Talent Rewards, Talent Uncertainty, and Career Tracks

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December 2020

ABSTRACT

I present a model in which (1) a more talent-demanding task increases both rewards for high talent and the penalty for low talent due to a greater fixed cost of production, and (2) individual talent is task-specific and talent updates occur only for tasks near the attempted task, which implies a task-sequence problem in which the initial task constrains subsequent task choices. Rising talent rewards and penalty stemming from a rising scale economy motivate young workers to choose a more talent-demanding task, raise the failure rate (i.e., the probability of the updated talent being lower than the exit threshold), and concentrate income gains in a diminishing fraction of high-talent workers. Rising talent rewards and penalty also increase the share of young workers subject to binding minimum current-income constraints, thus increasing the dispersion of tasks among young workers. The model sheds light on the rising stratification of careers among young workers and the rising polarization of the residual labor income distribution (i.e., the labor income distribution controlling for observable worker characteristics such as education and age).

JEL classification: D80; E20; J60; O30
Keywords: career track; talent reward; talent uncertainty; minimum-income constraint; income inequality; income polarization

1. Introduction

The steady rise in residual wage inequality (i.e., wage inequality controlling for observable worker characteristics such as education and age) has been documented extensively. Autor, Katz, and Kearney (2008) show that the rise in residual inequality in the United States became concentrated in the upper part of the wage distribution while residual inequality decreased in the lower part of the wage distribution since the 1990's (i.e., the polarization of the residual wage distribution). Relatedly, young people appear to have greater difficulty in finding stable careers now than in the past. They tend to study longer for a given degree and take longer to start a career.[1] Further, young people now appear to be more stratified in starting careers than in the past, with some having good prospects for upward mobility and others having few such prospects. The rise in non-regular employment (e.g., temporary or fixed-term employment) over the years, pronounced among young workers (OECD 2002, 2014), illustrates the increasing difficulty in finding a stable career and the contrasting prospects among young workers.[2] These changes predate the Great Recession, i.e., they are a secular trend rather than a business cycle event. Moreover, these changes appear to be global (The Economist 2016).

Skill-biased technological change is among the major factors that have been identified as causes of rising wage inequality. However, Autor, Levy, and Murnane (2003) and others have argued that a more nuanced version of technological changes, in particular the automation of middle jobs, is necessary to account for the polarization of the wage

[1] The mean duration from high school graduation to receiving a BA was 4.48 years for high school graduates of 1972, and 4.81 years for high school graduates of 1992 (Bound et. al 2012). The mean duration from initial enrollment to bachelor's degree attainment was 5.7 years for students who received their first bachelor's degree in 2014-2015 (Shapiro et. al 2016).

[2] In OECD countries, temporary employment as a share of dependent employment for 15 to 24 year-olds rose from 17.5 percent in 1980 to 25.7 percent in 2019, while for 25 to 54 year-olds it rose from 7.1 percent in 1980 to 9.9 percent in 2019 (OECD Labour Market Statistics available at https://www.oecd-ilibrary.org). The 15 to 24 age group does not fit the young college graduates on which this paper focuses, but the OECD do not provide a further break-down of employment by ages.

distribution. Although this is a plausible thesis in accounting for the polarization of the overall wage distribution, it is less clear whether automation of middle jobs explains the polarization of the residual wage distribution, in particular polarization among college graduates. This paper entertains the hypothesis that technological changes that reward skills or talent can also lead to the polarization of residual inequality and the stratification of young workers' career paths.

The outline of the argument is as follows. Young workers face uncertainty in their talent for various tasks and go through a process of trying different tasks until finding one that meets a threshold (see Topel and Ward, 1992, among others). Rising rewards for skills or talent can motivate young workers to attempt more talent-demanding tasks, while at the same time penalizing young workers who have not (yet) shown those skills or talent. These changes lead to a greater variation of the outcome and, in particular, a rise in the failure rate, thus polarizing the income distribution (i.e., income gains concentrated in a diminishing fraction of high-talent workers). The rising penalty for young workers may not only prevent those workers who need current income from attempting more talent-demanding tasks, but may also force them to attempt *less* talent-demanding tasks. Therefore, the heterogeneity among young workers in terms of current resources is a source of divergent career prospects among them.

The model builds on the individual production function used in Gibbons and Waldman (1999, 2006) among others. In Gibbons and Waldman (1999), worker i's output at job j at date t is $y_{ijt} = d_j + c_j(\theta_i f(x_{it}) + \epsilon_{ijt})$, where θ_i is the innate ability, x_{it} is the labor-market experience, ϵ_{ijt} is a noise term. The job-specific constants, d_j and c_j, satisfy $d_j > d_{j'}$ and $c_j < c_{j'}$ if $j < j'$. This implies that a more able or more experienced worker is assigned to a job with a higher j. The worker's innate ability θ_i is not known and is revealed over time by observing the sequence of outputs. The authors show that the model generates the wage and promotion dynamics observed inside firms. Gibbons and Waldman (2006) incorporate the notion of task-specific human capital. The worker's output is now

$y_{i1t} = d_1 + c_1(\theta_i f(x_{i1t}) + \epsilon_{i1t})$ and $y_{i2t} = d_2 + c_2(\theta_i f(x_{i2t} + \alpha x_{i1t}) + \epsilon_{i2t})$ where $\alpha < 1$. The authors interpret $\alpha < 1$ as task-specific human capital: The experience at job 1 is not fully transferable to job 2. Consider two ex-ante identical workers, one starting at job 1 and the other starting at job 2, but both doing job 2 at a later date. The worker who started at job 1 is expected to have a lower wage than the worker who started at job 2, due to the effective loss of human capital upon promotion to job 2. The authors use this model to explain the cohort effect observed in data, i.e., the average wage at which a cohort is hired is positively correlated with the cohort's average wage many years later, even after controlling for the usual worker characteristics. If a cohort is hired in a recession with a larger than usual fraction of the cohort hired in a lower-level job, this cohort will have a lower than the usual average wage due to the larger than usual effective loss of human capital as the cohort climbs up the job ladder.

In this paper, I interpret d_j as a fixed cost and assume that it takes on a negative value. I assume power functions for d_j and c_j, which deliver a well-defined elasticity, denoted by $\tilde{\gamma}$, of $y_i \equiv \max_j\{y_{ij}\}$ as θ_i changes. Instead of a task-specific human capital, I assume a task-specific talent θ_{ij}, which naturally leads to a failure as well as a success in a task. I also assume that working on a particular task j updates $\theta_{ij'}$ only for j' close enough to j. Under these assumptions, in sections 2 and 3, I characterize the optimal career track (i.e., the optimal initial task and the associated distribution of the optimal subsequent tasks). A rising $\tilde{\gamma}$ reflecting a rising scale economy motivates workers to increase the initial task level, lowering the initial income as a fraction of the maximum obtainable. Upon subsequent talent updates, it raises the failure rate (i.e., the probability of y_{ij} falling below the reservation income) and concentrates the income gains in a diminishing fraction of workers. In section 3, I also consider the current income constraints that workers may face. If workers are initially constrained by the minimum current income, the falling optimal initial income forces constrained workers to attempt less talent-demanding tasks in sharp contrast to unconstrained workers, increasing the dispersion of career tracks within cohorts at the beginning of careers. The failure rate falls for constrained workers in contrast with unconstrained workers, and the share of constrained workers rises. Thus, a rising $\tilde{\gamma}$ forces a larger share of workers to deviate from the optimal career track, thereby dispersing career tracks among workers.

In section 4, I consider an extension in which the upper and the lower limits of j' within which talent is updated are given by the fixed ratios of $c_{j'}/c_j$ or $d_{j'}/d_j$, which implies that a rising $\tilde{\gamma}$ narrows the task range on which talent is updated. I show that the narrowing of the range can discourage a worker from attempting a talent-demanding task. Nonetheless, as $\tilde{\gamma}$ rises in combination with the shrinking task range, the failure rate of the unconstrained workers rises and middle incomes fall, as in section 3. The above results hold as long as the optimal career track is well defined, and are independent of the talent distribution function. When necessary, I assume a Pareto distribution of talent that keeps the model tractable in combination with the power functions of d_j and c_j, and derive further results. In section 5, I show that rising talent rewards without a rising scale economy do not deliver the career-based task and income changes shown in the previous sections, and I outline whether and how education can alleviate income constraints. If workers manage to increase their talent through education, this will always alleviate income constraints, but not fully if workers' optimal initial incomes are negative.

The cohort effect discussed above, or more generally the importance of the first job as a determinant of a worker's career, has been emphasized by many authors. Using a survey conducted in 1996 and 1998 for Stanford MBA graduating classes of 1960-1995, Oyer (2008) shows that starting a career in an investment banking job, instrumented by the stock market index in the graduation year, has a persistent effect of continuing to work in an investment banking job throughout the career, with a large positive effect on income. In a sample of CEOs between 1992 and 2010, Schoar and Zuo (2017) find that CEOs who started their careers in a recession year become CEOs at smaller firms with less compensation than other CEOs. A significant part of this effect comes from starting careers at smaller and private firms due to the recession. While this paper is not about the effects of recessions, these studies are suggestive of the constraining effects of the first job on subsequent jobs. Thus, constraints in the choice of the first job may have a long-lasting effect on a worker's career.

That young workers may face current-income constraints in starting careers seems plausible. Rothstein and Rouse (2011) and Coffman, et. al. (2018) provide experimental evidence that some college graduates are credit-constrained, which affects their job choices in the US. In a sample of US college students, Minicozi (2005) shows that a higher college debt is associated with a higher initial wage rate after finishing college and a lower wage growth rate over the next several years, indicating a trade-off between current and future incomes in job choices. Perlin (2011) documents the prevalent and increasing use of internships with little or no pay as a means by which young workers enter while-collar professions, shutting out young workers with limited financial resources and connections. Tervio (2009) presents a model in which a worker may enter the industry even when the initial income is below the outside option, expecting the reward when his talent turns out to be high. The limited ability of the worker to tolerate low initial wages (e.g., financial constraints, minimum wage law) can inefficiently lower the entry of young workers into the industry. A similar point is made by Rosen (1972, p338): A minimum wage can inefficiently prevent young workers from working in jobs with learning opportunities but low wages net of learning costs. Garicano and Rayo (2017) and Fudenberg and Rayo (2019) argue that many apprenticeships are long in duration and require long working hours in order for the trainee to effectively make a payment to the trainer beyond the work amount necessary for training. Thus, current-income constraints may also reflect the ability to make effective payments, including long working hours, favors reciprocated, etc., which may differ across workers.

2. A Model of Career Tracks under Talent Uncertainty

Consider a worker in a trade with vertically differentiated tasks $\{m\}$. The worker's work output depends on the task and his talent in the task as follows.

Assumption 1. A worker whose talent in task m is $\tau(m)$, produces $y = \tau(m) \cdot am^{\alpha} - bm^{\gamma}$, where $\gamma > \alpha > 0$.

The expression am^{α} is the task-specific gross output per unit of talent or talent productivity, and bm^{γ} is a task-specific cost of production. Both are assumed to have constant elasticities, α and γ, with respect to m. Let $\tilde{\gamma} \equiv \gamma/(\gamma - \alpha)$; $\tilde{\alpha} \equiv \alpha/(\gamma - \alpha)$; $\tilde{m} \equiv m^{\gamma-\alpha}(b/a)(\tilde{\gamma}/\tilde{\alpha})$; and $\tilde{\tau}(\tilde{m}) \equiv \tau((\tilde{m}(a/b)(\tilde{\alpha}/\tilde{\gamma}))^{1/(\gamma-\alpha)})$. Then, $y = (\tilde{\tau}(\tilde{m}) \cdot \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}})\Upsilon$ where $\Upsilon \equiv (\tilde{\alpha}/b)^{\tilde{\alpha}}(a/\tilde{\gamma})^{\tilde{\gamma}}$ is a scale factor. If $\tilde{\tau}(\tilde{m})$ is constant at τ for all $\tilde{m}$, the maximum y is $\tau^{\tilde{\gamma}}\Upsilon$, obtained by choosing $\tilde{m} = \tau$. We can think of $\tilde{m}$ as the task variable normalized to the talent variable. The virtue of this normalization is that any rise in a worker's $\tilde{m}$ in response to changing production function parameter values can be interpreted as the worker choosing a more talent-demanding task. I call the normalized task $\tilde{m}$ simply 'the task' when the meaning is clear. The parameter $\tilde{\gamma}$ is a measure of talent rewards. A higher $\tilde{\gamma}$ raises the elasticity of the gross output with respect to the fixed cost $(\partial(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})/(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}}))/(\partial(\tilde{\alpha}\tilde{m}^{\tilde{\gamma}})/(\tilde{\alpha}\tilde{m}^{\tilde{\gamma}})) = \tilde{\alpha}/\tilde{\gamma}$, amplifying the income advantage of a high-talent worker from choosing a high $\tilde{m}$.[3] The rising elasticity captures the notion that the information technology and globalized markets create tasks that exploit the scale economy and demand a high talent. The elasticity $\tilde{\alpha}/\tilde{\gamma}$ is also the fixed cost share of the gross

[3] The advantage of a high talent worker can be decomposed into the advantage holding the task $(\partial y/y)/(\partial\tilde{\tau}/\tilde{\tau}) = 1/(1 - (\tilde{\alpha}/\tilde{\gamma})(\tilde{m}/\tilde{\tau}))$ that rises in $\tilde{\alpha}/\tilde{\gamma}$ for $\tilde{m} \leq \tilde{\tau}$, and the advantage from raising the task level $(\partial y/y)/(\partial\tilde{m}/\tilde{m})$. The latter can be further decomposed into the advantage from raising the fixed cost $(\partial y/y)/(\partial\tilde{\alpha}\tilde{m}^{\tilde{\gamma}}/\tilde{\alpha}\tilde{m}^{\tilde{\gamma}}) = (1 - \tilde{m}/\tilde{\tau})/(\tilde{\gamma}/\tilde{\alpha} - \tilde{m}/\tilde{\tau})$ that rises in $\tilde{\alpha}/\tilde{\gamma}$ for $\tilde{m} < \tilde{\tau}$, and the elasticity of the fixed cost $\tilde{\gamma}$: $(\partial y/y)/(\partial\tilde{m}/\tilde{m}) = \tilde{\gamma}(\partial y/y)/(\partial\tilde{\alpha}\tilde{m}^{\tilde{\gamma}}/\tilde{\alpha}\tilde{m}^{\tilde{\gamma}})$.

output under a uniform talent across tasks, which gives a sense of the plausible values of $\tilde{\gamma}$. For example, if the fixed cost share is one half, $\tilde{\gamma} = 2$.

Now suppose that $\tilde{\tau}(\tilde{m})$ is uncertain with its expected value $\bar{\tau}$ common to all $\tilde{m}$. Assume that the worker's income is the expected output $E[y]$ maximized by choosing $\tilde{m}$. The solution is $\tilde{m} = \bar{\tau}$ with the income $\bar{y} \equiv \bar{\tau}^{\tilde{\gamma}}\Upsilon$. I normalize the income levels by setting $\bar{\tau} = 1$ and $\Upsilon = 1$. Therefore, all incomes are measured in units of the unconditional maximum income $\bar{y}$. Now suppose that there are two periods. After one period of working on a task, talent uncertainty across tasks may be resolved, but possibly not fully. Since only the expected talent matters for income, with a slight abuse of notation, I write $\tilde{\tau}(\tilde{m})$ as a short-hand for the expected talent $E[\tilde{\tau}(\tilde{m})]$, and call it the talent for $\tilde{m}$ dropping the word 'expected' when the meaning is clear. Likewise, y and y' will be short-hands for the incomes $E[y]$ and $E[y']$ of the first and the second periods, respectively. The initial distribution of the second-period talent $\tilde{\tau}(\tilde{m})$ for the first-period task $\tilde{m}$ is assumed to be common to all $\tilde{m}$ and given by $F(\tilde{\tau})$ with $E[\tilde{\tau}] = \bar{\tau} = 1$: Talent is updated over time without an expected growth. In the second period, the worker can try a task in another trade with the same task-talent environment, but possibly a different value of talent rewards $\tilde{\gamma}$, not affected by the events in the initial trade, so the outside option is the unconditional maximum income $\bar{y} = 1$. The worker's maximization problem is

$$\max_{\tilde{m}} \left\{ \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} + \beta E\Big[\max_{\tilde{m}'} \Big\{ \max\left\{ \tilde{\tau}(\tilde{m}') \cdot \tilde{\gamma}(\tilde{m}')^{\tilde{\alpha}} - \tilde{\alpha}(\tilde{m}')^{\tilde{\gamma}},\ 1 \right\} \Big\} \Big] \right\}$$

where the parameter β is the weight of the second-period income, reflecting a combination of factors such as the degree of future orientation, the length of the second period relative to the first period, etc. In particular, it is possible to have $\beta > 1$. A career path is a sequence of tasks $(\tilde{m}, \tilde{m}')$. A career track is an initial task $\tilde{m}$ and the associated distribution of the subsequent task $\tilde{m}'$.

2.1 Talent Relation across Tasks

Consider the case of $\tilde{\tau}(\tilde{m}') = \tilde{\tau}(\tilde{m})$ for any second-period task $\tilde{m}'$ given the first-period task $\tilde{m}$: A worker's talent update at the initial task $\tilde{m}$ applies to any subsequent task $\tilde{m}'$. In this case, the worker simply maximizes his current income by choosing $\tilde{m} = 1$ in the first period, chooses $\tilde{m}'$ equal to the updated $\tilde{\tau}(1)$ if $\tilde{\tau}(1) > 1$, and moves to another trade if $\tilde{\tau}(1) < 1$. It is useful to think of this case of the uniform talent across tasks as a benchmark. Observe that a rising $\tilde{\gamma}$ does not affect the career track (i.e., $\tilde{m}$ and the distribution of $\tilde{m}'$). Further, observe that a rising $\tilde{\gamma}$ raises the second-period top incomes ($\tilde{\tau}(1) > 1$), and keeps incomes constant at one otherwise. These changes in the income distribution can be described as a *weak form of polarization*: The ratio of the top income and the median income rises faster than the ratio of the median income and the bottom income, but the ratio of any middle income and the bottom income does not fall.

