, this implies that:

Result 4. If $\phi\in[\bar{\phi}(2),\tilde{\phi})$ and $n_1+n_2=1$, the economy stagnates forever possibly after some periods of growth, i.e., there is $T\geq 1$ so that $q^t(n)=1$ for all $t<T$ and $q^t(n)=0$ for all $t\geq T$.

[14] We can show that $\hat{\phi}(2)\geq\tilde{\phi}$ iff $1-\beta\mu-\beta\lambda\mu\geq 0$. Consider the parameter range of $\tilde{\phi}\leq\phi<\hat{\phi}(2)$ and $1-\beta\mu-\beta\lambda\mu\geq 0$. We have: $\hat{S}=3$ and $\bar{S}=2$. Given (B8), we have:

$$\hat{W}(n) \geq \hat{W}(n|2) \equiv \max_{\tilde{n}}\{\hat{W}(n,\tilde{n}|2)\}, \tag{B14}$$

where

$$\hat{W}(n,\tilde{n}|2) \equiv Y(n,\tilde{n}) + \beta\lambda\mu(\hat{W}(\tilde{n}|1)-\beta\lambda\hat{V}_o(\bar{n})) + \beta\lambda(1-\mu)\cdot\bar{Y}(\tilde{n}). \tag{B15}$$

We have $g(n|1)=(\sum_{s=2}^{\infty}n_s, n_1,\ldots)$ since $\hat{\phi}(3)<\tilde{\phi}\leq\phi<\hat{\phi}(2)$. Then, $d\bar{W}(n|1)/dn_s \equiv \lim_{\epsilon\to 0}(\bar{W}((n_1+\epsilon,\ldots,n_s-\epsilon,\ldots)|1)-\bar{W}(n|1)/\epsilon = 1/\lambda-\phi/\lambda^s-(1-1/\lambda)(\beta\phi(1-\mu)+\beta\mu)/(1-\beta\mu)$ for all $s\geq 2$, and $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_s \equiv \lim_{\epsilon\to 0}(\hat{W}(n,(\tilde{n}_0+\epsilon,\ldots,\tilde{n}_s-\epsilon,\ldots)|2)-\hat{W}(n,\tilde{n}|2))/\epsilon = -(1-phi)/\lambda^s+\beta\phi(1-\mu)(1-1/\lambda^s)+\beta\lambda\mu\cdot d\bar{W}(\tilde{n}|1)/d\tilde{n}_s \geq 0$ iff $\phi\geq\check{\phi}(s)\equiv(1-\beta\mu-\beta\lambda^s\mu(1-\beta\lambda\mu))/((1-\beta)(1-\beta\mu)+\beta\lambda^s(1-\mu)(1-\beta\lambda\mu))$; We can show that $\check{\phi}(1)\geq\bar{\phi}(1)$. Then, $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_1<0$. We can show that $\check{\phi}(2)\leq\tilde{\phi}$ given that $1-\beta\mu-\beta\lambda\mu\geq 0$. Then, $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_s\geq 0$ for all $s\geq 2$. Then, we have

$g(n|2) = (\sum_{s\geq 2} n_s, n_1, 0, \ldots)$. Substituting the expression of $g(n|2)$ in (B14) and (B15), we have: $\hat{W}(n|2) = c(0|2) + \sum_{s=1}^{\infty} c(s|2)n_s$, where

$$\begin{aligned} c(0|2) &= -\beta^2\lambda^2\mu \cdot \hat{V}_o(\bar{n}) + \beta\lambda\mu \cdot c(0|1); \\ c(1|2) &= (1+\beta\phi(1-\mu))/\lambda + \beta\lambda\mu \cdot c(2|1); \\ c(s|2) &= \phi/\lambda^s + \beta\phi(1-\mu) + \beta\lambda\mu \cdot c(1|1) \quad \text{for } s \geq 2. \end{aligned} \tag{B16}$$

Comparing $c(\cdot|0)$, $c(\cdot|1)$, $c(\cdot|2)$, and $c(\cdot|\infty)$ in (B7), (B10), (B13), and (B16), we can show that:

$$\begin{aligned} \hat{W}(n|1) &\geq \hat{W}(n|t) \quad \text{for } t = 0, 2, \infty, \quad \text{if } n_1 \in [0, 1-\check{n}]; \\ \hat{W}(n|\infty) &\geq \hat{W}(n|t) \quad \text{for } t = 0, 1, 2, \quad \text{if } n_1 \in [1-\check{n}, \breve{n}]; \\ \hat{W}(n|2) &\geq \hat{W}(n|t) \quad \text{for } t = 0, 1, \infty, \quad \text{if } n_1 \in [\breve{n}, \tilde{x}(n_2)]; \\ \hat{W}(n|0) &\geq \hat{W}(n|t) \quad \text{for } t = 1, 2, \infty, \quad \text{if } n_1 \in [\tilde{x}(n_2), 1], \end{aligned}$$

where $\breve{n} \equiv (1-\phi+\beta\phi-\beta\lambda^2(\phi+\mu(1-\phi))(1+\beta^2\lambda\mu^2-\beta^2\lambda^2\mu^2)/(1-\beta\mu+\beta\lambda\mu))/((1-\beta\lambda\mu)(1-\phi+\beta\phi-\beta\lambda^2(\phi+\mu(1-\phi))))$; $\tilde{x}(n_2) \equiv \breve{n} \cdot n_2/(1-\breve{n}) + \check{n} \cdot (1-\breve{n}-n_2)/(1-\breve{n})$; and $\check{n} \equiv (\lambda-1)(1-\beta\mu-\beta^2\mu^2\lambda^2)(\mu+\phi(1-\mu))/(1-\beta\mu+\beta\lambda\mu)/(\lambda\mu(1-\beta\lambda\mu)-\phi(1-\beta\mu-\lambda(1-\mu)(1-\beta\lambda\mu))$. This motivates the following conjecture on the equilibrium when $\tilde{\phi} \leq \phi < \hat{\phi}(2)$ and $1-\beta\mu-\beta\lambda\mu \geq 0$:

$$\begin{aligned} &\hat{W}(n) = \hat{W}(n|1), \quad q(n) = 1, \text{ and } g(n) = (1-n_1, n_1, 0, \ldots) \text{ if } n_1 \in [0, 1-\check{n}]; \\ &\hat{W}(n) = \hat{W}(n|\infty), \quad q(n) = 1, \text{ and } g(n) = (1-n_1, n_1, 0, \ldots) \text{ if } n_1 \in [1-\check{n}, \breve{n}]; \\ &\hat{W}(n) = \hat{W}(n|2), \quad q(n) = 1, \text{ and } g(n) = (1-n_1, n_1, 0, \ldots) \text{ if } n_1 \in [\breve{n}, \tilde{x}(n_2)]; \\ &\hat{W}(n) = \hat{W}(n|0), \quad q(n) = 0, \text{ and } g(n) = (0, 1-n_2, n_2, \ldots) \text{ if } n_1 \in [\tilde{x}(n_2), 1], \end{aligned} \tag{B17}$$

with $\hat{V}_o(\bar{n})$ given by the low bound discussed in [11]: $\hat{V}_o(\bar{n}) = (\mu+\phi(1-\mu))/(\lambda(1-\beta\mu+\beta\lambda\mu))$. We can prove this conjecture by plugging in the conjectured value function $\hat{W}$ and $\hat{V}_o(\bar{n})$ on the right-hand side of (B1) to (B3), and then verify that the conjectured policy functions, q and g, solve the maximization problem. In summary, I have the following Result.