Now consider the case of $\tilde{\tau}(\tilde{m}') = 1$ for all $\tilde{m}' \neq \tilde{m}$: a worker's talent update at the first-period task $\tilde{m}$ does not affect his talent at any second-period task $\tilde{m}' \neq \tilde{m}$. In this case, given the first-period task $\tilde{m}$, the worker chooses $\tilde{m}' = \tilde{m}$ with the second-period income $y' = \tilde{\tau}(\tilde{m}) \cdot \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}}$ if $\tilde{\tau}(\tilde{m}) > \check{\tau}(\tilde{m}) \equiv (1 + \tilde{\alpha}\tilde{m}^{\tilde{\gamma}})/(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})$, and moves to another trade if $\tilde{\tau}(\tilde{m}) < \check{\tau}(\tilde{m})$. We can see that the expected second-period income rises as the first-period task $\tilde{m}$ rises from one since y' conditional on $\tilde{\tau}(\tilde{m}) > \check{\tau}(\tilde{m})$ rises with a higher $\tilde{m}$. Intuitively, raising $\tilde{m}$ lowers the talent-task gap if the talent turns out to be sufficiently high. On the other hand, as $\tilde{m}$ rises from one, the opposite force emerges: Raising $\tilde{m}$ increases the talent-task gap if the talent turns out to be above $\check{\tau}(\tilde{m})$ but below $\tilde{m}$. The third factor that affects the optimal $\tilde{m}$ is the first-period income y, which falls as $\tilde{m}$ rises from one but with the marginal fall equal to zero when $\tilde{m} = 1$. Therefore, the worker's choice of the first-period task $\tilde{m}$ is a balancing act of the three factors and the optimal $\tilde{m}$ is necessarily above one.

The left figure of Figure 1 shows the optimal task $\tilde{m}$, the optimal talent threshold for staying in the trade $\check{\tau}(\tilde{m})$, and the associated incomes y and y' for various $\tilde{\tau}(\tilde{m})$. It also shows that as $\tilde{\gamma}$ rises *holding* $\tilde{m}$, y falls and $\check{\tau}(\tilde{m})$ rises, lowering y' for $\tilde{\tau}(\tilde{m})$ above $\check{\tau}(\tilde{m})$ but below a threshold z_0 and raising y' for $\tilde{\tau}(\tilde{m})$ above z_0. Therefore, unlike the benchmark case of the uniform talent across tasks, a rising $\tilde{\gamma}$ lowers the first-period income y, can lower the second-period income y', and raises the failure rate $F(\check{\tau}(\tilde{m}))$. These changes in the income distribution can be described as a *strong form of polarization*: The ratio of a middle income and the bottom income falls. The right figure of Figure 1 shows the effect of a rising $\tilde{\gamma}$ *holding* y instead: The worker cannot tolerate the fall of the first-period income. Under this first-period income constraint, a rising $\tilde{\gamma}$ lowers $\tilde{m}$ and $\check{\tau}(\tilde{m})$, and does not lower y' for any $\tilde{\tau}(\tilde{m})$ in contrast with the effect on an unconstrained worker in the left figure.

Figure 1: The Career Track under Idiosyncratic Talent

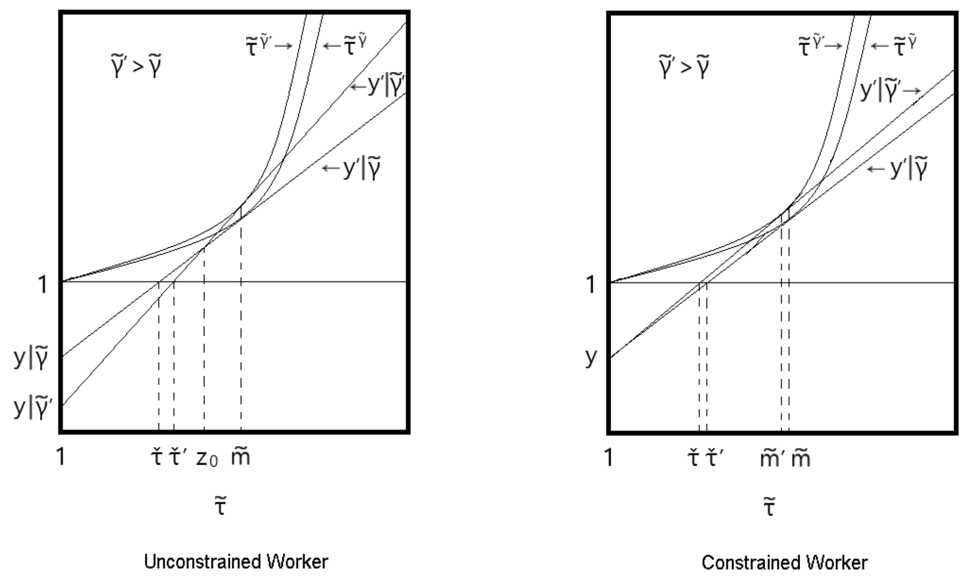


Figure 1 provides the intuition for how rising talent rewards affect the career track and incomes. It is of course unsatisfactory in arbitrarily holding $\tilde{m}$ for the unconstrained worker as $\tilde{\gamma}$ rises. It is also unsatisfactory in assuming that the talent update for a task attempted is entirely idiosyncratic and does not update talents for the other tasks. The remainder of the modeling exercise addresses these two points by relaxing the assumption of idiosyncratic talent and characterizing the optimal task choice. For the analytical tractability, I somewhat crudely incorporate the plausible notion that $\tilde{\tau}(\tilde{m}')$ is closer to $\tilde{\tau}(\tilde{m})$ for $\tilde{m}'$ that is closer to $\tilde{m}$ by the following assumption.

Assumption 2. Given the first-period task $\tilde{m}$, $\tilde{\tau}(\tilde{m}') = \tilde{\tau}(\tilde{m})$ if $\eta_l \tilde{m} \leq \tilde{m}' \leq \eta_h \tilde{m}$, and $\tilde{\tau}(\tilde{m}') = 1$ otherwise, where $\eta_l \leq 1 \leq \eta_h$. Further, $\eta_h < \inf\{\tau | F(\tau) = 1\}$.

The task range parameters η_l and η_h determine the task range on which talents are updated. The uniform-talent case of $\tilde{\tau}(\tilde{m}') = \tilde{\tau}(\tilde{m})$ for all $\tilde{m}'$ is obtained by setting $\eta_l = 0$ and $\eta_h = \infty$. The idiosyncratic-talent case of $\tilde{\tau}(\tilde{m}') = 1$ for all $\tilde{m}' \neq \tilde{m}$ is obtained by setting $\eta_l = \eta_h = 1$. The condition $\eta_h < \inf\{\tau | F(\tau) = 1\}$ ensures that the task-range constraint matters in the worker's task choice.

The standard learning models of career assume that a worker's talent is specific to, and constant across, a set of vertically ordered tasks (e.g., a firm, an occupation), which is equivalent to setting $\tilde{\tau}(\tilde{m}') = \tilde{\tau}(\tilde{m})$ for all $\tilde{m}'$ in this model. Non-trivial career paths are obtained by a gradual discovery of the talent, a task-specific human capital, a task-specific speed of learning, and other enrichments of the model (e.g., Gibbons and Waldman 1999, 2006; Antonovics and Golan 2012; Groes, Kircher, and Manovskii 2015; Pastorino 2015). In this paper, I effectively assume that talent updates are not only specific to a set of vertically ordered tasks, but also specific to a segment of the vertically ordered tasks around the attempted task. This assumption seems parsimonious and intuitive: A worker's talent at a low level of a hierarchy may give limited information about his talent at the top level, and vice-versa. The qualitative symmetry of upward and downward informational flows is distinct from the asymmetry of talent requirements: The minimum talent necessary to earn the reservation income rises as $\tilde{m}$ rises from one.

The task range parameters η_l and η_h may be affected by the production technology. In particular, a key property of a task is talent productivity $\tilde{\gamma}\tilde{m}^{\tilde{\alpha}}$. The ratio of talent productivities of any two tasks is a plausible measure of the technological difference between the two tasks. We can then consider $\tilde{\eta}_l \equiv \eta_l^{\tilde{\alpha}}$ and $\tilde{\eta}_h \equiv \eta_h^{\tilde{\alpha}}$ as the fundamental parameters of talent relation across tasks. An alternative measure is the ratio of the production costs $\tilde{\alpha}\tilde{m}^{\tilde{\gamma}}$ or equivalently the ratio of incomes $\tilde{m}^{\tilde{\gamma}}$ under the perfectly matched talent ($\tilde{\tau} = \tilde{m}$), in which case $\tilde{\eta}_l \equiv \eta_l^{\tilde{\gamma}}$ and $\tilde{\eta}_h \equiv \eta_h^{\tilde{\gamma}}$ would be the fundamental parameters. In either case, given $\tilde{\eta}_l$ and $\tilde{\eta}_h$, η_l rises and η_h falls as $\tilde{\gamma}$ rises. Intuitively, technological changes that create tasks exploiting the scale economy and demanding a high talent raise task differentiation across talent levels, reducing the task range on which a talent update is applicable. Section 4 adopts these specifications of $\tilde{\gamma}$-η relationship, and explores the implication.

2.2 The Optimal Career Track

Given Assumption 2, I write $\tilde{\tau}$ as a short-hand for the updated second-period talent $\tilde{\tau}(\tilde{m}')$ for $\tilde{m}' \in \{\eta_l\tilde{m}, \eta_h\tilde{m}\}$ when the meaning is clear. Figure 2 shows the optimal talent threshold for staying in the trade $\check{\tau}(\tilde{m})$ and the associated second-period income y' for various $\tilde{\tau}$, given the first-period task $\tilde{m} > 1$. The property of the optimal $\tilde{m} > 1$ holds for the same reason as in the case of the idiosyncratic-talent case discussed above: If $\tilde{m} \leq 1$, the worker can raise the second-period income for high enough $\tilde{\tau}$ by raising $\tilde{m}$ without lowering other incomes. The left figure, which I call *type 1 solution*, is the case of $\eta_l\tilde{m} < 1$. The worker stays in the trade if $\tilde{\tau} > \check{\tau}(\tilde{m}) = 1$, and attempts the task that is closest to his talent within the task range $[\eta_l\tilde{m}, \eta_h\tilde{m}]$: $\tilde{m}' = \tilde{\tau}$ if $\tilde{\tau} \in (1, \eta_h\tilde{m}]$ and $\tilde{m}' = \eta_h\tilde{m}$ if $\tilde{\tau} > \eta_h\tilde{m}$. The right figure, which I call *type 2 solution*, is the case of $\eta_l \tilde{m} > 1$. The worker stays in the trade if $\tilde{\tau} > \check{\tau}(\tilde{m}) \in (1, \eta_l \tilde{m})$ where

$$\check{\tau}(\tilde{m}) = \frac{1}{\tilde{\gamma}(\eta_l \tilde{m})^{\tilde{\alpha}}} + \frac{\tilde{\alpha}\eta_l \tilde{m}}{\tilde{\gamma}}, \tag{1}$$

$\tilde{m}' = \eta_l \tilde{m}$ if $\tilde{\tau} \in (\check{\tau}(\tilde{m}), \eta_l \tilde{m})$, $\tilde{m}' = \tilde{\tau}$ if $\tilde{\tau} \in [\eta_l \tilde{m}, \eta_h \tilde{m}]$, and $\tilde{m}' = \eta_h \tilde{m}$ if $\tilde{\tau} > \eta_h \tilde{m}$. Figure 3 shows $\check{\tau}(\tilde{m})$. In anticipation of the results that will come later, note that in (1) the first term is the talent level required to produce the net output equal to the outside option $y = 1$; The second term is the talent level required to produce the cost of production. A rise of $\tilde{m}$, $\tilde{\gamma}$, or η_l raises $\check{\tau}(\tilde{m})$ given $\eta_l \tilde{m} > 1$: The rise of talent required to cover the cost dominates the fall of talent required for the net output. Appendix A1 presents a proof of the optimal tasks discussed above.

Figure 2: The Optimal $\check{\tau}$ and y'

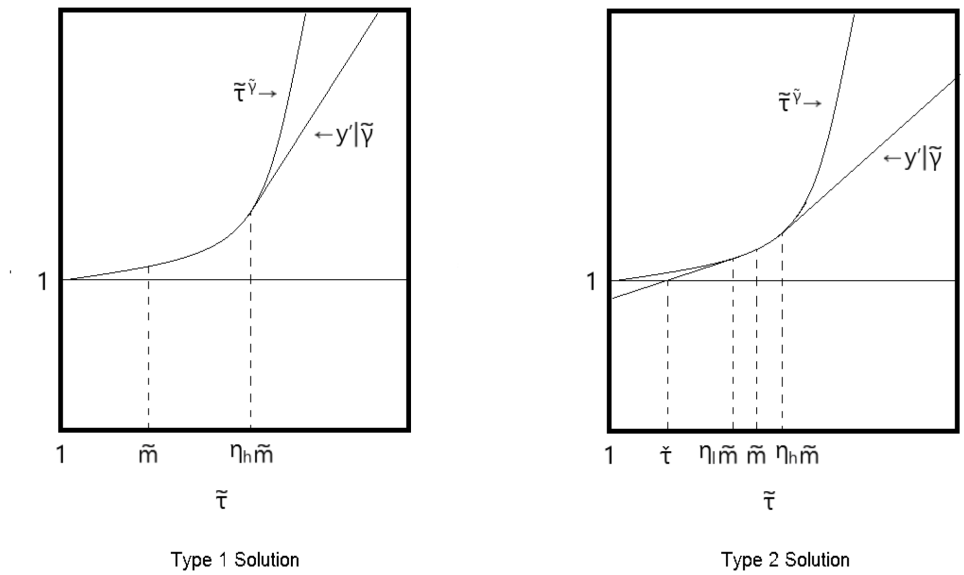
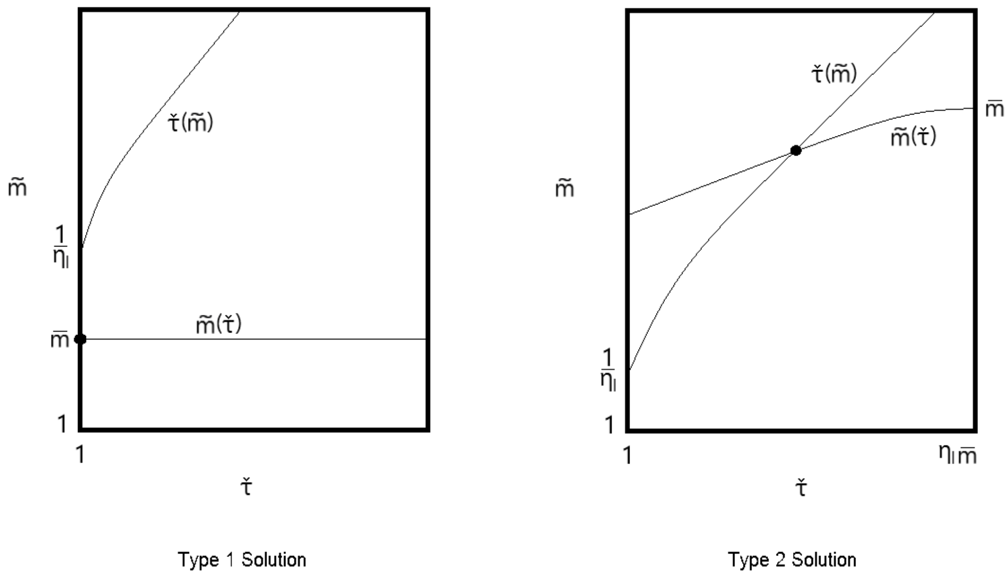


Figure 3: The Optimal $\tilde{m}$ and $\check{\tau}$



Given the above characterization of optimal tasks, the worker's maximization problem becomes

$$\max_{\tilde{m}} \left\{ \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} + \beta\Bigg(F(\check{\tau}(\tilde{m})) + \int_{\check{\tau}(\tilde{m})}^{\max\{\check{\tau}(\tilde{m}),\eta_l\tilde{m}\}} (\tilde{\tau}\tilde{\gamma}(\eta_l\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l\tilde{m})^{\tilde{\gamma}})dF(\tilde{\tau}) \right.$$
$$\left. + \int_{\max\{\check{\tau}(\tilde{m}),\eta_l\tilde{m}\}}^{\eta_h\tilde{m}} \tilde{\tau}^{\tilde{\gamma}} dF(\tilde{\tau}) + \int_{\eta_h\tilde{m}}^{\infty} (\tilde{\tau}\tilde{\gamma}(\eta_h\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_h\tilde{m})^{\tilde{\gamma}})dF(\tilde{\tau})\Bigg)\right\}.$$

This problem can be viewed as finding $\tilde{m}$ and $\check{\tau}$ with the property that given $\tilde{m}$, $\check{\tau} = \check{\tau}(\tilde{m}) = 1$ if $\eta_l\tilde{m} \leq 1$, and otherwise given by (1); and given $\check{\tau}$, $\tilde{m} = \tilde{m}(\check{\tau})$ solves the above problem where $\check{\tau}(\tilde{m})$ is replaced by $\check{\tau}$. That is, the optimal $(\tilde{m}, \check{\tau})$ is a crossing point of reaction functions $\tilde{m}(\check{\tau})$ and $\check{\tau}(\tilde{m})$.

If $\eta_l\tilde{m} < 1$, the first-order derivative of the worker's utility with respect to $\tilde{m}$ is positive iff

$$\tilde{m} - 1 < \beta\eta_h^{\tilde{\alpha}} \int_{\eta_h\tilde{m}}^{\infty} (\tilde{\tau} - \eta_h\tilde{m})dF(\tilde{\tau}). \tag{2}$$

The lefthand side is the first-period marginal cost of raising $\tilde{m}$, while the righthand side is the second-period marginal benefit of raising $\tilde{m}$, with both sides divided by a common term. Intuitively, the first-period marginal cost rises as $\tilde{m}$ moves away from the first-period utility-maximizing $\bar{\tau} = 1$. The gap $\tilde{m} - 1$ can be viewed as the degree to which talent is *overmatched* with the task in the first period. Similarly, the second-period marginal benefit falls as $\eta_h \tilde{m}$ moves closer to the updated talent $\tilde{\tau}$ when $\tilde{\tau}$ turns out to be greater than $\eta_h \tilde{m}$. The gap $\tilde{\tau} - \eta_h \tilde{m}$ can be viewed as the degree to which talent is *undermatched* with the task in the second period. The expression $\eta_h^{\tilde{\alpha}}$ on the righthand side is a sort of multiplier: As $\eta_h \tilde{m}$ rises from $\tilde{m}$ by the factor η_h holding the mismatch gap $\tilde{\tau} - \eta_h \tilde{m}$, the marginal benefit rises by the factor $\eta_h^{\tilde{\alpha}}$. This is a direct consequence of the advantage of a high talent matched with a high task level with the advantage determined by $\tilde{\gamma}$ as discussed earlier.[4]

If $\eta_l \tilde{m} > 1$, the first-order derivative of the worker's utility with respect to $\tilde{m}$ is positive iff

$$\tilde{m} - 1 + \beta \eta_l^{\tilde{\alpha}} \int_{\check{\tau}}^{\eta_l \tilde{m}} (\eta_l \tilde{m} - \tilde{\tau}) dF(\tilde{\tau}) < \beta \eta_h^{\tilde{\alpha}} \int_{\eta_h \tilde{m}}^{\infty} (\tilde{\tau} - \eta_h \tilde{m}) dF(\tilde{\tau}). \tag{3}$$

In comparison with (2), there is an additional second-period marginal cost of raising $\tilde{m}$. This second-period marginal cost occurs as $\eta_l \tilde{m}$ moves away from $\tilde{\tau}$ when $\tilde{\tau}$ turns out to be smaller than $\eta_l \tilde{m}$ but greater than $\check{\tau}$. The gap $\eta_l \tilde{m} - \tilde{\tau}$ can be viewed as the degree to which talent is overmatched with the task in the second period. The expression $\eta_l^{\tilde{\alpha}}$ on the lefthand side is again a (negative) multiplier: As $\eta_l \tilde{m}$ falls from $\tilde{m}$ by the factor η_l, the marginal cost falls by the factor $\eta_l^{\tilde{\alpha}}$. The multipliers $\eta_h^{\tilde{\alpha}}$ and $\eta_l^{\tilde{\alpha}}$ can be viewed as weights the worker attaches to the second-period undermatch and overmatch factors, $\int_{\eta_h \tilde{m}}^{\infty} (\tilde{\tau} - \eta_h \tilde{m}) dF(\tilde{\tau})$ and $\int_{\check{\tau}}^{\eta_l \tilde{m}} (\eta_l \tilde{m} - \tilde{\tau}) dF(\tilde{\tau})$, relative to the weight of the first-period undermatch factor $\tilde{m} - 1$.

Observe that given $\check{\tau}$, there is a unique $\tilde{m}(\check{\tau}) > 1$ that maximizes the worker's utility balancing the overmatch and undermatch factors in (2) and (3).[5] Suppose that (2) or (3)

[4] Condition (2) can be alternatively written as $1 - 1/\tilde{m} < \beta \eta_h^{\tilde{\gamma}} \int_{\eta_h \tilde{m}}^{\infty} (\tilde{\tau}/(\eta_h \tilde{m}) - 1) dF(\tilde{\tau})$ in which case $\eta_h^{\tilde{\gamma}}$ becomes the multiplier. Similarly, (3) can be written with multipliers $\eta_l^{\tilde{\gamma}}$ and $\eta_h^{\tilde{\gamma}}$.

[5] Conditions (2) and (3) implicitly restrict $(\tilde{m}, \check{\tau})$ to appropriate ranges: $\check{\tau} = 1$ in (2) and $1 < \check{\tau} < \eta_l \tilde{m}$ in (3). Generally, given any $\check{\tau} \geq 1$, the conditions for the positive derivative of the worker's utility with respect to $\tilde{m}$ are $\tilde{m} - 1 < \beta \eta_h^{\tilde{\alpha}} \int_{\check{\tau}}^{\infty} (\tilde{\tau} - \eta_h \tilde{m}) dF(\tilde{\tau})$ if $\check{\tau} > \eta_h \tilde{m}$; (2) if $\eta_l \tilde{m} \leq \check{\tau} \leq \eta_h \tilde{m}$; and (3) if $\check{\tau} < \eta_l \tilde{m}$.

holds with the opposite inequality when $\eta_l \tilde{m} = 1$ and $\check{\tau} = 1$:

$$\frac{1}{\eta_l} - 1 > \beta \eta_h^{\tilde{\alpha}} \int_{\eta_h/\eta_l}^{\infty} \left(\tilde{\tau} - \frac{\eta_h}{\eta_l} \right) dF(\tilde{\tau}). \quad (4)$$

Then, there is a unique type 1 solution, $\tilde{m} = \bar{m}$ and $\check{\tau} = 1$, where $\bar{m}$ solves (2) with equality. If (4) holds with the opposite inequality, $\tilde{m}(\check{\tau})$ solves (3) with equality, rising and becoming $\bar{m}$ when $\check{\tau} = \eta_l \bar{m}$. Figure 3 shows $\tilde{m}(\check{\tau})$. The two reaction functions $\tilde{m}(\check{\tau})$ and $\check{\tau}(\tilde{m})$ cross at least once, and a crossing point that delivers the maximum income is a type 2 solution. A sufficient condition for a unique crossing point is $\tilde{m}(\check{\tau})/\check{\tau}$ falling in $\check{\tau}$ and $\check{\tau}^{-1}(\check{\tau})/\check{\tau}$ rising in $\check{\tau}$. We have $\check{\tau}^{-1}(\check{\tau})/\check{\tau}$ rising in $\check{\tau}$ in (1) while $\tilde{m}(\check{\tau})/\check{\tau} > 1/\eta$ when $\check{\tau} = 1$, falling to $1/\eta_l$ when $\check{\tau} = \eta_l \bar{m}$ in (3). Whether $\tilde{m}(\check{\tau})/\check{\tau}$ falls at all $\check{\tau}$ between 1 and $\eta_l \bar{m}$ depends on the talent distribution function $F(\tilde{\tau})$. Appendix A2 shows that this property holds under the Pareto distribution of talent ($F(\tilde{\tau}) = 1 - k\tilde{\tau}^{-n}/n$ where $n > 1$). The Pareto distribution is widely used in describing the right tale of income distribution. In the remainder of the paper, I assume that there is a unique crossing point in the right figure in Figure 1, and assume the Pareto distribution for some additional results.

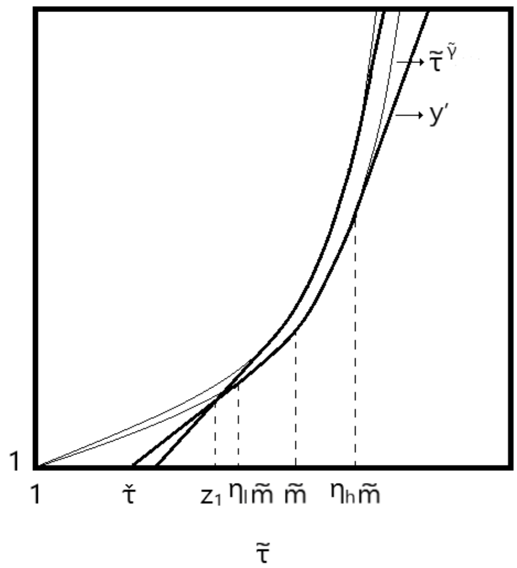
3. Rising Talent Rewards

Now we are ready to analyze how rising talent rewards (a rising $\tilde{\gamma}$) affect the career track. In (2) and (3), a higher $\tilde{\gamma}$ strengthens the multiplier effects (i.e., raises the undermatch factor weight $\eta_h^{\tilde{\alpha}}$ and lowers the overmatch factor weight $\eta_l^{\tilde{\alpha}}$), raising $\tilde{m}$: The benefit of raising $\tilde{m}$ in the case of drawing an undermatched talent ($\tilde{\tau} > \eta_h \tilde{m}$) becomes larger than the loss in the case of drawing an overmatched talent ($\tilde{\tau} \in (\check{\tau}, \eta_l \tilde{m})$). In (1), a higher $\tilde{\gamma}$ raises the optimal $\check{\tau}$ holding $\eta_l \tilde{m} > 1$: A rising $\tilde{\gamma}$ raises the fixed cost of production $\tilde{\alpha}(\eta_l \tilde{m})^{\tilde{\gamma}}$ as well as the talent productivity $\tilde{\gamma}(\eta_l \tilde{m})^{\tilde{\alpha}}$, with a negative net effect on the incomes of workers close to the talent threshold, as discussed in section 2.2. The rising $\check{\tau}$ lowers the overmatch factor $\int_{\check{\tau}}^{\eta_l \tilde{m}} (\eta_l \tilde{m} - \tilde{\tau}) dF(\tilde{\tau})$, reinforcing the rise of $\tilde{m}$ in a type 2 solution. In

Figure 3, the reaction function $\tilde{m}(\check{\tau})$ shifts up while the reaction function $\check{\tau}(\tilde{m})$ shifts to the right, raising the optimal $\tilde{m}$ in both types of solutions and raising the optimal $\check{\tau}$ in a type 2 solution. Figure 3 also shows that as $\tilde{\gamma}$ rises, a type 1 solution can transition to a type 2 solution, as can be seen in (4) as well: The talent threshold $\check{\tau}$ may stay constant at one and then begin to rise as $\tilde{\gamma}$ continues to rise.

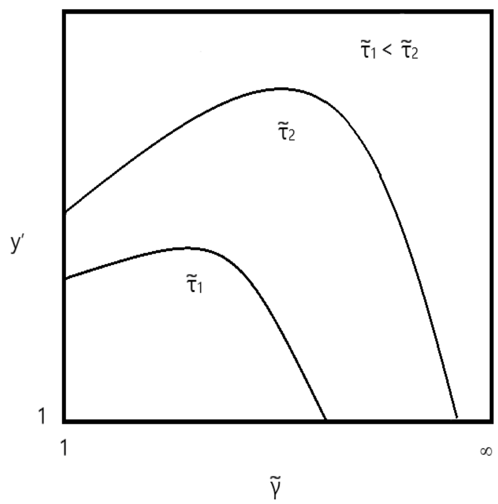
Now consider the effect of a rising $\tilde{\gamma}$ on incomes. The first-period income $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}}$ falls as $\tilde{\gamma}$ rises holding $\tilde{m} > 1$, as illustrated in the left figure of Figure 1. It also falls as $\tilde{m}$ rises holding $\tilde{\gamma}$. Therefore, a higher $\tilde{\gamma}$ lowers the first-period income by lowering the income holding $\tilde{m}$ and by raising $\tilde{m}$. Since the worker could earn $\bar{y} = 1$, the gap, $1 - y$, can be viewed as an investment for the second-period income. The above result shows that as $\tilde{\gamma}$ rises, the required investment $1 - y$ rises.

Figure 4: The Optimal y' in a Type 2 Solution



Now consider the effect of a rising $\tilde{\gamma}$ on the second-period income. Figure 4 illustrates the second-period income. The range of the second-period income y' can be broken down into four segments. The first segment ($\tilde{\tau} \leq \check{\tau}$) contains workers who exit from the trade and earn the reservation income. The second segment ($\tilde{\tau} \in (\check{\tau}, \eta_l \tilde{m})$) contains workers whose talents are overmatched with their task $\eta_l \tilde{m}$. The third segment ($\tilde{\tau} \in [\eta_l \tilde{m}, \eta_h \tilde{m}]$) contains workers whose talents are ideally matched with their tasks. The fourth segment ($\tilde{\tau} > \eta_h \tilde{m}$) contains workers whose talents are undermatched with their task $\eta_h \tilde{m}$. The second segment exists only in type 2 solutions (i.e., $1 < \check{\tau} < \eta_l \tilde{m}$). Figure 4 shows the changes of y' as $\tilde{\gamma}$ rises in a type 2 solution where the second segment is present: y' as a function of $\tilde{\tau}$ rotates on some value of $\tilde{\tau}$, denoted by z_1, on the second segment while $\check{\tau}$ and $\tilde{m}$ rise so that y' stays at one on the first segment, rises if $\tilde{\tau} > z_1$, but falls if $\tilde{\tau} \in (\check{\tau}, z_1)$. See Appendix A3 for details. The rise in top income combined with the fall in middle income implies a strong form of the polarization of the income distribution, as discussed in section 2.1. The changes in y' can be decomposed into two factors. As $\tilde{\gamma}$ rises holding $\tilde{m}$, a qualitatively same rotation occurs on a pivot $z_0 < z_1$ on the second segment as in the case of idiosyncratic talent in section 2.1. As $\tilde{m}$ rises holding $\tilde{\gamma}$, y' falls on the second segment, rises on the fourth segment, and stays constant on the other segments. Therefore, the rising $\tilde{m}$ raises z_0 to z_1 leading to a further polarization of the income distribution.

Figure 5: Second-Period Income y' as $\tilde{\gamma}$ Rises



In Figure 4, we see that as $\tilde{\gamma}$ rises, the second-period income y' rises for a diminishing fraction of workers on the right tail of the talent distribution, and falls to (or remains at) one for a rising fraction of workers on the left tail. This implies that holding $\tilde{\tau}$, y' can initially rise and eventually falls to one. Figure 5 shows the rising and then falling path of y' holding $\tilde{\tau}$ at $\tilde{\tau}_1$ or $\tilde{\tau}_2$ where $\tilde{\tau}_1 < \tilde{\tau}_2$. If the support of the talent distribution is unbounded above, we see in (1) and (3) that as $\tilde{\gamma}$ rises to ∞, the optimal $\tilde{m}$ and $\check{\tau}$ rise without a bound, lowering y' to one for any $\tilde{\tau}$ and concentrating income gains in a vanishingly small fraction of high-talent workers. In summary, we have the following proposition.

Proposition 1. As $\tilde{\gamma}$ rises, the optimal career tracks and incomes change as follows:

(a) The first-period task $\tilde{m}$ rises. The talent threshold $\check{\tau}$ rises in a type 2 solution. The first-period income falls.

(b) The second-period middle incomes ($\check{\tau} < \tilde{\tau} < z_1$) fall in a type 2 solution. The second-period top incomes rise.

(c) As $\tilde{\gamma}$ rises to ∞ holding $\tilde{\tau} > 1$, the second-period income eventually falls to one if the support of the talent distribution is unbounded above.

Proposition 1 paints the following picture of the effects of rising talent rewards. The worker attempts a more challenging task, motivated by rising top incomes that come with successful outcomes. Consequently, the income at the beginning of the career falls, the share of workers who fail in (leave) the trade rises, and the income gains are concentrated in a diminishing fraction of high-talent workers, polarizing the income distribution.

3.1 Constraints on the First-Period Income

The falling optimal first-period income y in response to a rising $\tilde{\gamma}$ indicates that any constraints on financial resources at the beginning of careers become more severe as $\tilde{\gamma}$ rises.

As a summary measure of financial resources, consider a minimum income that the worker must earn in the first-period: $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} \geq \omega$. The first-period task $\tilde{m}$ cannot exceed an upper limit given by this constraint. The parameter ω can be interpreted as a minimum consumption required minus the sum of the worker's initial wealth and the amount that the worker can borrow. The sum can be negative (e.g., student loan payments), pushing ω upward. The parameter ω may in part reflect policy and institutional constraints such as wage regulations, union wages, and wage norms. As mentioned in the introduction, ω may also reflect non-financial resources such as social networks in which favors are reciprocated, and the ability or willingness to endure long work hours as a means of paying for the fixed costs. Suppose that there are many workers who differ only in ω, and that $\omega < 1$ for all workers. The latter assumption ensures that all workers are able to satisfy the first-period income constraint by choosing $\tilde{m} = 1$ and thereby obtaining $y = 1$. Let $\bar{\omega}$ be the cutoff value of ω at which the first-period income constraint is just non-binding: $y = \bar{\omega}$ under the optimal $\tilde{m}$. Workers with $\omega \leq \bar{\omega}$ choose the optimal $\tilde{m}$ while workers with $\omega > \bar{\omega}$ are forced to choose $\tilde{m}$ lower than the optimal $\tilde{m}$.

As $\tilde{\gamma}$ rises, the optimal first-period task $\tilde{m}$ rises and the optimal first-period income y falls, lowering $\bar{\omega}$. The barely unconstrained worker (i.e., worker with $\omega = \bar{\omega}$) becomes constrained. Unconstrained workers (i.e., workers with $\omega < \bar{\omega}$) raise $\tilde{m}$ and raise $\check{\tau}$ if $\check{\tau} > 1$ (Proposition 1). On the other hand, constrained workers (i.e., workers with $\omega > \bar{\omega}$) are forced to lower $\tilde{m}$ in order to maintain y at ω, and lower $\check{\tau}$ if $\check{\tau} > 1$, as shown in the right figure of Figure 1 for the case of $\eta_l = 1$. Appendix A4 proves the falling $\check{\tau}$. Now consider the effect of a rising $\tilde{\gamma}$ on the second-period income. As $\tilde{\gamma}$ rises, an unconstrained worker's income falls for $\tilde{\tau} \in (\check{\tau}, z_1)$ if the second segment is present (i.e., $\check{\tau} > 1$). On the other hand, a constrained worker's income does not fall for any $\tilde{\tau}$. Unlike the unconstrained worker, a constrained worker's $\check{\tau}$ falls eliminating the falling part in the second segment ($\tilde{\tau} \in [\check{\tau}, \eta_l\tilde{m})$), and the income rises in the fourth segment ($\tilde{\tau} > \eta_h\tilde{m}$) despite a falling $\eta_h\tilde{m}$, as shown in the right figure of Figure 1 for the case of $\eta_l = 1$. Appendix A5 proves the rising income. In summary, we have the following proposition.