Result 5. If $\tilde{\phi} \leq \phi < \hat{\phi}(2)$ and $1-\beta\mu-\beta\lambda\mu \geq 0$, the equilibrium is (B17). If $n_1 \in [1-\check{n}, \breve{n}]$, $g_0(n) \in [1-\check{n}, \breve{n}]$; If $n_1 \in [\breve{n}, \tilde{x}(n_2)]$, $g_0(n) \in [0, 1-\check{n}]$; If $n_1 \in [0, 1-\check{n}]$, $q(n) = 0$. Therefore, the country either grows perpetually or stagnates after, at most, two periods of growth starting from any n.

[15] Now consider the parameter range of $\hat{\phi}(2) \le \phi \le \bar{\phi}(1)$ and $1-\beta\mu-\beta\lambda\mu \ge 0$. We have: $\hat{S}=3$ and $\bar{S}=2$. We have $g(n|1)=\bar{n}$ since $\phi \ge \hat{\phi}(2)$. Then, $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_s \equiv \lim_{\epsilon\to 0}(\hat{W}(n,(\tilde{n}_0+\epsilon,\ldots,\tilde{n}_s-\epsilon,\ldots)|2)-\hat{W}(n,\tilde{n}|2))/\epsilon = -(1-\phi)/\lambda^s+\beta\phi(1-1/\lambda^s) \ge 0$ iff $\phi \ge 1/(1-\beta+\beta\lambda^s)$. We have: $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_1 \le 0$ since $\phi \le \bar{\phi}(1) = 1/(1-\beta+\beta\lambda)$. We have: $\hat{\phi}(2)=(1-\beta\lambda\mu)/(1-\beta+\beta\lambda(1-\mu)) \ge 1/(1-\beta+\beta\lambda^2)$ iff $1-\beta\mu-\beta\lambda\mu \ge 0$. Then, $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_s \ge 0$ for all $s \ge 2$, and $g(n|2)=(\sum_{s\ge 2} n_s, n_1, 0, \ldots)$. Substituting the expression of $g(n|2)$ in (B14) and (B15), we obtain (B16). Comparing $c(\cdot|0)$, $c(\cdot|1)$, $c(\cdot|2)$, and $c(\cdot|\infty)$ in (B7), (B10), (B13), and (B16), we can show that:

$$
\begin{aligned}
\hat{W}(n|1) &\ge \hat{W}(n|t) \quad \text{for } t=0,2,\infty, \quad \text{if } n_1 \in [0,\hat{n}];\\
\hat{W}(n|\infty) &\ge \hat{W}(n|t) \quad \text{for } t=0,1,2, \quad \text{if } n_1 \in [\hat{n},1-\hat{n}];\\
\hat{W}(n|2) &\ge \hat{W}(n|t) \quad \text{for } t=0,1,\infty, \quad \text{if } n_1 \in [1-\hat{n},\tilde{x}];\\
\hat{W}(n|0) &\ge \hat{W}(n|t) \quad \text{for } t=1,2,\infty, \quad \text{if } n_1 \in [\tilde{x},1],
\end{aligned}
$$

where $\hat{n} \equiv \beta^2\lambda\mu^2/((1-\beta\lambda\mu)(1-\beta\mu+\beta\lambda\mu))$ and $\tilde{x} \equiv (1+\beta\lambda\mu-\mu(1/\phi-1)/(\lambda-1))/(1-\beta\mu+\beta\lambda\mu)$. This motivates the following conjecture on the equilibrium when $\hat{\phi}(2) \le \phi \le \bar{\phi}(1)$ and $1-\beta\mu-\beta\lambda\mu \ge 0$:

$$
\begin{aligned}
\hat{W}(n) &= \hat{W}(n|1), \quad q(n)=1, \text{ and } g(n)=(1, 0, 0,\ldots) \text{ if } n_1 \in [0,\hat{n}];\\
\hat{W}(n) &= \hat{W}(n|\infty), \quad q(n)=1, \text{ and } g(n)=(1-n_1, n_1, 0,\ldots) \text{ if } n_1 \in [\hat{n},1-\hat{n}];\\
\hat{W}(n) &= \hat{W}(n|2), \quad q(n)=1, \text{ and } g(n)=(1-n_1, n_1, 0,\ldots) \text{ if } n_1 \in [1-\hat{n},\tilde{x}];\\
\hat{W}(n) &= \hat{W}(n|0), \quad q(n)=0, \text{ and } g(n)=(0, 1, 0,\ldots) \text{ if } n_1 \in [\tilde{x},1],
\end{aligned}
\tag{B18}
$$

with $\hat{V}_o(\bar{n})$ given by the low bound discussed in [11]: $\hat{V}_o(\bar{n})=(\mu+\phi(1-\mu))/(\lambda(1-\beta\mu+\beta\lambda\mu))$. We can prove this conjecture by plugging in the conjectured value function $\hat{W}$ and $\hat{V}_o(\bar{n})$ on the right-hand side of (B1) to (B3), and then verify that the conjectured policy functions, q and g, solve the maximization problem. In summary, I have the following Result.

Result 6. If $\hat{\phi}(2) \le \phi \le \bar{\phi}(1)$ and $1-\beta\mu-\beta\lambda\mu \ge 0$, The equilibrium is (B18). If $n_1 \in [\hat{n},1-\hat{n}]$, $g_0(n) \in [\hat{n},1-\hat{n}]$; If $n_1 \in [1-\hat{n},\tilde{x}]$, $g_0(n) \in [0,\hat{n}]$; If $n_1 \in [0,\hat{n}]$, $q(n)=0$. Therefore, the country either grows perpetually or stagnates after, at most, two periods of growth starting from any n.

[16] Now consider the parameter range of $1-\beta\mu-\beta\lambda\mu < 0$ and $\tilde{\phi} \le \phi \le \bar{\phi}(1)$. Repeating the steps in [15], we have: $\hat{W}(n|1) > \hat{W}(n|\infty)$ if $n_1 \in [0,\hat{n})$. We have: $\tilde{\phi} > 1/(1-\beta+\beta\lambda^2)$ iff $1-\beta\mu-\beta\lambda\mu < 0$. Then, $d\hat{W}(n,\tilde{n}|2)/d\tilde{n}_s \ge 0$ for all $s \ge 2$, and $g(n|2)=(\sum_{s\ge 2} n_s, n_1, 0,\ldots)$.

Then, the expressions of $c(\cdot|2)$ in (B16) are valid and $\hat{W}(n|1) > \hat{W}(n|\infty)$ if $n_1 \in (1-\hat{n}, 1]$. We can show that $\hat{n} > 1/2$ when $1 - \beta\mu - \beta\lambda\mu < 0$. Therefore, $\hat{W}(n) > \hat{W}(n|\infty)$ for all n. Consulting Lemma 8 and Result 2, we have:

Result 7. If $1 - \beta\mu - \beta\lambda\mu < 0$ and $\tilde{\phi} \le \phi \le \bar{\phi}(1)$, the economy stagnates forever possibly after some periods of growth, i.e., there is $T \ge 1$ so that $q^t(n) = 1$ for all $t < T$ and $q^t(n) = 0$ for all $t \ge T$.

[17] Now consider the parameter range of $\phi > \bar{\phi}(1) = 1/(1-\beta+\beta\lambda)$. We have $\bar{S} = 1$. Then, $d\hat{W}_2(n, \tilde{n}|\infty)/d\tilde{n}_s = -(1-\phi)/\lambda^s + \beta\phi(1 - 1/\lambda^s) \ge 0$ for all $s \ge 1$. Consider the following conjecture on the equilibrium: $\hat{W}(n) = \hat{W}(n|\infty)$ and $g(n) = \bar{n}$ for all n if $\phi \ge \bar{\phi}(1)$. From (9) and (B3), I have $\hat{V}_o(\bar{n}) = \phi/\lambda$. We have: $d\hat{W}_1(n, \tilde{n})/d\tilde{n}_s \equiv \lim_{\epsilon \to 0}(\hat{W}_1(n, (\tilde{n}_0 + \epsilon, \ldots, \tilde{n}_s - \epsilon, \ldots)) - \hat{W}_1(n, \tilde{n}))/\epsilon = -(1-\phi)/\lambda^s + \beta\phi(1/\lambda - 1/\lambda^s) \ge 0$ for all $s \ge 2$. Then, $\max_{\tilde{n}_0 = 0}\{W_1(n, \tilde{n})\} = n_1/\lambda + \sum_{s=2}^{\infty} \phi n_s/\lambda^s + \beta\phi < \sum_{s=1}^{\infty} \phi n_s/\lambda^s + \beta\lambda\phi = \hat{W}(n|\infty)$. This verifies the conjecture. In summary, I have the following Result.

Result 8. If $\phi > \bar{\phi}(1)$, the equilibrium is:

$$\hat{W}(n) = \hat{W}(n|\infty), \;\; q(n) = 1, \;\; \text{and } g(n) = (1, 0, 0, \ldots). \text{ for all } n$$

Therefore, the country grows perpetually starting from any n.

Appendix 3: Proofs of Propositions in Section 5 (For Online Posting)

[1] Consider the following conjecture on the equilibrium in the parameter range of $\phi > \bar{\phi}(1) = 1/(1-\beta+\beta\lambda)$:

$$q^i(n) = 1 \text{ and } g^i(n) = (1, 0, 0, \ldots) \text{ for all } n. \tag{C1}$$

Given (C1), observe that $G^{-i}(n, \tilde{n}^i)$ is independent of n^i. Then, $\bar{V}_y^i(n)$ is independent of n^i in $(4)'$. From (C1) and $(8)'$, $\check{W}^i(n) = Y(n^i, \bar{n}) + \beta\lambda\check{V}_o^i(G^{-i}(n, \bar{n}))$. Then, from $(4)'$, (C1), and $(9)'$, $\bar{V}_y^i(n) = \max_{\tilde{n}^i}\{\beta\lambda\check{V}_o^i(G^{-i}(n, \tilde{n}^i))\} = \beta\lambda\check{V}_o^i(G^{-i}(n, \bar{n}))$. Then, from $(8)'$ and $(9)'$, $\check{W}_2^i(n, \tilde{n}^i) = Y(n^i, \tilde{n}^i) + \beta\lambda Y(\tilde{n}^i, \bar{n})$ for all $\tilde{n}^i$. I have $d\check{W}_2^i(n, \tilde{n}^i)/d\tilde{n}_s^i \equiv \lim_{\epsilon \to 0}(\check{W}_2^i(n, (\tilde{n}_0^i + \epsilon, ..., \tilde{n}_s^i - \epsilon, ...)) - \check{W}_2(n, \tilde{n}^i))/\epsilon = -(1-\phi)/\lambda^s + \beta\phi(1 - 1/\lambda^s) \ge 0$ for all $s \ge 1$ if $\phi \ge 1/(1-\beta+\beta\lambda)$. Then, $\check{W}^i(n) = \max_{\tilde{n}^i}\{\check{W}_2^i(n, \tilde{n}^i)\} = \check{W}_2^i(n, \bar{n})$. This validates (C1). In summary, I have the following Result.

Result 9. If $\phi > \bar{\phi}(1)$, an equilibrium of the world economy is (C1). Therefore, the perpetual growth in which all of the young people adopt the frontier technology in both countries in each period is an equilibrium.