Proposition 2. As $\tilde{\gamma}$ rises, the following changes occur to constrained workers:

(a) The share of constrained workers rises.

(b) The first-period task $\tilde{m}$ of a constrained worker falls and the talent threshold $\check{\tau}$ of a constrained worker also falls if $\check{\tau} > 1$.

(c) The second-period income of a constrained worker rises for some $\tilde{\tau}$ and does not fall for any $\tilde{\tau}$.

The changes in Proposition 2 can be characterized as constrained workers becoming more cautious and less risk-taking, even though workers are assumed to be risk-neutral. In sharp contrast, unconstrained workers raise $\tilde{m}$ and raise $\check{\tau}$ if $\check{\tau} > 1$. The first-period income falls and the second-period income may fall depending on $\tilde{\tau}$. In comparison with constrained workers, these changes can be characterized as unconstrained workers becoming more aggressive and more risk-taking.

4. Rising Task Differentiation

We now consider how the career track and incomes change as the degree of task differentiation rises (a falling η_h and a rising η_l), which may accompany rising talent rewards (a rising $\tilde{\gamma}$), as discussed in section 2.1.

4.1 Rising Task Differentiation Only

Consider the effects of a falling η_h and a rising η_l, holding $\tilde{\gamma}$. A falling η_h or a rising η_l shrinks the range of tasks to which a worker can move up or down based on his talent update at a given task. In (2) and (3), the shrinking range weakens the multiplier effects (i.e., lowers the undermatch factor weight $\eta_h^{\tilde{\alpha}}$ or raises the overmatch factor weight $\eta_l^{\tilde{\alpha}}$), lowering $\tilde{m}$ in contrast with a rising $\tilde{\gamma}$ in section 3. However, mismatch factors can work in the opposite direction: a falling η_h raises the undermatch factor $\int_{\eta_h \tilde{m}}^{\infty}(\tilde{\tau} - \eta_h \tilde{m}) dF(\tilde{\tau})$, and a rising η_l raises the overmatch factor $\int_{\check{\tau}}^{\eta_l \tilde{m}}(\eta_l \tilde{m} - \tilde{\tau}) dF(\tilde{\tau})$ holding $\check{\tau}$ but lowers it by raising the talent threshold $\check{\tau}$ in (1). Appendix A6 details the conflicting forces of these changes and proves the following proposition.

Proposition 3. Assume the Pareto distribution of talent. As η_h falls, the optimal $\eta_h \tilde{m}$ falls. As η_l rises, the optimal $\eta_l \tilde{m}$ rises. As η_h falls and η_l rises simultaneously, $\eta_h \tilde{m}$ falls.

The last statement about the falling $\eta_h \tilde{m}$ is in sharp contrast with the effect of a rising $\tilde{\gamma}$. Also in contrast with a rising $\tilde{\gamma}$, a rising η_l or a falling η_h reduces the opportunity for vertical mobility ($\tilde{m}'$ rising or falling from $\tilde{m}$) and thereby has an overall negative effect on incomes. In particular, a rising η_l or a falling η_h lowers income on the top segment ($\tilde{\tau} > \eta_h \tilde{m}$) by lowering $\eta_h \tilde{m}$.

4.2 Joint Effects of Rising Talent Rewards and Rising Task Differentiation

We will now assume that the bounds of the task range are given by the fixed ratio of talent productivity ($\tilde{\eta}_l = \eta_l^{\tilde{\alpha}} < 1$ and $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}} > 1$) or the fixed ratio of the production cost ($\tilde{\eta}_l = \eta_l^{\tilde{\gamma}} < 1$ and $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}} > 1$), as discussed in section 2.1. The results below apply to either version of the bounds. Rising task differentiation associated with rising talent rewards means that holding the talents of two workers, A and B, the task optimized for worker A, call it task A, becomes more different from the task optimized for worker B, call it task B, such that for any worker, his talent for task A becomes less indicative of his talent for task B.[6] As mentioned, technological changes that create tasks exploiting the

[6] This notion of task differentiation is mechanically similar to the notion of technological distance in Violante (2002): The author assumes that the fraction of skills of a worker that can be transferred as the worker switches from an old to a new machine is determined by the ratio of productivities of the two machines.

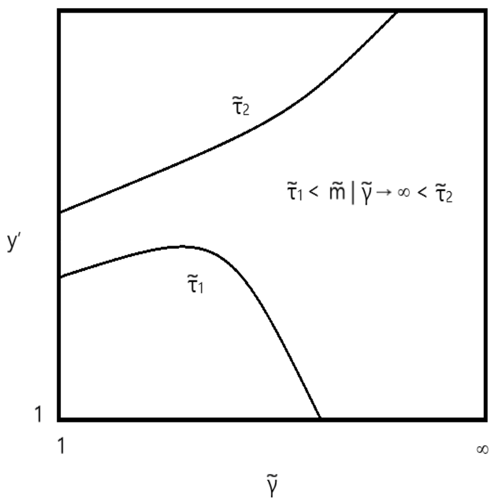
scale economy and demanding a high talent would lead to more differentiated tasks across talent levels.

In (4), there is a threshold $\bar{\gamma} > 1$ such that a type 1 solution exists if $\tilde{\gamma} < \bar{\gamma}$, and a type 2 solution exists if $\tilde{\gamma} > \bar{\gamma}$. In a type 1 solution ($\tilde{\gamma} < \bar{\gamma}$), the combined effects of rising talent rewards (i.e., a rising $\tilde{\gamma}$ holding η_h and η_l) and rising task differentiation (i.e., a falling η_h and a rising η_l holding $\tilde{\gamma}$) raise the optimal $\tilde{m}$ in (2). In a type 2 solution ($\tilde{\gamma} > \bar{\gamma}$), however, the optimal $\tilde{m}$ may fall due to a possibly rising overmatch factor $\int_{\check{\tau}}^{\eta_l \tilde{m}}(\eta_l \tilde{m} - \tilde{\tau})dF(\tilde{\tau})$ in (3). The fall of the optimal $\tilde{m}$ requires a rising η_l acting on a large enough overmatch factor. Appendix A7 details the condition. If $\tilde{\gamma} \approx \bar{\gamma}$, $\eta_l \tilde{m} \approx \check{\tau}$, so the overmatch factor is negligible and the optimal $\tilde{m}$ rises as $\tilde{\gamma}$ rises. In (1) and (3), as $\tilde{\gamma} \to \infty$, $\check{\tau} \to \eta_l \tilde{m} \to \tilde{m} \to \lim_{\tilde{\gamma}\to\infty} \bar{m}$ where $\bar{m} > \tilde{m}$ solves (2) and rises in $\tilde{\gamma}$ (see section 2.2): The overmatch factor asymptotically disappears in (3), raising $\tilde{m}$ to $\lim_{\tilde{\gamma}\to\infty} \bar{m}$. Intuitively, as $\tilde{\gamma}$ rises, the penalty for a talent overmatched with a task (i.e. $\tilde{\tau} < \eta_l \tilde{m}$) rises so that the chance of staying in the trade with overmatched talents eventually becomes negligible, which motivates workers to attempt a more talent-demanding task. Therefore, even though the optimal $\tilde{m}$ may fall at some $\tilde{\gamma} > \bar{\gamma}$, it eventually rises to a higher value as $\tilde{\gamma}$ continues to rise: The positive effect of rising talent rewards eventually dominates any negative effects of rising task differentiation.

Although the response of the optimal $\tilde{m}$ to a rising $\tilde{\gamma}$ is ambiguous in a type 2 solution, both the optimal lower task bound $\eta_l \tilde{m}$ and the optimal talent threshold $\check{\tau}$ rise robustly. In order to see this, consider (1) and (3) as relations involving $\eta_l \tilde{m}$ instead of $\tilde{m}$, setting $\tilde{m} = (1/\eta_l) \cdot \eta_l \tilde{m}$ and $\eta_h \tilde{m} = (\eta_h/\eta_l) \cdot \eta_l \tilde{m}$. We can see that rising talent rewards and rising task differentiation in combination raise $\eta_l \tilde{m}$ robustly holding $\check{\tau}$ in (3). In the right figure in Figure 3 with the vertical axis $\eta_l \tilde{m}$ instead of $\tilde{m}$, the $\eta_l \tilde{m}$ reaction function shifts up and the $\check{\tau}$ reaction function shifts to the right, which implies a rising $\eta_l \tilde{m}$ and a rising $\check{\tau}$. Therefore, even when the optimal $\tilde{m}$ falls, the optimal $\eta_l \tilde{m}$ and the optimal $\check{\tau}$ rise: The positive effect of rising talent rewards dominates a possibly negative effect of rising task differentiation.

Now consider the effects on incomes. In comparison with section 3, the shrinking task range adds an overall negative effect on incomes, and redistributes incomes across periods and talent segments. By possibly lowering the optimal $\tilde{m}$, it may raise the first-period income at the expense of the second-period income. In particular, the second-period income on the fourth segment ($\tilde{\tau} > \eta_h \tilde{m}$) may fall. Appendix A8 decomposes the income changes, and shows that top incomes rise as long as the optimal $\tilde{m}$ does not fall. Since the optimal $\tilde{m}$ eventually rises to a higher value as the optimal $\tilde{\gamma}$ continues to rise, any rise of the first-period income and any fall of top incomes are eventually reversed. Appendix A8 also shows that even when the optimal $\tilde{m}$ falls, top incomes rise if the talent is distributed by the Pareto distribution. Importantly, the effect on the first three segments ($\tilde{\tau} \leq \eta_h \tilde{m}$) of the second period income is qualitatively the same as in section 3 since $\eta_l \tilde{m}$ robustly rises, implying a falling middle income ($\check{\tau} < \tilde{\tau} < z_1$) in a type 2 solution.

Figure 6: Second-Period Income y' as $\tilde{\gamma}$ Rises with Changing $\tilde{\eta}_h$ and $\tilde{\eta}_l$



As discussed above, as $\tilde{\gamma}$ rises from $\bar{\gamma}$ to ∞, $\check{\tau}$ rises from 1 to $\lim_{\tilde{\gamma}\to\infty} \bar{m} > 1$. This implies that as $\tilde{\gamma}$ rises from (ϵ above) one to ∞ holding $\tilde{\tau}$ at a value $\tilde{\tau}_1 \in (1, \lim_{\tilde{\gamma}\to\infty} \bar{m})$, the second-period income y' initially rises but eventually falls to one. As $\tilde{\gamma}$ rises to ∞ holding $\tilde{\tau}$ at another value $\tilde{\tau}_2 > \lim_{\tilde{\gamma}\to\infty} \bar{m}$, on the other hand, y' rises without a bound. Figure 6 shows the divergent paths of y' under $\tilde{\tau}_1$ and $\tilde{\tau}_2$. As $\tilde{\gamma}$ becomes extremely large, workers are divided by the talent threshold $\lim_{\tilde{\gamma}\to\infty} \bar{m}$. Workers below this threshold all fail and obtain the reservation income while workers above this threshold succeed with their incomes rising without a bound. This result is in contrast with section 3 in which without the narrowing of the task range, the optimal $\tilde{m}$ and $\check{\tau}$ could rise without a bound, lowering the second-period income for any $\tilde{\tau} > 1$ to one (Figure 5) and concentrating income gains on a vanishingly small fraction of workers. In summary, we have the following proposition.

Proposition 4. Assume that $\tilde{\eta}_l = \eta_l^{\tilde{\alpha}} < 1$ and $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}} > 1$, or $\tilde{\eta}_l = \eta_l^{\tilde{\gamma}} < 1$ and $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}} > 1$. As $\tilde{\gamma}$ rises, the optimal career track and incomes change as follows:

(a) The task low bound $\eta_l \tilde{m}$ rises. The talent threshold $\check{\tau}$ rises in a type 2 solution. The first-period task $\tilde{m}$ may fall but eventually rises to a higher value. The first-period income may rise but eventually falls to a lower value.

(b) The second-period middle incomes ($\check{\tau} < \tilde{\tau} < z_1$) fall in a type 2 solution. The second-period top incomes may fall but eventually rise to higher values, and, if the talent is distributed by the Pareto distribution, always rise.

(c) As $\tilde{\gamma}$ rises from $\bar{\gamma}$ to ∞ holding $\tilde{\tau} \in (1, \lim_{\tilde{\gamma}\to\infty} \bar{m})$, the second-period income initially rises but eventually falls to one. As $\tilde{\gamma}$ rises to ∞ holding $\tilde{\tau} > \lim_{\tilde{\gamma}\to\infty} \bar{m}$, the second-period income rises without a bound.

Proposition 4 shows that rising task differentiation can dampen the effects of rising talent rewards, possibly demotivating the worker from attempting a challenging task. Notably, even when the worker attempts a less talent-demanding task in the first period, the worker attempts a sufficiently talent-demanding task so as to raise the low bound of the second-period task and the failure rate. Relatedly, the fall of middle incomes occurs unconditionally in a type 2 solution, with a polarized income distribution dividing workers by a talent threshold as $\tilde{\gamma}$ becomes extremely large.

4.3 Joint Effects under First-Period Income Constraints

Now consider the effects of a rising $\tilde{\gamma}$ on constrained workers. Since the changes of η_l and η_h do not affect $\tilde{m}$ for a constrained worker, a constrained $\tilde{m}$ falls as when $\tilde{\gamma}$ was rising without changing η_l and η_h in section 3.1. However, recall that with the changes in η_l and η_h, the unconstrained $\tilde{m}$ can fall too. If the unconstrained $\tilde{m}$ falls fast enough, the unconstrained first-period income y rises, lowering the share of constrained workers. The effect of a rising $\tilde{\gamma}$ on $\check{\tau}$ now includes the positive effect through a rising η_l in addition to the negative effect through a falling $\tilde{m}$. We can show that under the fixed ratio of talent productivity ($\tilde{\eta}_l = \eta_l^{\tilde{\alpha}}$), the positive effect can dominate, raising a constrained worker's $\check{\tau}$ from one as $\tilde{\gamma}$ rises. This implies that a constrained worker's second-period income may fall for a segment of $\tilde{\tau}$, a qualitatively same result as for the unconstrained worker in sections 3 and 4.2. See Appendices A9 and A10 for details.

Recall that as $\tilde{\gamma} \to \infty$, the unconstrained optimal $\tilde{m}$ rises toward $\lim_{\tilde{\gamma}\to\infty} \bar{m} > 0$. Then, the unconstrained optimal first-period income y falls without a bound. Since every worker's ω is finite, every worker becomes constrained as $\tilde{\gamma} \to \infty$ with any worker's $\tilde{m}$ falling toward one, which implies that any worker's $\check{\tau}$ falls toward one as well. Therefore, any fall in the share of constrained workers or any rise in a constrained $\check{\tau}$ and the associated income falls are reversed as $\tilde{\gamma}$ continues to rise, and the results in proposition 2 are valid in this asymptotic sense.

5. Effective Talent Dispersion and Initial Talent Upgrade

Talent rewards may rise for reasons unrelated to a rising scale economy modeled in previous sections. Consider a production function in which $\tau(m)$ is replaced by $\tau(m)^\rho$: $y = \tau(m)^\rho \cdot am^\alpha - bm^\gamma = \tilde{\tau}(\tilde{m})^\rho \cdot \tilde{\gamma}\tilde{m}^{\rho\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\rho\tilde{\gamma}}$ where $\rho > 0$ and $\tilde{m} \equiv (m^{\gamma-\alpha}(b/a)(\tilde{\gamma}/\tilde{\alpha}))^{1/\rho}$ is the task variable normalized to the talent variable: If $\tilde{\tau}(\tilde{m})$ is constant at τ for all $\tilde{m}$, the maximum y is $\tilde{\tau}^{\rho\tilde{\gamma}}$ obtained by choosing $\tilde{m} = \tau$. This normalization maintains the virtue that a worker raising $\tilde{m}$ in response to changing values of production function parameters (including ρ) indicates a climb-up over the fixed task ranks given by the talent distribution of workers $F(\tau)$. A rising ρ amplifies the ratios of the *effective* talent $\tau(m)^\rho$ and thereby the income ratios between workers. Appendix A11 shows that a rising ρ raises the optimal task levels as a rising $\tilde{\gamma}$ does. However, a rising ρ raises the convexity of effective talent, advantaging young workers with talent uncertainty (i.e., $E[\tau^\rho]/E[\tau]^\rho$ rises). Consequently, the current-income maximizing task level $E[\tau^\rho]^{1/\rho}$ rises as well, and the effects on incomes in units of the maximum current income, $y/\bar{y}$ and $y'/\bar{y}$, are ambiguous. Therefore, rising talent rewards due to a dispersion of the effective talent across workers generally motivate workers to choose more challenging tasks, but does not necessarily enhance the career-based motivation to attempt challenging tasks nor the career-based income effects.

The focus of this paper is on the career impact of rising talent rewards on workers whose initial talent expectation is identical. In the remainder, consider the impact of an initial talent upgrade for a worker: The initial talent $\bar{\tau}$ rises from 1 to a scale-up factor $\lambda > 1$, and the talent update distribution scales up from $F_{initial}$ to F_{new} where $F_{new}(\lambda\tilde{\tau}) = F_{initial}(\tilde{\tau})$ for all $\tilde{\tau}$. The talent upgrade may reflect, among other factors, education that has endowed the worker with greater talent or has screened the worker for a high talent. In the worker's problem in section 2, we see that optimal tasks scale up by the factor λ and optimal incomes scale up by the factor $\lambda^{\tilde{\gamma}}$: $\tilde{m}_{new} = \lambda\tilde{m}_{initial}$, $\tilde{m}'_{new}(\lambda\tilde{\tau}) = \lambda\tilde{m}'_{initial}(\tilde{\tau})$, $y_{new} = \lambda^{\tilde{\gamma}}y_{initial}$, and $y'_{new}(\lambda\tilde{\tau}) = \lambda^{\tilde{\gamma}}y'_{initial}(\tilde{\tau})$. The income changes do not necessarily mean a higher income: If the first-period income is negative, education amplifies the negative income. Appendix A12 derives the conditions for a negative first-period income. As noted in section 4.3, as $\tilde{\gamma}$ continues to rise, the optimal first-period income eventually becomes negative. This suggests that as talent rewards rise due to a rising scale economy, the effect of education on incomes at the beginning of careers may become weaker and even negative.

5.1 Initial Talent Upgrade under First-Period Income Constraints

The results in the previous section imply that, assuming that a worker's minimum first-period income ω is not affected by education, education turns a marginally constrained worker (i.e., a worker with ω epsilon below $\bar{\omega}$) into an unconstrained worker, lowering the share of constrained workers if the optimal first-period income is positive; it turns a marginally unconstrained worker (i.e., a worker with $\omega = \bar{\omega}$) into a constrained worker, raising the share of constrained workers if the optimal first-period income is negative. For a worker who is constrained by the minimum income ω and who has upgraded his talent by the factor λ, the first-period income is $y = \lambda\tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} = \omega$. As λ rises, y rises for each $\tilde{m}$, which allows the worker to raise $\tilde{m}$ while earning the minimum income ω. Therefore, education allows a constrained worker to attempt a more talent-demanding task regardless of whether the optimal first-period income is positive or negative. If the optimal first-period income is positive, the worker's $\tilde{m}$ becomes the optimal $\tilde{m}$ if λ rises enough: The worker can escape the minimum income constraint by receiving enough education. If the optimal first-period income is negative, the optimal $\tilde{m}$ stays above the worker's $\tilde{m}$ indefinitely as λ rises: The worker cannot escape the minimum income constraint. Since the optimal first-period income eventually becomes negative as $\tilde{\gamma}$ continues to rise, this result suggests that as talent rewards rise due to a rising scale economy, education as a means of overcoming income constraints becomes weaker.[7]

6. Discussion

The individual production function (Assumption 1) embodies the basic talent-task matching problem: Given a talent level, the income falls as the task level deviates from the one corresponding to the talent level. The task-range constraint (Assumption 2) provides reasons for career-concerned workers deviating from the current-income maximizing task. Together, these two assumptions lead to non-trivial career paths that respond to rising talent rewards. The analysis of the model delineates the relevant factors. The mismatch factors ($\int_{\eta_h \tilde{m}}^{\infty}(\tilde{\tau} - \eta_h \tilde{m})dF(\tilde{\tau})$ and $\int_{\check{\tau}}^{\eta_l \tilde{m}}(\eta_l \tilde{m} - \tilde{\tau})dF(\tilde{\tau})$) are the gaps between the career track and possible talent draws. The multipliers ($\eta_h^{\tilde{\alpha}}$ and $\eta_l^{\tilde{\alpha}}$) are the weights in balancing the mismatch factors. As talent rewards rise due to a rising scale economy (a rising $\tilde{\gamma}$), the weighted undermatch factor ($\eta_h^{\tilde{\alpha}} \int_{\eta_h \tilde{m}}^{\infty}(\tilde{\tau} - \eta_h \tilde{m})dF(\tilde{\tau})$) rises by a rising undermatch factor weight (section 3) or a rising undermatch factor (section 4.2), increasing the worker's concern for a career track adequate for high talent draws. The weighted overmatch factor ($\eta_l^{\tilde{\alpha}} \int_{\check{\tau}}^{\eta_l \tilde{m}}(\eta_l \tilde{m} - \tilde{\tau})dF(\tilde{\tau})$) falls by a falling overmatch factor (section 3), but may rise by a possibly rising overmatch factor (section 4.2), increasing the worker's concern for a career track adequate for low talent draws. Nonetheless, the lower bound of the task range $\eta_l \tilde{m}$ and the talent threshold $\check{\tau}$ robustly rise, and any increase in the overmatch factor is reversed as talent rewards continue to rise. These results amount to an analytic argument for why workers may become more aspirational in choosing a career track as talent rewards rise, for which both Assumptions 1 and 2 are essential.

[7] Any positive effects of education in alleviating income constraints is of course useful only if the worker has access to education. Workers with high current income needs may also be constrained in financing education. In addition, the effect of education on talent upgrading may differ across workers. In particular, if education screens workers for talent as opposed to upgrading talent for all workers, education can disperse career tracks of young workers by differentiating them in terms of talent at the beginning of careers.

Other results follow. Incomes of workers at the beginning of a career may fall with the subsequent income gains concentrated in a diminishing segment of high talent draws, polarizing the income distribution. On the other hand, falling optimal initial incomes may force workers facing current-income constraints to become less aspirational with moderate income gains in comparison with unconstrained workers. Therefore, rising talent rewards may disperse the career tracks of workers, acting on their diverse income constraints. Further, rising talent rewards, by lowering the initial income gains from education and training, may weaken their role as a means of overcoming income constraints.

A 'trade' in the model is a set of tasks ordered in the corresponding talent level (i.e., the talent level under which the task maximizes the worker's income), from which a career path (i.e., a sequence of tasks) is constructed. I have avoided labels such as 'industry' or 'occupation' since a career path may very well involve crossing over industries and occupations. Using Danish data from 1980 to 2002, Groes, Kircher, and Manovskii (2015) show that workers receiving high wages in an occupation tend to move to a new occupation that pays more, while workers receiving low wages in an occupation tend to move to a new occupation that pays less. Thus, a career path may involve moving over an occupation ladder. It may also involve moving across firms ordered with respect to the firm characteristics, such as the size, the productivity, the degree of trade exposure, etc. See Haltiwanger et al. (2018a, 2018b) for evidence of this. There are potentially many dimensions in which a career path can be constructed, which makes empirically identifying a career progression (as opposed to a career change) challenging.

'Talent' in the model is essentially any factors that determine the worker's productivity in a task with the property that this is uncertain initially and resolved over time, and any updates apply to the adjacent tasks as well. Uncertainty of talent may not be an important factor in manual tasks but may be an important factor in managerial tasks. The bounds of the talent-update task range limits the upward and the downward movement on the task ladder. Since the upward movement follows favorable talent updates (i.e., $\tilde{\tau} > \tilde{m}$) and the downward movement follows unfavorable talent updates (i.e., $\tilde{\tau} < \tilde{m}$), the essential assumption is the upper bound following favorable talent updates and the lower bound following unfavorable talent updates.

The limits of movement on the task ladder can be moderated in a version where there are more than two periods. Having moved up or down on the task ladder, the worker would update his talent around the new task attempted on the task ladder. This would generate a gradual climb up or down the task ladder. Appendix B presents a three-period model that allows for two movements on the task ladder, and conducts numerical exercises. As expected, the model generates a gradual rise in income for workers with positive talent updates. It also provides insights unobtainable in a two-period model. A worker who received a low-talent update may stay in the initial trade for a second chance at obtaining a high-talent update, receiving a prolonged low income (i.e., income lower than the maximum obtainable for two consecutive periods). These patterns become stronger as talent rewards rise. Consequently, the lack of mobility out of the initial trade may disguise the low current-income stayers, and a rise in the final-period income may disguise a possibly large income loss along the career path. As talent rewards rise, workers with the current-income constraint may not be able to tolerate the prolonged low income and may exit from the initial trade after a negative talent update, foregoing the upside income potential and polarizing the final-period income distribution. Generally, exiting from the initial trade reflects career concern and income constraints unlike in the two-period model, and the exit rate is decoupled from the 'failure' rate.

As discussed in section 2.1, rising talent rewards in the model stems from the exploitation of the scale economy that becomes easier with the advancement of information technology and globalization. As mentioned, a related property of the rising scale economy is that the fixed-cost share of the gross output of a worker ideally matched with a task $\tilde{\alpha}/\tilde{\gamma}$ rises as $\tilde{\gamma}$ rises. Conceptually, the fixed cost includes the costs of capital, intermediate goods, and labor inputs by coworkers that are wasted when the worker does not perform his duties. The rising fixed-cost share of the gross output seems plausible (e.g., the rising complexity of global supply chains that amplifies a managerial failure in each step of production). Given the advancement of information technology and globalization that create tasks that exploit the scale economy and demand a high talent, the associated rise of vertical task differentiation seems intuitive. However, rising task differentiation qualifies key results rather than enhances them: The model without rising task differentiation (section 3) generates a rising failure rate, a rising polarization of income distribution, and the contrasting effects of financial constraints in response to rising talent rewards more robustly than the model with rising task differentiation (section 4).

Beaudry et al. (2014, 2016) document worsening labor market outcomes for young workers since 2000 in the Unites States. Figure 13 in Beaudry et al. (2016) shows falling wages of young college workers (aged 25 to 35) and falling wages of older college workers (aged 36 to 54) earning low incomes, with their share rising over time. The authors interpret these changes as a falling demand for cognitive jobs associated with the maturity of information technology around 2000. However, falling wages of young workers and an increasingly concentrated income growth of older workers are also broadly consistent with a career-motivated supply response to rising talent rewards and penalty modeled in this paper.

The model abstracts from many realistic elements. There is no human capital accumulation. If human capital is general and rises over time for all workers regardless of career tracks, the model would change little by adding human capital. If human capital is task-specific and rises only for the tasks around the attempted task, it would reinforce the importance of the career track in determining the labor market outcome. Thus, qualitatively similar results are likely to obtain. The model assumes that there are no long-term contracts that can alleviate financial constraints. This is a common assumption in career models and reflects that the firm cannot practically prevent the worker from leaving for higher pay at a competing firm. The difficulty of enforcing a long-term contract is likely to become more severe as the optimal first-period income falls and the optimal second-period incomes become more divergent following talent updates, as can happen with rising talent rewards in the model. The model has some policy implications. If technological and other forces raise talent rewards but lower the current incomes of young workers who have not yet shown their talents, possible policy responses include providing support to young workers with limited financial resources working in talent-demanding jobs during the trial period (e.g., loans payable upon a successful career) and subsidizing education and training that help young workers to acquire and demonstrate their talents in these jobs.

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Appendix A1: Proof of the Characterization of Optimal Tasks in Section 2.2

Let $\hat{\tau}(\tilde{m}')$ denote the cutoff talent below which the second period income from trying the task $\tilde{m}'$ is lower than the outside option $\bar{y} = 1$: $\hat{\tau}(\tilde{m}') \cdot \tilde{\gamma}(\tilde{m}')^{\tilde{\alpha}} - \tilde{\alpha}(\tilde{m}')^{\tilde{\gamma}} = 1$. Some properties of $\hat{\tau}(\tilde{m}')$ are $\hat{\tau}(1) = 1$; $\hat{\tau}(\tilde{m}')$ rises as $\tilde{m}'$ moves away from 1; and $1 < \hat{\tau}(\tilde{m}') < \tilde{m}'$ if $\tilde{m} > 1$. This implies that if the updated talent for the initial task $\tilde{\tau} \leq 1$, the worker moves to another trade, assuming the tie-breaking rule that the worker moves when staying and moving deliver the same future income. If $\tilde{\tau} > 1$, there are $\hat{m}_l(\tilde{\tau}) < 1$ and $\hat{m}_h(\tilde{\tau}) > \tilde{\tau}$ such that $\tilde{\tau} > \hat{\tau}(\tilde{m}')$ iff $\hat{m}_l(\tilde{\tau}) < \tilde{m}' < \hat{m}_h(\tilde{\tau})$. Then, the worker stays in the initial trade iff the set $S \equiv \{\tilde{m}' | \, \tilde{m}' \in (\hat{m}_l(\tilde{\tau}), \hat{m}_h(\tilde{\tau})) \cap [\eta_l \tilde{m}, \eta_h \tilde{m}]\}$ is not empty. Within this set, the second-period income y' rises in $\tilde{m}'$ if $\tilde{m}' < \tilde{\tau}$, and falls in $\tilde{m}'$ if $\tilde{m}' > \tilde{\tau}$. Therefore, if $\tilde{\tau} > 1$ and S is not empty, the second-period income is maximized by choosing $\tilde{m}' = \tilde{\tau}$ if $\tilde{\tau} \in [\eta_l \tilde{m}, \eta_h \tilde{m}]$; $\tilde{m}' = \eta_l \tilde{m}$ if $\tilde{\tau} < \eta_l \tilde{m}$; and $\tilde{m}' = \eta_h \tilde{m}$ if $\tilde{\tau} > \eta_h \tilde{m}$.

If the initial task $\tilde{m} < 1$, the worker can raise the first-period income y by raising $\tilde{m}$ without lowering the second-period income y' conditional on any $\tilde{\tau}$. If $\tilde{m} = 1$, the worker can raise the second-period income y' conditional on $\tilde{\tau} > \eta_h \tilde{m}$ by raising $\tilde{m}$ without lowering the first-period income y or the second-period income y' conditional on $\tilde{\tau} \leq \eta_h \tilde{m}$. Therefore, we can set $\tilde{m} > 1$ without a loss of generality. Then, the set S is not empty iff $\eta_l \tilde{m} < \hat{m}_h(\tilde{\tau})$, which defines the talent threshold $\check{\tau}(\tilde{m})$ for the first-period task $\tilde{m}$ above which the worker stays in the initial trade. If $\eta_l \tilde{m} \leq 1$, $\check{\tau}(\tilde{m}) = 1$. If $\eta_l \tilde{m} > 1$, $\check{\tau}(\tilde{m}) \in (1, \eta_l \tilde{m})$ solves $\eta_l \tilde{m} = \hat{m}_h(\check{\tau}(\tilde{m}))$, that is, $\check{\tau}(\tilde{m}) \cdot \tilde{\gamma}(\eta_l \tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l \tilde{m})^{\tilde{\gamma}} = 1$ or (1) in the main text.

Appendix A2: Proof of a Unique Optimal $(\tilde{m}, \check{\tau})$ under the Pareto Distribution of Talent in Section 2.2

Substituting $F(\tilde{\tau}) = 1 - k\tilde{\tau}^{-n}/n$ in (2) and rearranging terms using $\bar{\tau} = n/(n-1) \cdot (k/n)^{1/n}$ $= 1$, we obtain

$$\check{\tau}^n - \beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1} - \frac{k}{n}\right)\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}} - \frac{\check{\tau}}{\eta_l\tilde{m}}\cdot\eta_l\check{\tau}^{n-1} < 0. \tag{A1}$$

Similarly, from (3), we obtain

$$\check{\tau}^n+\beta\eta_l^{\tilde{\gamma}}\frac{k}{n}+\beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1}-\frac{k}{n}\right)\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)-\frac{\check{\tau}}{\eta_l\tilde{m}}\left(\eta_l\check{\tau}^{n-1}+\beta\eta_l^{\tilde{\gamma}}\frac{k}{n-1}\right) < 0. \tag{A2}$$

Let LH and $\tilde{LH}$ denote the lefthand sides of (A1) and (A2), respectively. We can think of LH and $\tilde{LH}$ as functions of two variables, $\check{\tau}$ and $\check{\tau}/(\eta_l\tilde{m})$. We can show that in (A2), holding $\check{\tau}$, $\tilde{LH}$ is positive when $\check{\tau}/(\eta_l\tilde{m}) = 0$, and declines as $\check{\tau}/(\eta_l\tilde{m})$ rises while $\check{\tau}/(\eta_l\tilde{m}) \leq 1$. In (A1), $LH = \tilde{LH}$ when $\check{\tau}/(\eta_l\tilde{m}) = 1$, declines as $\check{\tau}/(\eta_l\tilde{m})$ rises, and becomes negative before $\check{\tau}/(\eta_l\tilde{m}) = \check{\tau}/\eta_l$. Therefore, given $\check{\tau}$, there is a unique value of $\eta_l\tilde{m}/\check{\tau}$ that solves (A1) or (A2) with equality, as expected. In (A2), holding $\check{\tau}/(\eta_l\tilde{m})$, $\tilde{LH}$ rises in $\check{\tau}$ if $1 < \check{\tau} < \eta_l\tilde{m}$. This implies that if (4) does not hold, $\check{\tau}/(\eta_l\tilde{m})$ that solves (A2) with equality rises in $\check{\tau}$ or equivalently, $\tilde{m}/\check{\tau}$ falls in $\check{\tau}$. This proves that if talent is distributed by a Pareto distribution and if (4) does not hold, there is a unique crossing point $(\tilde{m}, \check{\tau})$ that solves $\tilde{m} = \tilde{m}(\check{\tau})$ and $\check{\tau} = \check{\tau}(\tilde{m})$.