[2] Consider the parameter range of $\phi > \bar{\phi}(1) = 1/(1-\beta+\beta\lambda)$. Let $q(n,t) \equiv \max_j\{q^j(n, t)\}$, $t \geq 0$, denote the indicator of technology advancement in the world in period t on the equilibrium path, given n at $t = 0$. Let $\check{V}_y^i(n;t)$, $t \geq 0$, denote the utility of the young generation at date t. We have $\check{V}_y^i(n;t) \geq \lambda^{Q(n,t)}\beta\phi$, where $Q(n,0) = 0$ and $Q(n,t) \equiv \sum_{s=0}^{t-1} q(n,s)$ for $t \geq 1$, from (3)′ and (4)′. We also have: $\check{V}_o^i(n^i) \geq Y(n^i, \bar{n})$ from (3)′. Then, $\Theta^i(n) \equiv \check{V}_o^i(n^i) + \sum_{t=0}^{\infty} \beta^t \check{V}_y^i(n;t) \geq \bar{\Theta}^i(n) \equiv Y(n^i,\bar{n}) + \sum_{t=0}^{\infty} \beta^t \lambda^{Q(n,t)}\beta\phi$. Now, consider the problem of maximizing the discounted sum of utilities across generations in country i, denoted by $\tilde{\Theta}^i(n)$, by choosing $\{g^i(n,t)\}$ given n and $\{q(n,t)\}$. We can show that the solution to this problem is: $g^i(n,t) = \bar{n}$ if $q(n,t) = 1$, and $g^i(n,t) = (0,\bar{n})$ if $q(n,t) = 0$, if $\phi > \bar{\phi}(1)$. Now, suppose that $q(n,0) = 0$ for some n. We have: $\Theta^i(n) \leq \tilde{\Theta}^i(n) = Y(n^i,(0,\bar{n})) + \beta\phi/\lambda + \sum_{t=1}^{\infty} \beta^t \lambda^{Q(n,t)}(I_{q(n,t)=0}\cdot(1-\phi)/\lambda + \beta\phi\lambda^{q(n,t)}/\lambda) < Y(n^i,\bar{n}) + \beta\phi + \sum_{t=1}^{\infty} \beta^t\lambda^{Q(n,t)}\beta\phi$ $= \bar{\Theta}^i(n) \leq \check{\Theta}^i(n)$ if $\phi > \bar{\phi}(1)$. This is a contradiction. Therefore, we have:

Result 10. If $\phi > \bar{\phi}(1)$, an equilibrium in which every country stagnates for some n does not exist.

[3] Now consider the following conjecture on the equilibrium when $\tilde{\phi} \leq \phi \leq \bar{\phi}(1)$:

$$q^i(n) = 1 \text{ and } g^i(n) = (1-n_1^i, n_1^i, 0, \ldots) \text{ for all } n. \tag{C2}$$

Since $q^i(n)$ is independent of n, conjecture that $\check{W}^i(n)$ and $\check{V}_o^i(n)$ are independent of n^j for $j \neq i$. We can then write: $\check{W}^i(n) = \check{W}^i(n^i|\infty)$, $\check{W}_2^i(n,\tilde{n}^i) = \check{W}_2^i(n^i,\tilde{n}^i|\infty)$, and $\check{V}_o^i(n) = \check{V}_o^i(n^i|\infty)$. Further, conjecture that $\check{W}^i(n^i|\infty)$ and $\check{V}_o^i(n^i|\infty)$ are increasing in n^i so that $\bar{V}_y^i(n) = \max_{\tilde{n}^i}\{\beta\lambda\check{V}_o^i(n^i|\infty)\} = \beta\lambda\check{V}_o^i(\bar{n}|\infty)$. From (6)′ to (9)′, $\check{W}^i(n^i|\infty) = Y(n^i,(1-n_1^i,n_1^i,\ldots)) + \beta\lambda\mu(\check{W}^i((1-n_1^i,n_1^i,\ldots)|\infty) - \beta\lambda\check{V}_o^i(\bar{n}|\infty)) + \beta\lambda(1-\mu)\cdot Y((1-n_1^i,n_1^i,\ldots),\bar{n})$. Consulting (B11) and (B12), observe that $\check{W}^i(n^i|\infty)$ is equal to $\hat{W}(n|\infty)$ with $\bar{S} = 2$ and with a change in $c(0|\infty)$: $\check{W}^i(n^i|\infty) = \check{c}(0|\infty) + \sum_{s=1}^{\infty} \check{c}(s|\infty)n_s^i$ where

$$\check{c}(0|\infty) = -\beta^2\lambda^2\mu \cdot \check{V}_o^i(\bar{n}|\infty)/(1-\beta\lambda\mu);$$

$$\check{c}(s|\infty) = 1/\lambda^s + \beta\phi(1-\mu)/\lambda^s + \beta\lambda\mu\cdot\check{c}(s+1|\infty) \quad \text{for } s = 1;$$

$$\check{c}(s|\infty) = \phi/\lambda^s + \beta\phi(1-\mu) + \beta\lambda\mu\cdot\check{c}(1|\infty) \quad \text{for } s \geq 2.$$

Repeating the steps in [9] in Appendix 2, we have: $d\check{W}_2^i(n^i,\tilde{n}^i|\infty)/d\tilde{n}_s^i \equiv \lim_{\epsilon\to 0}(\check{W}_2^i(n^i, (\tilde{n}_0^i+\epsilon,\ldots,\tilde{n}_s^i-\epsilon,\ldots)|\infty) - \check{W}_2^i(n^i,\tilde{n}^i|\infty))/\epsilon = -(1-\phi)/\lambda^s + \beta\phi(1-\mu)(1-1/\lambda^s) +$

$\beta\lambda\mu(\check{c}(1|\infty) - \check{c}(s+1|\infty)) > 0$ if $s \geq 2$ and $d\check{W}_2^i(n^i, \tilde{n}^i|\infty)/d\tilde{n}_1^i \leq 0$ since $\bar{\phi}(2) < \tilde{\phi} \leq \phi \leq \bar{\phi}(1)$. Then, $g^i(n) = (1 - n_1^i, n_1^i, \ldots)$. Also, $c(s|\infty) - c(s+1|\infty) > 0$ for all s. Then, $\check{W}^i(n^i|\infty)$ and $\check{V}_o^i(n^i|\infty)$ are increasing in n^i, verifying that $\bar{V}_y^i(n) = \beta\lambda\check{V}_o^i(\bar{n}|\infty)$. Further, $q^i(n) = 1$ is (trivially) optimal. This validates (C2). In summary, I have the following Result.

Result 11. If $\tilde{\phi} \leq \phi \leq \bar{\phi}(1)$, an equilibrium of the world economy is (C2). Therefore, the world economy may grow perpetually starting from any n.

[4] Now consider the following conjecture on the equilibrium when $\hat{\phi}(2) \leq \phi \leq \bar{\phi}(1)$ and $1 - \beta\mu - \beta\lambda\mu \geq 0$:

$$\begin{aligned}
&q^i(n) = 0 \text{ and } g^i(n) = (0, 1, 0, \ldots) \text{ for } n \in \Upsilon_{stag1}, \\
&q^i(n) = 1 \text{ and } g^i(n) = (1, 0, 0, \ldots) \text{ for } n \in \Upsilon_{stag2}, \\
&q^i(n) = 1 \text{ and } g^i(n) = (1 - n_1^i, n_1^i, 0, \ldots) \text{ for } n \in \Upsilon_{stag3} \cup \Upsilon_{grow1} \cup \Upsilon_{grow2}^{ij}, \text{ and} \\
&q^i(n) = 1 \text{ and } g^i(n) = (\hat{m}, 1 - \hat{m}, 0, \ldots) \text{ for } n \in \Upsilon_{grow2}^{ii}
\end{aligned} \tag{C3}$$