Appendix A3: The Effects of a Rising $\tilde{\gamma}$ on y' in Section 3

As $\tilde{\gamma}$ rises, y' stays at one if $\tilde{\tau} \leq \check{\tau}$ (the first-segment), and $y' = \tilde{\tau}^{\tilde{\gamma}}$ rises if $\tilde{\tau} \in [\eta_l\tilde{m}, \eta_h\tilde{m}]$ (the third segment). If $\tilde{\tau} > \eta_h\tilde{m}$ (the fourth segment), a rising $\tilde{\gamma}$ raises $y' = \tilde{\tau}\tilde{\gamma}(\eta_h\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_h\tilde{m})^{\tilde{\gamma}}$ by raising it holding $\eta_h\tilde{m}$ and by raising $\eta_h\tilde{m}$. If $\tilde{\tau} \in (\check{\tau}, \eta_l\tilde{m})$ (the second segment), $y' = \tilde{\tau}\tilde{\gamma}(\eta_l\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l\tilde{m})^{\tilde{\gamma}}$ and $\partial y'/\partial\tilde{\gamma} \geq 0$ if and only if $\tilde{\tau} \geq z_0 \equiv \eta_l\tilde{m}\cdot(1 + \tilde{\alpha}\log(\eta_l\tilde{m}))/(1 + \tilde{\gamma}\log(\eta_l\tilde{m}))$ where $z_0 \in (\check{\tau}, \eta_l\tilde{m})$. Further, $\partial y'/\partial\tilde{\gamma}$ rises in $\tilde{\tau}$; $\partial y'/\partial(\eta_l\tilde{m})$ rises in $\tilde{\tau}$ and is equal to zero when $\tilde{\tau} = \eta_l\tilde{m}$. Then, $dy'/d\tilde{\gamma} = \partial y'/\partial\tilde{\gamma} + (d(\eta_l\tilde{m})/d\tilde{\gamma})\cdot(\partial y'/\partial(\eta_l\tilde{m})) = 0$ for some $z_1 \in (z_0, \eta_l\tilde{m})$, and $dy'/d\tilde{\gamma}$ rises in $\tilde{\tau}$. Therefore, in the second segment, y' falls if $\tilde{\tau} \in (\check{\tau}, z_1)$, and rises if $\tilde{\tau} \in (z_1, \eta_l\tilde{m})$. Figure 4 visualizes the changes of y'.

Appendix A4: Proof that $d\tilde{m}/d\tilde{\gamma} < 0$ and $d\check{\tau}/d\tilde{\gamma} < 0$ if $\check{\tau} > 1$ in Proposition 2

Given $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} = \omega$, we have $dy/d\tilde{\gamma} = \partial y/\partial\tilde{\gamma} + (d\tilde{m}/d\tilde{\gamma}) \cdot (\partial y/\partial\tilde{m}) = 0$, from which we can derive

$$\frac{d\tilde{m}}{d\tilde{\gamma}} = -\frac{\tilde{m}}{\tilde{\alpha}\tilde{\gamma}} - \frac{\tilde{m}}{\tilde{\gamma}}\log\tilde{m} + \frac{1}{\tilde{\alpha}\tilde{\gamma}} \cdot \frac{\tilde{m}\log\tilde{m}}{\tilde{m}-1} < 0. \tag{A3}$$

If $\check{\tau} > 1$ or equivalently $\eta_l\tilde{m} > 1$, using (1), we can derive

$$\frac{d\check{\tau}}{d\tilde{\gamma}} = \frac{\partial\check{\tau}}{\partial\tilde{\gamma}} + \frac{d\tilde{m}}{d\tilde{\gamma}} \cdot \frac{\partial\check{\tau}}{\partial\tilde{m}} = \frac{1}{\tilde{\gamma}^2} \cdot \eta_l\tilde{m} \cdot \Omega \tag{A4}$$

where

$$\begin{aligned}\Omega &\equiv (\eta_l\tilde{m})^{-\tilde{\gamma}}\log(\eta_l\tilde{m})^{-\tilde{\gamma}} + \left(\frac{\log\tilde{m}}{\tilde{m}-1} - \tilde{\alpha}\log\tilde{m}\right)(1-(\eta_l\tilde{m})^{-\tilde{\gamma}}) \\ &\leq (\eta_l\tilde{m})^{-\tilde{\gamma}}\log(\eta_l\tilde{m})^{-\tilde{\gamma}} + \left(\frac{\log\eta_l\tilde{m}}{\eta_l\tilde{m}-1} - \tilde{\alpha}\log\eta_l\tilde{m}\right)(1-(\eta_l\tilde{m})^{-\tilde{\gamma}}) \\ &< 0.\end{aligned} \tag{A5}$$

Appendix A5: Proof that $dy'/d\tilde{\gamma} \geq 0$ at any $\tilde{\tau}$ in Proposition 2

As $\tilde{\gamma}$ rises, y' stays at one if $\tilde{\tau} < \check{\tau}$ (the first-segment), and $y' = \tilde{\tau}^{\tilde{\gamma}}$ rises if $\tilde{\tau} \in [\eta_l\tilde{m}, \eta_h\tilde{m}]$ (the third segment). If $\tilde{\tau} \in [\check{\tau}, \eta_l\tilde{m})$ (the second segment), $y' = \tilde{\tau}\tilde{\gamma}(\eta_l\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_l\tilde{m})^{\tilde{\gamma}}$ rises both when $\tilde{\tau} = \eta_l\tilde{m}$ and when $\tilde{\tau} = \check{\tau}$ given the falling $\check{\tau}$, which implies that y' rises for all $\tilde{\tau}$ between them. If $\tilde{\tau} > \eta_h\tilde{m}$ (the fourth segment), $y' = \tilde{\tau}\tilde{\gamma}(\eta_h\tilde{m})^{\tilde{\alpha}} - \tilde{\alpha}(\eta_h\tilde{m})^{\tilde{\gamma}}$ and rises when $\tilde{\tau} = \eta_h\tilde{m}$. Given $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} = \omega$, we can show

$$\frac{dy}{d\tilde{\gamma}} = \frac{d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})}{d\tilde{\gamma}} - \frac{d(\tilde{\alpha}\tilde{m}^{\tilde{\gamma}})}{d\tilde{\gamma}} = -(\tilde{m}-1)\frac{d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})}{d\tilde{\gamma}} + m^{\tilde{\gamma}}\log\tilde{m} = 0, \tag{A6}$$

which implies that $d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})/d\tilde{\gamma} > 0$. Then, $y'|_{\tilde{\tau}>\eta_h\tilde{m}} - y'|_{\tilde{\tau}=\eta_h\tilde{m}}$ rises as $\tilde{\gamma}$ rises. Therefore, y' rises for all $\tilde{\tau} > \eta_h\tilde{m}$.

Appendix A6: The Effects of the Changes of η_h and η_l in Section 4.1

First, consider how the optimal $(\tilde{m}, \check{\tau})$ changes as η_h changes. In (2) and (3), the weighted undermatch factor $\beta\eta_h^{\tilde{\alpha}}\int_{\eta_h\tilde{m}}^{\infty}(\tilde{\tau}-\eta_h\tilde{m})dF(\tilde{\tau})$ rises as η_h falls iff

$$\frac{E[\tilde{\tau}|\tilde{\tau}\geq\eta_h\tilde{m}]}{\eta_h\tilde{m}}<\frac{\tilde{\gamma}}{\tilde{\gamma}-1}. \tag{A7}$$

As discussed in section 2.2, the undermatch factor $\int_{\eta_h\tilde{m}}^{\infty}(\tilde{\tau}-\eta_h\tilde{m})dF(\tilde{\tau})$ falls in η_h holding $\tilde{m}$, while the undermatch factor weight $\eta_h^{\tilde{\alpha}}$ rises in η_h. The first effect dominates the second effect iff $\tilde{\gamma}=1+\tilde{\alpha}$ is small enough. Assuming the Pareto distribution of talent, $E[\tilde{\tau}|\tilde{\tau}\geq\eta_h\tilde{m}]/\eta_h\tilde{m}=n/(n-1)<\tilde{\gamma}/(\tilde{\gamma}-1)$ iff $\tilde{\gamma}<n$. In figure 3, the reaction function $\tilde{m}(\check{\tau})$ shifts up if $\tilde{\gamma}<n$, and shifts down if $\tilde{\gamma}>n$, while the reaction function $\check{\tau}(\tilde{m})$ stays constant. Therefore, as η_h falls, the optimal $\tilde{m}$ rises if $\tilde{\gamma}<n$, and falls if $\tilde{\gamma}>n$. The optimal $\check{\tau}$ moves in the same direction in a type 2 solution.

We can also show that $\eta_h\tilde{m}$ falls as η_h falls. This is easy to see in (2) for a type 1 solution. For a type 2 solution, assuming the Pareto distribution of talent, Appendix A6.1 shows that $\eta_h\tilde{m}$ falls as η_h falls even if $\tilde{\gamma}<n$: The direct effect of a falling η_h dominates the indirect effect of a rising $\tilde{m}$. In summary, we have the following proposition.

Proposition A1. Assume the Pareto distribution of talent. As η_h falls, the optimal $\tilde{m}$ rises if $\tilde{\gamma}<n$, and falls if $\tilde{\gamma}>n$. As η_h falls while $\check{\tau}>1$ (type 2 solution), the optimal $\check{\tau}$ rises if $\tilde{\gamma}<n$, and falls if $\tilde{\gamma}>n$. As η_h falls, the optimal $\eta_h\tilde{m}$ falls.

Now, consider how the optimal $(\tilde{m},\check{\tau})$ changes as η_l rises. In (3), both the overmatch factor $\int_{\check{\tau}}^{\eta_l\tilde{m}}(\eta_l\tilde{m}-\tilde{\tau})dF(\tilde{\tau})$ and the overmatch factor weight $\eta_l^{\tilde{\alpha}}$ rise in η_l. In (1), holding $\tilde{m}$, as η_l rises holding $\tilde{m}$, $\check{\tau}$ rises. In figure 3, the optimal $\tilde{m}$ and $\check{\tau}$ do not change in a type 1 solution while $\tilde{m}(\check{\tau})$ shifts down while $\check{\tau}(\tilde{m})$ shifts to the right in a type 2 solution, leaving the possibility of a rising $\tilde{m}$ or a falling $\check{\tau}$. Assuming the Pareto distribution of talent, we can show that the overall effect is a falling $\tilde{m}$ and a rising $\check{\tau}$. We can also show that $\eta_l\tilde{m}$ rises as η_l rises: the direct effect of rising η_l dominates any indirect effects of falling $\tilde{m}$. See Appendices A6.2 and A6.3. In summary, we have the following proposition.

Proposition A2. As η_l rises, the optimal $\tilde{m}$ and $\check{\tau}$ stay constant in a type 1 solution. Assume the Pareto distribution of talent for the following statements. As η_l rises, the optimal $\tilde{m}$ falls and the optimal $\check{\tau}$ rises in a type 2 solution. As η_l rises, the optimal $\eta_l\tilde{m}$ rises.

Proposition 3 follows from Propositions A1 and A2.

Appendix A6.1: Proof that $d(\eta_h\tilde{m})/d\eta_h > 0$ in a Type 2 Solution in Proposition A1

Rewrite (A2) with equality as follows.

$$\begin{aligned}\Phi(\eta_h, \check{\tau}, \tilde{m}) \equiv{}& 1 - \frac{1}{\tilde{m}} + \beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1} - \frac{k}{n}\right)\left(\frac{1}{\eta_l\tilde{m}}\right)^n\left(1 - \left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right) \\ &- \beta\eta_l^{\tilde{\gamma}}\frac{1}{\check{\tau}^n}\left(\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}} - \frac{k}{n}\right) \\ ={}& 0.\end{aligned} \tag{A8}$$

We have

$$\frac{\partial\Phi}{\partial\eta_h} = \beta\eta_l^{\tilde{\gamma}}k\cdot\frac{n-\tilde{\gamma}}{n(n-1)}\left(\frac{1}{\eta_l\tilde{m}}\right)^n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\frac{1}{\eta_h}; \tag{A9}$$

$$\frac{\partial\Phi}{\partial\check{\tau}} = -\beta\eta_l^{\tilde{\gamma}}k\cdot\frac{1}{\check{\tau}^{n+1}}\left(1 - \frac{\check{\tau}}{\eta_l\tilde{m}}\right); \tag{A10}$$

and

$$\frac{\partial\Phi}{\partial\tilde{m}} = \frac{1}{\tilde{m}^2} + \beta\eta_l^{\tilde{\gamma}}k\cdot\frac{1}{n-1}\cdot\frac{1}{\check{\tau}^n}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}\cdot\frac{1}{\tilde{m}}\left(1 - \left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^{n-1}\left(1 - \left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\right). \tag{A11}$$

Dividing both sides of (1) by $\eta_l\tilde{m}$ and using $\tilde{\alpha} = \tilde{\gamma} - 1$, we have

$$\tilde{\gamma}\left(1 - \frac{\check{\tau}(\tilde{m})}{\eta_l\tilde{m}}\right) = 1 - \left(\frac{1}{\eta_l\tilde{m}}\right)^{\tilde{\gamma}}. \tag{A12}$$

Using (A12), we have

$$d\check{\tau} = \frac{\tilde{\alpha}}{\tilde{\gamma}}\left(1 - \left(\frac{1}{\eta_l\tilde{m}}\right)^{\tilde{\gamma}}\right)d(\eta_l\tilde{m}) = (\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l\tilde{m}}\right)\eta_l d\tilde{m}. \tag{A13}$$

Combining (A8), (A9), (A10), (A11), and (A13), we have

$$d\Phi = d\check{\tau}\frac{\partial\Phi}{\partial\check{\tau}} + d\eta_h\frac{\partial\Phi}{\partial\eta_h} + d\tilde{m}\frac{\partial\Phi}{\partial\tilde{m}} = d\eta_h\frac{\partial\Phi}{\partial\eta_h} + d\tilde{m}\left(\frac{\beta\eta_l^{\tilde{\gamma}}k}{\check{\tau}^n\tilde{m}}\Psi_{\tilde{m}} + \frac{1}{\tilde{m}^2}\right) = 0 \tag{A14}$$

where

$$\Psi_{\tilde{m}} = -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^2\frac{\eta_l\tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}\left(1-\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^{n-1}\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\right). \tag{A15}$$

Now, we have $\check{\tau}/\eta_l\tilde{m} > (\tilde{\gamma}-1)/\tilde{\gamma}$ from (A12). In (A2) with equality, noting the two terms without the factor $\beta\eta_l^{\tilde{\gamma}}$ on the lefthand side sum to a positive value, we have

$$\frac{\check{\tau}}{\eta_l\tilde{m}} > \frac{n-1}{n} + \frac{1}{n}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right). \tag{A16}$$

Then, if $\tilde{\gamma} < n$, $\check{\tau}/\eta_l\tilde{m} > (n-1)/n > (\tilde{\gamma}-1)/\tilde{\gamma}$. If $\tilde{\gamma} > n$, $\check{\tau}/\eta_l\tilde{m} > (\tilde{\gamma}-1)/\tilde{\gamma} > (n-1)/n$.

Using (A15) and (A16), we can show

$$\Psi_{\tilde{m}} > -(\tilde{\gamma}-1)\left(\frac{\eta_l\tilde{m}}{\check{\tau}}-1\right) + \tilde{\gamma}\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right) > 0 \tag{A17}$$

for all $\check{\tau}/(\eta_l\tilde{m}) \in ((\tilde{\gamma}-1)/\tilde{\gamma}, 1)$. Then, from (A14), $d(\eta_h\tilde{m})/d\eta_h = \tilde{m} + \eta_h d\tilde{m}/d\eta_h > 0$ iff

$$\frac{d\tilde{m}}{d\eta_h} = -\frac{\partial\Phi/\partial\eta_h}{\beta\eta_l^{\tilde{\gamma}}k/(\check{\tau}^n\tilde{m})\cdot\Psi_{\tilde{m}} + 1/\tilde{m}^2} > -\frac{\tilde{m}}{\eta_h}. \tag{A18}$$

The inequality in (A18) holds if

$$\Lambda \equiv \Psi_{\tilde{m}} - \frac{n-\tilde{\gamma}}{n(n-1)}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}} > 0. \tag{A19}$$

We have $\Lambda > 0$ if $\tilde{\gamma} \geq n$. If $\tilde{\gamma} < n$, consulting (A15) and (A16), we have

$$\begin{aligned}\Lambda =& -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^2\frac{\eta_l\tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}\left(1-\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^{n-1}\left(1-\frac{\tilde{\gamma}}{n}\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\right)\\ >& -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^2\frac{\eta_l\tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}} - \frac{1}{n-1}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\\ &+\frac{\tilde{\gamma}}{n}\left(1+\frac{1}{n-1}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n - \frac{n}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}\right)\\ \equiv&\ \tilde{\Lambda}.\end{aligned} \tag{A20}$$

Consider $\tilde{\Lambda}$ as a function of $\check{\tau}/(\eta_l\tilde{m})$. We can show that $\tilde{\Lambda} > 0$ when $\check{\tau}/(\eta_l\tilde{m}) = (n-1)/n$; $\tilde{\Lambda} = 0$ when $\check{\tau}/(\eta_l\tilde{m}) = 1$; and $\partial\tilde{\Lambda}/\partial(\check{\tau}/\eta_l\tilde{m})$ falls as $\check{\tau}/(\eta_l\tilde{m})$ rises. These properties of $\tilde{\Lambda}$ imply that $\tilde{\Lambda} > 0$ for all $\check{\tau}/(\eta_l\tilde{m}) \in ((n-1)/n, 1)$, which in turn implies that $d(\eta_h\tilde{m})/d\eta_h > 0$.