where $\Upsilon_{stag1} \equiv \{n : n_1^j = 1 \text{ for both } j\}$; $\Upsilon_{stag2} \equiv \{n : n_1^j \leq \hat{m} \text{ for both } j\}$; $\Upsilon_{stag3} \equiv \{n : n_1^j \geq 1 - \hat{m} \text{ for both } j\} - \Upsilon_{stag1} - \Upsilon_{stag2}$; $\Upsilon_{grow1} \equiv \{n : n_1^j > \hat{m} \text{ for some } j \text{ and } n_1^j < 1 - \hat{m}$ for some $j\}$; $\Upsilon_{grow2} \equiv \tilde{\Upsilon}_{grow2}^{ii} + \tilde{\Upsilon}_{grow2}^{ij}$ where $i \neq j$; $\Upsilon_{grow2}^{ii} \equiv \{n : 1 - \hat{m} \leq n_1^i < 1 - \hat{\rho}\hat{m}$ and $1 - \hat{m} \leq n_1^j < \tilde{y}$ for $j \neq i\} \cap \{n : n_1^i < n_1^j$, or $n_1^A = n_1^B$ and $i = A\} - \Upsilon_{stag2}$; $\Upsilon_{grow2}^{ij} \equiv \{n : 1 - \hat{m} \leq n_1^i < \tilde{y}$ and $1 - \hat{m} \leq n_1^j < 1 - \hat{\rho}\hat{m}$ for $j \neq i\} \cap \{n : n_1^j < n_1^i$, or $n_1^A = n_1^B$ and $i = B\} - \Upsilon_{stag2}$; $\hat{m} \equiv \beta^2\lambda\mu^2/(1 - \beta\mu - \beta^2\lambda^2\mu^2)$; and $\hat{\rho} \equiv 1/(1 + \beta\lambda\mu)$.

Result 12. If $\hat{\phi}(2) \leq \phi \leq \bar{\phi}(1)$ and $1 - \beta\mu - \beta\lambda\mu \geq 0$, an equilibrium of the world economy is (C3). If $n \in \Upsilon_{grow1}$, $g(n) \in \Upsilon_{grow1}$; If $n \in \Upsilon_{grow2}$, $g(n) \in \Upsilon_{grow1}$; If $n \in \Upsilon_{stag3}$, $g(n) \in \Upsilon_{stag2}$; If $n \in \Upsilon_{stag2}$, $g(n) \in \Upsilon_{stag1}$. Therefore, the world economy either grows perpetually or stagnates after, at most, two periods of growth starting from any n.

Below I will sketch the steps of the proof. Given the expressions of $\hat{m}$ and $\hat{\rho}$, there are 4 possible orderings of $\hat{m}$, $1 - \hat{m}$, $\hat{\rho}\hat{m}$, and $1 - \hat{\rho}\hat{m}$: (1) $0 \leq \hat{\rho}\hat{m} \leq \hat{m} \leq 1 - \hat{m} \leq 1 - \hat{\rho}\hat{m} \leq 1$; (2) $0 \leq \hat{\rho}\hat{m} \leq 1 - \hat{m} \leq \hat{m} \leq 1 - \hat{\rho}\hat{m} \leq 1$; (3) $0 \leq 1 - \hat{m} \leq \hat{\rho}\hat{m} \leq 1 - \hat{\rho}\hat{m} \leq \hat{m} \leq 1$; (4) $0 \leq 1 - \hat{m} \leq 1 - \hat{\rho}\hat{m} \leq \hat{\rho}\hat{m} \leq \hat{m} \leq 1$;

Consider Case (1). Let zz' denote the state of the world economy from the perspective of country i: z characterizes the human capital distribution of country i and z' that of country $j \neq i$ as follows. For all z', subscript $\bar{l}z'$ means $n_1^i = 0$; $\tilde{l}z'$ means $n_1^i \in (0, \hat{\rho}\hat{m}]$; lz' means $n_1^i \in [\hat{\rho}\hat{m}, \hat{m}]$; cz' means $n_1^i \in [\hat{m}, 1 - \hat{m}]$; hz' means $n_1^i \in [1 - \hat{m}, 1 - \hat{\rho}\hat{m}]$; $\tilde{h}z'$

means $n_1^i \in [1-\hat{\rho}\hat{m}, 1)$; and $\bar{h}z'$ means $n_1^i = 1$. Similarly, for all z, subscript $z\bar{l}$ means $n_1^j = 0$; $z\tilde{l}$ means $n_1^j \in (0, \hat{\rho}\hat{m}]$; zl means $n_1^j \in [\hat{\rho}\hat{m}, \hat{m}]$; zc means $n_1^j \in [\hat{m}, 1-\hat{m}]$; zh means $n_1^j \in [1-\hat{m}, 1-\hat{\rho}\hat{m}]$; $z\tilde{h}$ means $n_1^j \in [1-\hat{\rho}\hat{m}, 1)$; and $z\bar{h}$ means $n_1^j = 1$.

The evolution of the world economy from the perspective of country i is summarized in the following table of transition paths, that can be derived from (C3). The row indexes the current period z; the column indexes the current period z'; and each cell shows the next period zz'. The cell cl when $zz' = hh$ is valid if $n_1^i < n_1^j$ or if $n_1^A = n_1^B$ and $i = A$; the cell becomes lc otherwise.

	$\bar{l}$	$\tilde{l}$	l	c	h	$\tilde{h}$	$\bar{h}$
$\bar{l}$	$\bar{h}\bar{h}$	$\bar{h}\bar{h}$	$\bar{h}\bar{h}$	$\bar{h}c$	$\bar{h}l$	$\bar{h}\tilde{l}$	$\bar{h}\bar{l}$
$\tilde{l}$	$\bar{h}\bar{h}$	$\bar{h}\bar{h}$	$\bar{h}\bar{h}$	$\tilde{h}c$	$\tilde{h}l$	$\tilde{h}\tilde{l}$	$\tilde{h}\bar{l}$
l	$\bar{h}\bar{h}$	$\bar{h}\bar{h}$	$\bar{h}\bar{h}$	hc	hl	$h\tilde{l}$	$h\bar{l}$
c	$c\bar{h}$	$c\tilde{h}$	ch	cc	cl	$c\tilde{l}$	$c\bar{l}$
h	$l\bar{h}$	$l\tilde{h}$	lh	lc	cl	$c\tilde{l}$	$l\bar{l}$
$\tilde{h}$	$\tilde{l}\bar{h}$	$\tilde{l}\tilde{h}$	$\tilde{l}h$	$\tilde{l}c$	$\tilde{l}c$	$\tilde{l}\tilde{l}$	$\tilde{l}\bar{l}$
$\bar{h}$	$\bar{l}\bar{h}$	$\bar{l}\tilde{h}$	$\bar{l}h$	$\bar{l}c$	$\bar{l}l$	$\bar{l}\tilde{l}$	$\bar{h}\bar{h}$

Let $\zeta(zz') \equiv (\zeta_z(zz'), \zeta_{z'}(zz'))$ denote the next period state of the world given this period state zz', defined according to the above table. The value functions are:

$$\check{W}^i(\bar{n}|\bar{h}\bar{h}) = Y(\bar{n}, (0, \bar{n})) + \beta \check{V}_o^i(\bar{n}|\bar{h}\bar{h}); \tag{C4}$$

$$\check{W}^i(n^i|zz') = Y(n^i, g^i(n)) + \beta\lambda \check{V}_o^i(g^i(n)|\zeta(zz')) \tag{C5}$$

for all $zz' \neq \bar{h}\bar{h}$;

$$\check{V}_o^i(n^i|zz') = \mu \cdot (\check{W}^i(n^i|zz') - \bar{V}_y^i(n^i|zz')) + (1-\mu) \cdot Y(n^i, \bar{n}); \tag{C6}$$

for all zz';

$$\bar{V}_y^i(\bar{n}|\bar{h}\bar{h}) = \beta\lambda \check{V}_o^i(\bar{n}|\bar{h}\bar{l}); \tag{C7}$$

$$\bar{V}_y^i(n^i|zz') = \beta\lambda \check{V}_o^i(\bar{n}|(\bar{h}, \zeta_{z'}(zz'))) \tag{C8}$$

for all $zz' \neq \bar{h}\bar{h}$. Equations (C7) and (C8) assume that the outside option of the young generation is an all-out-technology-adoption (i.e., the first argument with $\tilde{n}^i = \bar{n}$ in $(4)'$), which will be verified. Since the outside option is on an off-equilibrium path, in (C7) the outside option of the young generation in country i leaves the other country remain stagnating so that the state of the world economy becomes $\bar{h}\bar{l}$ in the next period.

From $\{\zeta(zz')\}$ and (C4) to (C8), we can derive the expressions of $\check{W}^i(n^i|zz')$, $\check{V}_o^i(n^i|zz')$, and $\bar{V}_y^i(n^i|zz')$ in terms of β, λ, μ, and ϕ only. Substituting these value functions in $(7)'$ to $(9)'$, we can verify the policy functions in (C3). Further, we can verify (C7) and (C8) from $(4)'$. This completes the proof for Case (1).

For each of Cases (2) to (4), we can repeat the above steps: construct $\{\zeta(zz')\}$ from (C3); derive $\check{W}^i(n^i|zz')$, $\check{V}_o^i(n^i|zz')$, and $\bar{V}_y^i(n^i|zz')$ using (C4) to (C8); and verify (C3), (C7), and (C8) from $(6)'$ to $(9)'$ and $(4)'$.

[5] Now consider the following conjecture on the equilibrium when $\tilde{\phi} \le \phi < \hat{\phi}(2)$ and $1 - \beta\mu - \beta\lambda\mu \ge 0$:

$$\begin{aligned} &q^i(n) = 0 \text{ and } g^i(n) = (0, 1 - n_2^i, n_2^i, \ldots) \text{ for } n \in \Omega_{stag1} \\ &q^i(n) = 1 \text{ and } g^i(n) = (1 - n_1^i, n_1^i, 0, \ldots) \text{ for } n \in \Omega_{stag2} \cup \Omega_{stag3}^{ij} \cup \Omega_{stag4} \cup \Omega_{grow} \\ &q^i(n) = 1 \text{ and } g^i(n) = (\check{m}, 1 - \check{m}, 0, \ldots) \text{ for } n \in \Omega_{stag3}^{ii} \end{aligned} \tag{C9}$$

where $\Omega_{grow} \equiv \{n : n_1^j > 1 - \check{m}$ for some j and $n_1^j < \check{m}$ for some $j\}$; $\Omega_{stag1} \equiv \{n : n_1^j \ge \tilde{y}(n_2^j)$ for both $j\}$; $\Omega_{stag2} \equiv \{n : n_1^j \le 1 - \check{m}$ for both $j\} - \Omega_{stag1}$; $\Omega_{stag3} \equiv \Omega_{stag3}^{ii} + \Omega_{stag3}^{ij}$ where $i \ne j$; $\Omega_{stag3}^{ii} \equiv \{n : 1 - \check{m} \le n_1^i \le 1 - \check{\rho}\check{m}$ and $\check{m} \le n_1^j \le 1 - \check{m}$ for $j \ne i\} - \Omega_{stag1}$; $\Omega_{stag3}^{ij} \equiv \{n : \check{m} \le n_1^i \le 1 - \check{m}$ and $1 - \check{m} \le n_1^j \le 1 - \check{\rho}\check{m}$ for $j \ne i\} - \Omega_{stag1}$; $\Omega_{stag4} \equiv \{n : n_1^j \ge \check{m}$ for both $j\} - \Omega_{stag1} \cup \Omega_{stag2} \cup \Omega_{stag3}$; $\check{m} \equiv (1+\beta\lambda\mu)(1-\beta\mu-\beta\lambda^2\mu(1-\beta\lambda\mu)-\phi((1-\beta)(1-\beta\mu)+\beta\lambda^2(1-\mu)(1-\beta\lambda\mu)))/(1-\beta\lambda^2\mu-\phi(1-\beta+\beta\lambda^2(1-\mu)))/(1-\beta\mu-\beta^2\lambda^2\mu^2)$; $\check{m}$ is increasing in ϕ; $\check{m} = 1 - \hat{m}$ when $\phi = \hat{\phi}(2)$; $\check{\rho} \equiv (1 - \beta\mu)(1 - \phi(1 - \beta + \beta\lambda))/(1 + \beta\lambda\mu)/(1 - \beta\mu - \beta\lambda\mu + \beta^2\lambda^2\mu^2 - \phi((1 - \beta)(1 - \beta\mu) + \beta\lambda(1 - \mu)(1 - \beta\lambda\mu)))$; $\tilde{y}(n_2) \equiv \check{m} \cdot n_2/(1 - \check{m}) + \breve{m} \cdot (1 - \check{m} - n_2)/(1 - \check{m})$; $\breve{m} \equiv (\mu(1 - \beta\lambda\mu)((\lambda - 1)(1 - \beta\mu) - \beta^2\lambda^3\mu^2) + \phi((\lambda - 1)(1 - \mu)(1 - \beta\mu - \beta\lambda\mu) + \beta^2\lambda\mu^2(\mu - \beta\mu - \lambda(1 - \mu)(\lambda - 1 - \beta\lambda^2\mu))))/((1 - \beta\mu - \beta^2\lambda^2\mu^2)(\lambda\mu(1-\beta\lambda\mu)-\phi(1-\beta\mu-\lambda(1-\mu)(1-\beta\lambda\mu)))) > \check{m}$; $\breve{m}$ is increasing in ϕ; $\breve{m} = 1-\hat{m}$ when $\phi = \hat{\phi}(2)$.