Appendix A6.2: Proof that $d\tilde{m}/d\eta_l < 0$ in a Type 2 solution in Proposition A2

Consider Φ in (A8) as a function of η_l, $\check{\tau}$, and $\tilde{m}$: $\Phi = \Phi(\eta_l, \check{\tau}, \tilde{m}) = 0$. Then, (A10) and (A11) are valid;

$$\frac{\partial\Phi}{\partial\eta_l} = -\beta\eta_l^{\tilde{\alpha}}k(n-\tilde{\gamma})\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(\frac{1}{\eta_l\tilde{m}}\right)^n + \beta\eta_l^{\tilde{\alpha}}k \cdot \frac{1}{\check{\tau}^n}\left(\frac{\tilde{\gamma}}{n} - \frac{\tilde{\gamma}-1}{n-1} \cdot \frac{\check{\tau}}{\eta_l\tilde{m}}\right); \quad \text{(A21)}$$

and (A13) becomes

$$d\check{\tau} = \frac{\tilde{\alpha}}{\tilde{\gamma}}\left(1 - \left(\frac{1}{\eta_l\tilde{m}}\right)^{\tilde{\gamma}}\right)d(\eta_l\tilde{m}) = (\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l\tilde{m}}\right)(\tilde{m}d\eta_l + \eta_l d\tilde{m}). \quad \text{(A22)}$$

Combining (A10), (A11), (A21), (A22), and $\Phi(\eta_l, \check{\tau}, \tilde{m}) = 0$, we have

$$d\Phi = d\check{\tau} \cdot \frac{\partial\Phi}{\partial\check{\tau}} + d\eta_l \cdot \frac{\partial\Phi}{\partial\eta_l} + d\tilde{m} \cdot \frac{\partial\Phi}{\partial\tilde{m}} = d\eta_l \cdot \frac{\beta\eta_l^{\tilde{\alpha}}k}{\check{\tau}^n} \cdot \Psi_{\eta_l} + d\tilde{m}\left(\frac{\beta\eta_l^{\tilde{\gamma}}k}{\check{\tau}^n\tilde{m}} \cdot \Psi_{\tilde{m}} + \frac{1}{\tilde{m}^2}\right) = 0 \quad \text{(A23)}$$

where

$$\begin{aligned}\Psi_{\eta_l} &= -(\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l\tilde{m}}\right)^2\frac{\eta_l\tilde{m}}{\check{\tau}} - (n-\tilde{\gamma})\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n + \frac{\tilde{\gamma}}{n} - \frac{\tilde{\gamma}-1}{n-1} \cdot \frac{\check{\tau}}{\eta_l\tilde{m}} \\ &= (\tilde{\gamma}-1)\left(1 - \frac{\check{\tau}}{\eta_l\tilde{m}}\right)\left(\frac{n}{n-1} - \frac{\eta_l\tilde{m}}{\check{\tau}}\right) + (n-\tilde{\gamma})\left(\frac{1}{n-1} - \frac{1}{n}\right)\left(1 - \left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\right)\end{aligned} \quad \text{(A24)}$$

and $\Psi_{\tilde{m}}$ is as in (A15). Since $\eta_l\tilde{m}/\check{\tau} < n/(n-1)$ (see Appendix A6.1), $\Psi_{\eta_l} > 0$ given $\check{\tau}/(\eta_l\tilde{m}) < 1$ if $\tilde{\gamma} \leq n$. If $\tilde{\gamma} > n$, using $\eta_l\tilde{m}/\check{\tau} < \tilde{\gamma}/(\tilde{\gamma}-1)$, we have

$$\Psi_{\eta_l} > \frac{\tilde{\gamma}-n}{n-1}\left(1 - \frac{\check{\tau}}{\eta_l\tilde{m}} - \frac{1}{n}\left(1 - \left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\right)\right) > 0 \quad \text{(A25)}$$

given $\check{\tau}/\eta_l\tilde{m} < 1$. Using (A17) and (A23), we have $d\tilde{m}/d\eta_l < 0$.

Appendix A6.3: Proof that $d\check{\tau}/d\eta_l > 0$ and $d(\eta_l\tilde{m})/d\eta_l > 0$ in a Type 2 Solution in Proposition A2

Rewrite Φ in (A8) as a function of η_l, $\check{\tau}$, and $\eta_l\tilde{m}$.

$$\begin{aligned}\Phi(\eta_l,\check{\tau},\eta_l\tilde{m}) &\equiv 1-\frac{\eta_l}{\eta_l\tilde{m}}+\beta\eta_l^{\tilde{\gamma}}\left(\frac{k}{n-1}-\frac{k}{n}\right)\left(\frac{1}{\eta_l\tilde{m}}\right)^n\left(1-\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)\\ &\quad-\beta\eta_l^{\tilde{\gamma}}\frac{1}{\check{\tau}^n}\left(\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}-\frac{k}{n}\right)\\ &=0.\end{aligned}\tag{A26}$$

We can see that when (A26) holds, Φ becomes negative as η_l rises if $\tilde{\gamma}\le n$, since the sum of the last two terms is negative. If $\tilde{\gamma}>n$, we have

$$\begin{aligned}\frac{\partial\Phi}{\partial\eta_l} &= -\frac{1}{\eta_l\tilde{m}}+\frac{\beta\eta_l^{\tilde{\alpha}}}{\check{\tau}^n}\left(\frac{k}{n(n-1)}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n\left(\tilde{\gamma}-n\left(\frac{\eta_l}{\eta_h}\right)^{n-\tilde{\gamma}}\right)-\tilde{\gamma}\left(\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}-\frac{k}{n}\right)\right)\\ &< -\frac{1}{\eta_l\tilde{m}}+\frac{\beta\eta_l^{\tilde{\alpha}}}{\check{\tau}^n}\left(\frac{k}{n(n-1)}\left(\frac{\check{\tau}}{\eta_l\tilde{m}}\right)^n(\tilde{\gamma}-n)-\frac{k}{n(n-1)}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}(\tilde{\gamma}-n)\right.\\ &\qquad\left.-\frac{k}{n-1}\cdot\frac{\check{\tau}}{\eta_l\tilde{m}}+\tilde{\gamma}\frac{k}{n}\left(1-\frac{\check{\tau}}{\eta_l\tilde{m}}\right)\right)\\ &< 0,\end{aligned}\tag{A27}$$

where the first inequality uses $(\eta_l/\eta_h)^{n-\tilde{\gamma}}>1$, and the second inequality uses $\check{\tau}/\eta_l\tilde{m}>(n-1)/n$ and $\check{\tau}/\eta_l\tilde{m}>(\tilde{\gamma}-1)/\tilde{\gamma}$ (see Appendix A6.1). Therefore, when (A26) holds, Φ becomes negative as η_l rises for all $\tilde{\gamma}>1$. We can also see that Φ rises in $\eta_l\tilde{m}$ given $\check{\tau}/(\eta_l\tilde{m})<1$. Then, holding $\check{\tau}$, $\eta_l\tilde{m}$ rises as η_l rises in order for (A26) to hold. Now consider (A26) as defining the optimal $\eta_l\tilde{m}$ as a function of $\check{\tau}$, and consider (1) as defining the optimal $\check{\tau}$ as a function of $\eta_l\tilde{m}$. In the right figure in Figure 3 with the vertical axis $\eta_l\tilde{m}$ instead of $\tilde{m}$, the effect of a rising η_l is to shift up the $\eta_l\tilde{m}$ reaction function around the crossing point raising both the optimal $\check{\tau}$ and the optimal $\eta_l\tilde{m}$. Therefore, we have $d\check{\tau}/d\eta_l>0$ and $d(\eta_l\tilde{m})/d\eta_l>0$.

Appendix A7: The Condition for $d\tilde{m}/d\tilde{\gamma} > 0$ in a Type 2 Solution in Section 4.2

If $\tilde{\eta}_l = \eta_l^{\tilde{\alpha}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}}$, differentiating (3) with respect to $\tilde{\gamma}$, we can see that $\tilde{m}(\check{\tau})$ shifts up iff

$$\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}(-\log \tilde{\eta}_l)\big(F(\eta_l \tilde{m}) - F(\check{\tau})\big) < \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}(\log \tilde{\eta}_h)\big(1 - F(\eta_h \tilde{m})\big). \tag{A28}$$

Intuitively, the optimal $\tilde{m}$ rises iff the fraction of workers overmatched with tasks ($F(\eta_l \tilde{m})$ $- F(\check{\tau})$)) who lose by raising $\tilde{m}$ is small relative to the fraction of workers undermatched with tasks $(1 - F(\eta_h \tilde{m}))$ who gain by raising $\tilde{m}$. Condition (A28) is sufficient for both the optimal $\tilde{m}$ and the optimal $\check{\tau}$ to rise as $\tilde{\gamma}$ rises in a type 2 solution. As discussed in the main text, if $\tilde{\gamma} = \bar{\gamma}$, $\check{\tau} = \eta_l \tilde{m}$, so (A28) holds; As $\tilde{\gamma} \to \infty$, (A28) holds as well since $\check{\tau} \to \eta_l \tilde{m} \to \tilde{m} \to \lim_{\tilde{\gamma}\to\infty} \bar{m}$. However, (A28) may not hold for some values of $\tilde{\gamma}$. In Appendix A7.1, assuming the Pareto distribution of talent, I show that $d\tilde{m}/d\tilde{\gamma} > 0$ iff

$$-\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \frac{n\tilde{\alpha}}{\check{\tau}} \left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right) \frac{\partial \check{\tau}}{\partial \tilde{\gamma}} - \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \left(\log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right) \left(1 - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right)^n\right) - \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \left(\log \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right) \left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right)^n < 0. \tag{A29}$$

where $\partial \check{\tau}/\partial \tilde{\gamma} > 0$ is given by (A31). If $\tilde{\eta}_l = \eta_l^{\tilde{\gamma}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}} > 1$, alternative versions of (A28) and (A29) can be derived by following the same steps, with the same interpretations.

In order to assess the plausibility of (A29), consider the case of symmetric talent relation (i.e., $\tilde{\eta}_h \tilde{\eta}_l = 1$). Given $\tilde{\gamma}$, n, and $\tilde{\eta}_l$, we can find the range of $\check{\tau}/(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m})$ under which (A29) does not hold subject to the lower bound of $\check{\tau}/(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m})$ given by (A12) and (A16), and use (1) and (A2) to compute the value of the remaining parameter β corresponding to each value of $\check{\tau}/(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m})$ in the range. For example, consider $n = 2$, which implies that the talent Gini is $1/(2n - 1) = 1/3$, which also approximates the income Gini on the right tale of the second-period income distribution, and $\tilde{\gamma} = 2$, which implies that the fixed cost share of the gross output under a uniform talent across tasks is $\tilde{\alpha}/\tilde{\gamma} = 1/2$, as discussed in section 2. With $n = 2$ and $\tilde{\gamma} = 2$, there are no values of $\tilde{\eta}_l$ and β under which (A29) does not hold. With $n = 2$ and $\tilde{\gamma} = 1.5$, the lowest value of β under which (A29) does not hold, is about 18 and requires $\tilde{\eta}_l \approx 0.8$. With $n = 2$ and $\tilde{\gamma} = 1.1$, the lowest value of β under which (A29) does not hold, is about 4.6 and requires $\tilde{\eta}_l \approx 0.99$. In order for (A29) not to hold, a very high value of β (i.e., a very large weight of the second-period income) or a very low value of $\tilde{\gamma}$ (i.e., a very low fixed cost as a fraction of the gross output) coupled with a very high value of $\tilde{\eta}_l$ (i.e., a very small range of tasks across which talent is transferable) is needed.

Appendix A7.1: Proof of Condition (A29)

Given $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$, (1) becomes

$$\check{\tau}(\tilde{m}) = \frac{\tilde{\alpha}}{\tilde{\gamma}} \cdot \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m} + \frac{1}{\tilde{\gamma}} \cdot \frac{1}{\tilde{\eta}_l \tilde{m}^{\tilde{\alpha}}}. \tag{A30}$$

We have

$$\frac{\partial \check{\tau}}{\partial \tilde{\gamma}} = \frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}{\tilde{\gamma}^2} \left(1 - \left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right)^{\tilde{\gamma}} \left(1 - \tilde{\gamma} \log\left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right) + \tilde{\gamma}\left(\left(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}\right)^{\tilde{\gamma}} - 1\right) \log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\right) > 0 \tag{A31}$$

for all $\eta_l \tilde{m} > 1$. Adapting (A13), we have

$$d\check{\tau} = d\tilde{\gamma} \frac{\partial \check{\tau}}{\partial \tilde{\gamma}} + d\tilde{m} \frac{\partial \check{\tau}}{\partial \tilde{m}} = d\tilde{\gamma} \frac{\partial \check{\tau}}{\partial \tilde{\gamma}} + d\tilde{m}(\tilde{\gamma} - 1)\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right) \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}. \tag{A32}$$

Given $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} = 1/\eta_h$, rewrite Φ in (A8) as a function of $\tilde{\gamma}$, $\check{\tau}$, and $\tilde{m}$.

$$\begin{aligned} \Phi(\tilde{\gamma}, \check{\tau}, \tilde{m}) \equiv\ & 1 - \frac{1}{\tilde{m}} + \beta \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \left(\frac{k}{n-1} - \frac{k}{n}\right) \left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}}\right)^n \left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right) \\ & - \beta \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} \frac{1}{\check{\tau}^n} \left(\frac{k}{n-1} \cdot \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} \tilde{m}} - \frac{k}{n}\right) \\ =\ & 0. \end{aligned} \tag{A33}$$

Adapting (A14), we have

$$d\Phi = d\check{\tau} \frac{\partial \Phi}{\partial \check{\tau}} + d\tilde{\gamma} \frac{\partial \Phi}{\partial \tilde{\gamma}} + d\tilde{m} \frac{\partial \Phi}{\partial \tilde{m}} = d\tilde{\gamma} \frac{\beta k}{\check{\tau}^n n \tilde{\alpha}} \cdot \Psi_{\tilde{\gamma}} + d\tilde{m}\left(\frac{\beta \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} k}{\check{\tau}^n \tilde{m}} \cdot \tilde{\Psi}_{\tilde{m}} + \frac{1}{\tilde{m}^2}\right) = 0 \tag{A34}$$

where $\partial\Phi/\partial\check{\tau}$ and $\partial\Phi/\partial\tilde{m}$ are as in (A10) and (A11);

$$\frac{\partial\Phi}{\partial\tilde{\gamma}} = \frac{\beta k}{\check{\tau}^n n\tilde{\alpha}}\left(\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n - 1\right) - \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right); \quad \text{(A35)}$$

(A15) becomes

$$\tilde{\Psi}_{\tilde{m}} = -(\tilde{\gamma}-1)\left(1-\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^2\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}} + \frac{1}{n-1}\cdot\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\left(1-\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^{n-1}\left(1-\left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right); \quad \text{(A36)}$$

and

$$\Psi_{\tilde{\gamma}} = -\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\frac{n\tilde{\alpha}}{\check{\tau}}\left(1-\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\frac{\partial\check{\tau}}{\partial\tilde{\gamma}} - \tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(1-\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right) - \tilde{\eta}_h^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n. \quad \text{(A37)}$$

Since $\tilde{\Psi}_{\tilde{m}} > 0$, $d\tilde{m}/d\tilde{\gamma} > 0$ in (A34) iff $\Psi_{\tilde{\gamma}} < 0$.

Appendix A8: The Effects of a Rising $\tilde{\gamma}$ on y and y' in Section 4.2

As in section 3, the effect of a rising $\tilde{\gamma}$ on the first period income y is composed of the effect holding $\tilde{m}$, which lowers y, and the effect via $\tilde{m}$, which lowers y iff the optimal $\tilde{m}$ rises. Therefore, as $\tilde{\gamma}$ rises, y falls if $\tilde{m}$ does not fall, but may rise if $\tilde{m}$ falls. As $\tilde{\gamma}$ rises, the second-period income y' stays at one if $\tilde{\tau} < \check{\tau}$ (the first-segment), and $y' = \tilde{\tau}^{\tilde{\gamma}}$ rises if $\tilde{\tau} \in [\eta_l\tilde{m}, \eta_h\tilde{m}]$ (the third segment). If $\tilde{\tau} \in [\check{\tau}, \eta_l\tilde{m})$ (the second segment), given $d(\eta_l\tilde{m})/d\tilde{\gamma} > 0$, we can repeat the steps in Appendix A3 to show that there is $z_1 \in (z_0, \eta_l\tilde{m})$ such that $dy'/d\tilde{\gamma} < 0$ if $\tilde{\tau} \in (\check{\tau}, z_1)$ and $dy'/d\tilde{\gamma} > 0$ if $\tilde{\tau} \in (z_1, \eta_l\tilde{m})$. If $\tilde{\tau} > \eta_h\tilde{m}$ (the fourth segment) and if $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}}$, $y' = \tilde{\tau}\cdot\tilde{\gamma}\tilde{\eta}_h\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{\eta}_h^{\tilde{\gamma}/\tilde{\alpha}}\tilde{m}^{\tilde{\gamma}}$ and

$$\begin{aligned}\frac{\partial y'}{\partial\tilde{\gamma}} &= \tilde{\eta}_h\tilde{m}^{\tilde{\alpha}}\left(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m} + (\tilde{\gamma}\tilde{\tau} - \tilde{\alpha}\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\log(\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}) - \tilde{\gamma}(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}})(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\right)\\ &= \tilde{\eta}_h\tilde{m}^{\tilde{\alpha}}\left(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m} + \tilde{\gamma}(\tilde{\tau} - \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\log\tilde{m} + \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}\log(\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m})\right) \quad \text{(A38)}\\ &> 0.\end{aligned}$$

If $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}}$, $y' = \tilde{\tau}\cdot\tilde{\gamma}^{\frac{\tilde{\alpha}}{\tilde{\gamma}}}\tilde{\eta}_h\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{\eta}_h\tilde{m}^{\tilde{\gamma}}$ and $\partial y'/\partial\tilde{\gamma} > 0$ holds by a variation of (A38). In comparison with the fourth segment in Appendix A3, a higher $\tilde{\gamma}$ raises y' directly and lowers y' by lowering $\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}$, but the former effect dominates the latter effect. The overall effect of a rising $\tilde{\gamma}$ includes the third effect through changing $\tilde{m}$: $dy'/d\tilde{\gamma} = \partial y'/\partial\tilde{\gamma} + (d\tilde{m}/d\tilde{\gamma})(\partial y'/\partial\tilde{m})$. Therefore, $dy'/d\tilde{\gamma} > 0$ unconditionally in a type 1 solution, or if the optimal $\tilde{m}$ does not fall in a type 2 solution. We can also show that $dy'/d\tilde{\gamma} > 0$ regardless of the possibly falling $\tilde{m}$ if the talent is distributed by the Pareto distribution. See Appendix A8.1.