Result 13. If $\tilde{\phi} \le \phi < \hat{\phi}(2)$ and $1 - \beta\mu - \beta\lambda\mu \ge 0$, an equilibrium of the world economy is (C9). If $n \in \Omega_{grow}$, $g(n) \in \Omega_{grow}$; If $n \in \Omega_{stag4}$, $g(n) \in \Omega_{stag2}$; If $n \in \Omega_{stag2}$, $g(n) \in \Omega_{stag1}$; If $n \in \Omega_{stag3}$, $g(n) \in \Omega_{stag1}$. Therefore, the world economy either grows perpetually or stagnates after, at most, two periods of growth starting from any n.

The steps of the proof are similar to [4]. Given the expressions of $\check{m}$, $\breve{m}$, and $\check{\rho}$, there are 7 possible orderings of $\check{m}$, $1 - \check{m}$, $\tilde{y}(n_2^i)$, $\check{\rho}\check{m}$, and $1 - \check{\rho}\check{m}$: (1) $0 \le 1 - \check{m} \le \check{m} \le \tilde{y}(n_2^j) \le 1$ for $j = A, B$; (2) $0 \le \check{\rho}\check{m} \le \check{m} \le \tilde{y}(n_2^j) \le 1 - \check{m} \le 1 - \check{\rho}\check{m} \le 1$ for $j = A, B$; (3) $0 \le \check{\rho}\check{m} \le \check{m} \le \tilde{y}(n_2^i) \le 1 - \check{m} \le 1 - \check{\rho}\check{m} \le 1$ and $0 \le \check{\rho}\check{m} \le \check{m} \le 1 - \check{m} \le \tilde{y}(n_2^j) \le 1 - \check{\rho}\check{m} \le 1$ where $i \ne j$;

(4) $0 \leq \check{\rho}\check{m} \leq \check{m} \leq \tilde{y}(n_2^i) \leq 1 - \check{m} \leq 1 - \check{\rho}\check{m} \leq 1$ and $0 \leq \check{\rho}\check{m} \leq \check{m} \leq 1 - \check{m} \leq 1 - \check{\rho}\check{m} \leq \tilde{y}(n_2^j) \leq 1$ where $i \neq j$; (5) $0 \leq \check{\rho}\check{m} \leq \check{m} \leq 1 - \check{m} \leq \tilde{y}(n_2^j) \leq 1 - \check{\rho}\check{m} \leq 1$ for $j = A, B$; (6) $0 \leq \check{\rho}\check{m} \leq \check{m} \leq 1 - \check{m} \leq 1 - \check{\rho}\check{m} \leq \tilde{y}(n_2^j) \leq 1$ and $0 \leq \check{\rho}\check{m} \leq \check{m} \leq 1 - \check{m} \leq \tilde{y}(n_2^i) \leq 1 - \check{\rho}\check{m} \leq 1$ where $i \neq j$; (7) $0 \leq \check{\rho}\check{m} \leq \check{m} \leq 1 - \check{m} \leq 1 - \check{\rho}\check{m} \leq \tilde{y}(n_2^j) \leq 1$ for $j = A, B$.

Consider Case (1). Let zz' denote the state of the world economy from the perspective of country i: z characterizes the human capital distribution of country i and z' that of country $j \neq i$ as follows. For all z', subscript lz' means $n_1^i \in [0, 1 - \check{m}]$; cz' means $n_1^i \in [1 - \check{m}, \check{m}]$; $\tilde{h}z'$ means $n_1^i \in [\check{m}, \tilde{y}(n_2^i)]$; and $\bar{h}z'$ means $n_1^i \in [1 - \hat{\rho}\hat{m}, 1]$. Similarly, for all z, subscript zl means $n_1^j \in [0, 1 - \check{m}]$; zc means $n_1^j \in [1 - \check{m}, \check{m}]$; $z\tilde{h}$ means $n_1^j \in [\check{m}, \tilde{y}(n_2^j)]$; and $z\bar{h}$ means $n_1^j \in [\check{m}, 1]$.

	l	c	$\tilde{h}$	$\bar{h}$
l	$\bar{h}\bar{h}$	$\bar{h}c$	$\bar{h}l$	$\bar{h}l$
c	$c\bar{h}$	cc	cl	cl
$\tilde{h}$	$l\bar{h}$	lc	ll	ll
$\bar{h}$	$l\bar{h}$	lc	ll	$\bar{h}\bar{h}$

The evolution of the world economy from the perspective of country i is summarized in the above table of transition paths, that can be derived from (C9). The row indexes the current period z; the column indexes the current period z'; and each cell shows the next period zz'. Let $\zeta(zz') \equiv (\zeta_z(zz'), \zeta_{z'}(zz'))$ denote the next period state of the world given this period state zz', defined according to the above table. The value functions are (C5), (C6), (C8), and a variation of (C4) and (C7):

$$\check{W}^i(n^i|\bar{h}\bar{h}) = Y(n^i, (0, 1 - n_2^i, n_2^i, \ldots)) + \beta \check{V}_o^i(1 - n_2^i, n_2^i, \ldots |\bar{h}\bar{h}); \tag{C4$'$}$$

$$\bar{V}_y^i(n^i|\bar{h}\bar{h}) = \beta\lambda \check{V}_o^i(\bar{n}|\bar{h}\bar{l}). \tag{C7$'$}$$

From $\{\zeta(zz')\}$, (C4)$'$, (C5), (C6), (C7)$'$, and (C8), we can derive the expressions of $\check{W}^i(n^i|zz')$, $\check{V}_o^i(n^i|zz')$, and $\bar{V}_y^i(n^i|zz')$ in terms of β, λ, μ, and ϕ only. Substituting these value functions in (7)$'$ to (9)$'$, we can verify the policy functions in (C9). Further, we can verify (C7)$'$ and (C8) from (4)$'$. This completes the proof for Case (1).

For each of Cases (2) to (7), we can repeat the above steps with a slight modification. Note that Cases (2) to (7) can be considered as six different regimes of a single ordering of parameters with the current regime depending on the current n. If the initial regime is (2), the regime remains to be (2) in all subsequent periods. If the initial regime is (3) to (7), the regime switches to (2) in all subsequent periods. Thus, in any of Cases (3) to (7), the possible states of the world $\{zz'\}$ needs to be expanded to include the states of Case (2). With this expansion of the states of the world, we can construct $\{\zeta(zz')\}$ from (C9); derive $\check{W}^i(n^i|zz')$, $\check{V}_o^i(n^i|zz')$, and $\bar{V}_y^i(n^i|zz')$ using (C4)′, (C5), (C6), (C7)′, and (C8); and verify (C9), (C7)′, and (C8) from (6)′ to (9)′ and (4)′.

Appendix 4: Proofs of Propositions in Section 7 (For Online Posting)

We have $\check{W}^i(n) = \check{W}^i(n^i|\infty)$ under Unconditional Growth World Equilibrium (i.e., $\check{W}^i(n)$ given any $\{n^j\}_{j\neq i}$), as shown in [3] in Appendix 3. I can show that:

$$\begin{aligned} &\hat{W}(n^i) \geq \check{W}^i(n^i|\infty) \quad \text{for all } n^i \text{ with } n_1^i + n_2^i = 1 \text{ and } n_1^i \in [\check{x}, 1]; \\ &\hat{W}(n^i) \leq \check{W}^i(n^i|\infty) \quad \text{for all } n^i \text{ with } n_1^i + n_2^i = 1 \text{ and } n_1^i \in [\hat{x}, \check{x}]; \\ &\hat{W}(n^i) \geq \check{W}^i(n^i|\infty) \quad \text{for all } n^i \text{ with } n_1^i + n_2^i = 1 \text{ and } n_1^i \in [0, \hat{x}], \end{aligned} \tag{D1}$$

where $\hat{x} < \check{x}$; $\hat{x} = \beta^2\lambda\mu^2/(1-\beta\mu+\beta\lambda\mu)$ and $\check{x} = ((\phi+\mu(1-\phi))(1-\beta^2\lambda^2\mu^2)/(\lambda(1-\beta\mu+\beta\lambda\mu)) - ((\phi+\mu(1-\phi))(1-\beta\lambda\mu)+\beta\mu\phi)+\beta^2\lambda\mu^2\phi)/(\phi/\lambda-((\phi+\mu(1-\phi))(1-\beta\lambda\mu)+\beta\mu\phi))$ if $\hat{\phi}(2) \leq \phi \leq \bar{\phi}(1)$; $\hat{x} = \beta\lambda\mu(1-\phi(1-\beta+\beta\lambda))(1-\beta\mu)/(1-\beta\mu+\beta\lambda\mu)/(\phi(1-\beta+\beta\lambda^2(1-\mu))-(1-\beta\lambda^2\mu))$ and $\check{x} = ((1-\beta\mu+\beta\lambda\mu)(1+\beta\lambda\mu-\beta\lambda^2\mu)-\beta^3\lambda^2\mu^3-\phi(\beta\lambda(1+\beta\lambda\mu)(\mu(1-\beta)+\lambda(1-\mu))+(1-\beta\mu)((1-\beta)(1+\beta\lambda\mu(1+\beta\lambda\mu))+\beta^3\lambda^3\mu^2)))/(1-\beta\mu+\beta\lambda\mu)/(1-\beta\lambda^2\mu-\phi(1-\beta+\beta\lambda^2-\beta\lambda^2\mu))$ if $\tilde{\phi} \leq \phi \leq \hat{\phi}(1)$. These properties imply Proposition 5.

Consulting [14] and [15] in Appendix 2, I can show that $\hat{x} \leq \hat{n}$ and $\tilde{x} \leq \check{x} \leq 1$ if $\hat{\phi}(2) \leq \phi \leq \bar{\phi}(1)$; and $\hat{x} \leq 1-\check{n}$ and $\check{n} = \tilde{x}(1-\check{n}) \leq \check{x} \leq 1-\hat{x}$ if $\tilde{\phi} \leq \phi \leq \hat{\phi}(2)$. These properties, combined with Proposition 5, imply Corollary 5.

Proposition 6 and Corollary 6 follow from Proposition 5 and Corollary 5, since the aggregate utility of a political union is the same as a country in autarky.

The proof of Proposition 7 is as follows. According to Proposition 6, the political union would form if and only if $n_1^u \in [0, \hat{x}) \cup (\check{x}, 1]$. Let $\{\xi^t(n^i)\}$ denote the evolution of human capital distribution in country i, for any i, starting from n^i in a politically fragmented world: $\xi^0(n^i) \equiv n^i$ and $\xi^{t+1}(n^i) = g^i(\xi^t(n^i))$. From (11) and (D1), I have

$$\hat{W}\left(\sum\nolimits_i \eta^i \xi^t(n^i)\right) \leq \sum\nolimits_i \eta^i \check{W}\left(\xi^t(n^i)\right) \tag{D2}$$

for all t if $n^u = \sum_i \eta^i n^i \in [\max\{\hat{x}, 1-\check{x}\}, \min\{\check{x}, 1-\hat{x}\}]$. Thus, under this condition there will never be an incentive to form a political union. On the other hand, (D2) will be violated at least once every two periods if $n_1^u \in [0, \max\{\hat{x}, 1-\check{x}\}) \cup (\min\{\check{x}, 1-\hat{x}\}, 1]$. Under this condition, the political union will form eventually.

Now let $\{\tilde{\xi}^t(n^u)\}$ denote the evolution of human capital distribution in the political union starting from $n^u \in [0, \hat{x}) \cup (\check{x}, 1]$: $\tilde{\xi}^0(n^u) = n^u$; $\tilde{\xi}^{t+1}(n^u) = g_{-0}(\tilde{\xi}^t(n^u))$ if $q(n^u) = 0$; and $\tilde{\xi}^{t+1}(n^u) = g(\tilde{\xi}^t(n^u))$ if $q(n^u) = 1$. I have: $\tilde{\xi}_1^t(n^u) = \tilde{\xi}_1^1(n^u) > \check{x}$ for all $t \geq 1$, from (B17), (B18), and the above-mentioned property that $\check{x} \leq 1 - \hat{x}$ if $\tilde{\phi} \leq \phi \leq \hat{\phi}(2)$. This implies that the political union stagnates after at most one period of growth, as stated in Corollary 6. Further, it implies that $\hat{W}(\tilde{\xi}^t(n^u)) \geq \check{W}(\tilde{\xi}^t(n^u))$ for all t. Combined with (11), I have

$$\hat{W}(\tilde{\xi}^t(n^u)) \geq \sum_i \tilde{\eta}_t^i \check{W}(m_t^i)$$

for all t and for any $\{\tilde{\eta}_t^i\}$ and $\{m_t^i\}$ that satisfy $\tilde{\xi}^t(n^u) = \sum_i \tilde{\eta}_t^i m_t^i$. Thus, there is no incentive to break up a political union once it is formed.