Appendix A8.1: Proof that $dy'/d\tilde{\gamma} > 0$ when $\tilde{\tau} > \eta_h\tilde{m}$ under the Pareto Distribution in Appendix A8

The following proof is under $\tilde{\eta}_l = \eta_l^{\tilde{\alpha}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\alpha}}$. For the proof under $\tilde{\eta}_l = \eta_l^{\tilde{\gamma}}$ and $\tilde{\eta}_h = \eta_h^{\tilde{\gamma}}$, we can use the steps in Appendix 7.1 and below with minor variation of expressions. The overall effect of rising $\tilde{\gamma}$ is $dy'/d\tilde{\gamma} = \partial y'/\partial\tilde{\gamma} + (d\tilde{m}/d\tilde{\gamma})(\partial y'/\partial\tilde{m})$, which is positive if $d\tilde{m}/d\tilde{\gamma} \geq 0$. Suppose that $d\tilde{m}/d\tilde{\gamma} < 0$. Then, $d\tilde{m}/d\tilde{\gamma} > -(\tilde{m}\Psi_{\tilde{\gamma}})/(\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\alpha}\tilde{\Psi}_{\tilde{m}})$ in (A34), and

$$\begin{aligned}\frac{dy'}{d\tilde{\gamma}} &> \frac{\partial y'}{\partial\tilde{\gamma}} - \frac{\tilde{m}\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\alpha}\tilde{\Psi}_{\tilde{m}}} \cdot \frac{\partial y'}{\partial\tilde{m}} \\ &= \frac{\partial y'}{\partial\tilde{\gamma}} - \frac{\tilde{m}\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\alpha}\tilde{\Psi}_{\tilde{m}}} \cdot \tilde{\eta}_h\tilde{m}^{\tilde{\alpha}} \cdot \tilde{\alpha}\tilde{\gamma}\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\left(\frac{\tilde{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - 1\right).\end{aligned} \tag{A39}$$

When $\tilde{\tau}(\tilde{m}) = \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}$, $dy'/d\tilde{\gamma} > 0$, so $dy'/d\tilde{\gamma} > 0$ for all $\tilde{\tau} \geq \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}$ if the derivative of the righthand side of (A39) with respect to $\tilde{\tau}$ is not negative; that is,

$$-\frac{\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n\tilde{\Psi}_{\tilde{m}}} \geq -\frac{1}{\tilde{\gamma}} - \log\tilde{m} = -\frac{1}{\tilde{\gamma}} - \log\left(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}\right) + \log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}. \tag{A40}$$

Using (A31),(A37), and (A12), we have

$$\begin{aligned}-\frac{\Psi_{\tilde{\gamma}}}{\tilde{\eta}_l^{\frac{\tilde{\gamma}}{\tilde{\alpha}}} n} = \frac{\tilde{\alpha}}{\tilde{\gamma}}\left(\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}} - 1\right)\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} + \left(-\tilde{\alpha} + \tilde{\gamma}\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\log\left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right) - \tilde{\gamma}\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right) \\ + \frac{1}{n}\left(\log\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\right)\left(1 - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\right) + \frac{1}{n}\left(\frac{\tilde{\eta}_h}{\tilde{\eta}_l}\right)^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n.\end{aligned} \tag{A41}$$

Using (A41) and (A36), after some algebra, we can write (A40) as

$$A \log\left(\frac{1}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right) + B \log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}} + C \geq 0 \tag{A42}$$

where

$$A \equiv \frac{\tilde{\alpha}}{\tilde{\gamma}}\left(\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}} - 1\right) - \frac{1}{n-1}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right), \tag{A43}$$

$$B \equiv \frac{1}{n} - \frac{1}{n}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n - \frac{1}{n-1}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right), \tag{A44}$$

and

$$C \equiv \frac{1}{n}\left(\frac{\tilde{\eta}_h}{\tilde{\eta}_l}\right)^{\frac{\tilde{\gamma}}{\tilde{\alpha}}}\left(\log \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\right)\left(\frac{\check{\tau}}{\tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n + \frac{1}{\tilde{\gamma}(n-1)}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - \left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n\left(1 - \left(\frac{\tilde{\eta}_l}{\tilde{\eta}_h}\right)^{\frac{n-\tilde{\gamma}}{\tilde{\alpha}}}\right)\right) > 0. \tag{A45}$$

Using (A16), we have

$$A < \frac{\tilde{\alpha}}{\tilde{\gamma}}\frac{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}{\check{\tau}}\left(1 - \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right) - \frac{1}{n-1} \cdot \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} + \frac{n}{n-1} \cdot \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - 1 < 0 \tag{A46}$$

where the last inequality uses $(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})/\check{\tau} < \tilde{\gamma}/\tilde{\alpha}$ (see Appendix A6.1), and

$$B < \frac{1}{n} - \frac{1}{n}\left(\frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}}\right)^n - \frac{1}{n-1} \cdot \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} + \frac{n}{n-1} \cdot \frac{\check{\tau}}{\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m}} - 1 < 0. \tag{A47}$$

Therefore, (A40) holds and $dy'/d\tilde{\gamma} > 0$ for all $\tilde{\tau} > \tilde{\eta}_h^{\frac{1}{\tilde{\alpha}}}\tilde{m}$.

Appendix A9: The Effects of a Rising $\tilde{\gamma}$ on the Constrained $\check{\tau}$ in Section 4.3

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}$, (A4) is valid with Ω replaced by

$$\tilde{\Omega}_{\tilde{\gamma}} \equiv \Omega + \tilde{\alpha}(1 - (\tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}\tilde{m})^{-\tilde{\gamma}})(-\log \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}). \tag{A48}$$

The additional term reflects the positive effect of a rising η_l. Taking the same steps as in (A5), we have $d\check{\tau}/d\tilde{\gamma} < 0$.

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$, (A4) is valid with Ω replaced by

$$\tilde{\Omega} \equiv \Omega + \tilde{\gamma}(1 - (\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})^{-\tilde{\gamma}})(-\log \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}). \tag{A49}$$

We have $\tilde{\Omega} = 0$ if $\eta_l = 1/\tilde{m}$. We can derive

$$\frac{\partial \tilde{\Omega}}{\partial \eta_l} = \frac{\tilde{\gamma}\tilde{\eta}_l^{-\frac{1}{\tilde{\alpha}}}}{\tilde{m} - 1}\Big(- (\tilde{m} - 1) + (\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}\tilde{m})^{-\tilde{\gamma}}\tilde{m}\log \tilde{m}\Big). \tag{A50}$$

We can show that the expression inside the large bracket is positive when $\eta_l = 1/\tilde{m}$ given $\tilde{m} > 1$, falls as η_l rises, and becomes zero when $\eta_l = \hat{\eta}_l(\tilde{m}) \equiv (\log \tilde{m}/(\tilde{m}^{\tilde{\alpha}}(\tilde{m} - 1)))^{1/\tilde{\gamma}}$. For a constrained worker, therefore, a sufficient condition for $d\check{\tau}/d\tilde{\gamma} > 0$ is $\eta_l \in (1/\tilde{m}, \hat{\eta}_l(\tilde{m}))$. The bound $\hat{\eta}_l(\tilde{m})$ falls from 1 to 0 as $\tilde{m}$ rises from 1 to ∞. Then, given $\tilde{\gamma} > 1$, an alternative expression of the sufficient condition for $d\check{\tau}/d\tilde{\gamma} > 0$ is $\tilde{m} \in (1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}, \hat{\eta}_l^{-1}(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}))$. If $\tilde{\gamma} \leq \bar{\gamma}$, $\tilde{m} \leq 1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$ for any ω, so the condition $\tilde{m} \in (1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}, \hat{\eta}_l^{-1}(\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}))$ does not hold.

In order to proceed further, let $\bar{\omega}(\tilde{\gamma})$ denote the threshold ω for which the first-period income constraint is just non-binding given $\tilde{\gamma}$: $y = \bar{\omega}(\tilde{\gamma})$ under the unconstrained optimal $\tilde{m}$ given $\tilde{\gamma}$. Let $\hat{\omega}(\tilde{\gamma})$ denote the first-period income y given $\tilde{m} = 1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$: $\hat{\omega}(\tilde{\gamma}) = \tilde{\gamma}/\tilde{\eta}_l - \tilde{\alpha}/\tilde{\eta}_l^{\tilde{\gamma}/\tilde{\alpha}}$. We have $\bar{\omega}(\bar{\gamma}) = \hat{\omega}(\bar{\gamma})$ and $\bar{\omega}(\tilde{\gamma}) < \hat{\omega}(\tilde{\gamma})$ for $\tilde{\gamma} > \bar{\gamma}$ since the unconstrained optimal $\tilde{m} > 1/\tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$ for $\tilde{\gamma} > \bar{\gamma}$. Therefore, for $\tilde{\gamma} > \bar{\gamma}$, $(\bar{\omega}(\tilde{\gamma}), \hat{\omega}(\tilde{\gamma}))$ defines the range of ω under which the worker is constrained and the constrained $\check{\tau} > 1$. We can show that $d\hat{\omega}(\tilde{\gamma})/d\tilde{\gamma} > 0$ for all $\tilde{\gamma} \geq \bar{\gamma}$ and $\hat{\omega}(\tilde{\gamma}) \to (1 + \log \tilde{\eta}_l)/\tilde{\eta}_l < 1$ as $\tilde{\gamma} \to \infty$. Then, if $\omega \geq (1 + \log \tilde{\eta}_l)/\tilde{\eta}_l$, $\check{\tau} = 1$ for all $\tilde{\gamma} > 1$. If $\omega \in (\bar{\omega}(\bar{\gamma}), (1 + \log \tilde{\eta}_l)/\tilde{\eta}_l)$, $\check{\tau} = 1$ for $\tilde{\gamma} \leq \hat{\omega}^{-1}(\omega)$; $\check{\tau} > 1$ for $\tilde{\gamma} > \hat{\omega}^{-1}(\omega)$; and $\check{\tau} \to 1$ as $\tilde{\gamma} \to \infty$, where the last property holds since for any $\omega < 1$, the worker becomes constrained with the constrained $\tilde{m} \to 1$ as $\tilde{\gamma} \to \infty$. If $\omega \leq \bar{\omega}(\bar{\gamma})$, $\check{\tau} = 1$ for $\tilde{\gamma} \leq \bar{\gamma}$; $\check{\tau} > 1$ for $\tilde{\gamma} > \bar{\gamma}$; and $\check{\tau} \to 1$ as $\tilde{\gamma} \to \infty$. Further, the above property of $\tilde{\Omega}$ implies that given $\tilde{\gamma} > \bar{\gamma}$, there is an $\epsilon > 0$ such that $d\check{\tau}/d\tilde{\gamma} > 0$ if $\omega \in (\hat{\omega}(\tilde{\gamma}) - \epsilon, \hat{\omega}(\tilde{\gamma}))$. It also implies that given $\tilde{\gamma} > 1$, there is an $\epsilon > 0$ such that $d\check{\tau}/d\tilde{\gamma} > 0$ for a constrained worker with $\check{\tau} \in (1, 1+\epsilon)$.

Appendix A10: The Effects of a Rising $\tilde{\gamma}$ on the Constrained y' in Section 4.3

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}$, the constrained $\check{\tau}$ falls as $\tilde{\gamma}$ rises (see Appendix A9), so y' stays constant in the first segment ($\tilde{\tau} < \check{\tau}$), and rises in the second and the third segments ($\tilde{\tau} \in [\check{\tau}, \eta_h \tilde{m}]$) as in section 3.1. In the fourth segment ($\tilde{\tau} > \eta_h \tilde{m}$), a rise of $\tilde{\gamma}$ lowers a constrained worker's $\eta_h \tilde{m}$ by lowering $\tilde{m}$ as in section 3.1 and also by lowering η_h. Nonetheless, y' rises. In order to see this, note that in the fourth segment, $y' = \tilde{\tau} \cdot \tilde{\gamma}\tilde{\eta}_h \tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{\eta}_h^{\tilde{\gamma}/\tilde{\alpha}} \tilde{m}^{\tilde{\gamma}}$ and rises when $\tilde{\tau} = \eta_h \tilde{m}$. By (A6), $d(\tilde{\gamma}\tilde{m}^{\tilde{\alpha}})/d\tilde{\gamma} > 0$ for all $\tilde{m} > 1$. Then, $y'|_{\tilde{\tau} > \eta_h \tilde{m}} - y'|_{\tilde{\tau} = \eta_h \tilde{m}}$ rises as $\tilde{\gamma}$ rises. Therefore, y' rises for all $\tilde{\tau} > \eta_h \tilde{m}$, as when $\tilde{\gamma}$ was rising without changing η_l and η_h in section 3.1.

If $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\alpha}}}$, as $\tilde{\gamma}$ rises, y' stays constant in the first segment ($\tilde{\tau} < \check{\tau}$), and rises in the third and fourth segments ($\tilde{\tau} > \eta_l \tilde{m}$) as when $\eta_l = \tilde{\eta}_l^{\frac{1}{\tilde{\gamma}}}$ above: Repeat the same steps of reasoning for the fourth segment. In the second segment, y' may fall as $\tilde{\gamma}$ rises: For a constrained worker whose $\check{\tau}$ is above but close enough to one, $\check{\tau}$ rises as $\tilde{\gamma}$ rises (see Appendix A9), lowering y' toward one for $\tilde{\tau}$ above but close enough to $\check{\tau}$. This result is qualitatively the same as for the unconstrained worker in sections 3 and 4.2. Since $\check{\tau}$ falls back toward one as $\tilde{\gamma}$ continues to rise, however, holding $\tilde{\tau}$ this fall is reversed and y' rises as $\tilde{\gamma}$ continues to rise.

Appendix A11: The Effects of a Rising ρ in Section 5

Given $y = \tau(m)^\rho \cdot am^\alpha - bm^\gamma$, we can write $y = \tilde{\tau}(\tilde{m})^\rho \cdot \tilde{\gamma}\tilde{m}^{\rho\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\rho\tilde{\gamma}}$ where $\tilde{m} \equiv (m^{\gamma-\alpha}(b/a)(\tilde{\gamma}/\tilde{\alpha}))^{1/\rho}$, $\tilde{\tau}(\tilde{m}) \equiv \tau((\tilde{m}^\rho (a/b)(\tilde{\alpha}/\tilde{\gamma}))^{1/(\gamma-\alpha)})$, and $\Upsilon \equiv (\tilde{\alpha}/b)^{\tilde{\alpha}}(a/\tilde{\gamma})^{\tilde{\gamma}} = 1$ by normalization. If $\tilde{\tau}(\tilde{m})$ is constant at τ for all $\tilde{m}$, the maximum y is $\tau^{\rho\tilde{\gamma}}$ obtained by choosing $\tilde{m} = \tau$. The maximum obtainable income is $\bar{y} = \max_{\tilde{m}}\{E[\tau^\rho]\cdot\tilde{\gamma}\tilde{m}^{\rho\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\rho\tilde{\gamma}})\} = (E[\tau^\rho])^{\tilde{\gamma}}$. We will assume that talent uncertainty for the first-period task is fully resolved after one period, so we can write $E[\tilde{\tau}(\tilde{m})^\rho] = \tilde{\tau}(\tilde{m})^\rho$ for the first-period task $\tilde{m}$ after one period. However, the initial expectation of effective talent $E[\tilde{\tau}^\rho] = \int_0^\infty \tilde{\tau}^\rho dF(\tilde{\tau}) \neq (\int_0^\infty \tilde{\tau} dF(\tilde{\tau}))^\rho = E[\tilde{\tau}]^\rho$ generally. Relations (1) to (4) are replaced by

$$\check{\tau}^\rho = \frac{(E[\tilde{\tau}^\rho])^{\tilde{\gamma}}}{\tilde{\gamma}(\eta_l\tilde{m})^{\rho\tilde{\alpha}}} + \frac{\tilde{\alpha}(\eta_l\tilde{m})^\rho}{\tilde{\gamma}}, \tag{A51}$$

$$\int_0^\infty (\tilde{m}^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau}) < \beta\eta_h^{\rho\tilde{\alpha}}\int_{\eta_h\tilde{m}}^\infty (\tilde{\tau}^\rho - (\eta_h\tilde{m})^\rho)dF(\tilde{\tau}), \tag{A52}$$

$$\int_0^\infty (\tilde{m}^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau}) + \beta\eta_l^{\rho\tilde{\alpha}}\int_{\check{\tau}}^{\eta_l\tilde{m}} ((\eta_l\tilde{m})^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau}) < \beta\eta_h^{\rho\tilde{\alpha}}\int_{\eta_h\tilde{m}}^\infty (\tilde{\tau}^\rho - (\eta_h\tilde{m})^\rho)dF(\tilde{\tau}), \tag{A53}$$

and

$$\frac{1}{\eta_l^\rho} - 1 \geq \beta\eta_h^{\rho\tilde{\alpha}}\int_{(\eta_h/\eta_l)(E[\tau^\rho])^{1/\rho}}^\infty \left(\frac{\tilde{\tau}^\rho}{E[\tau^\rho]} - \frac{\eta_h^\rho}{\eta_l^\rho}\right)dF(\tilde{\tau}). \tag{A54}$$

In (A51), $\check{\tau}$ rises in ρ. Write (A53) with equality as

$$\int_0^\infty\left(1-\left(\frac{\tilde{\tau}}{\tilde{m}}\right)^\rho\right)dF(\tilde{\tau}) + \beta\eta_l^{\rho\tilde{\gamma}}\int_{\check{\tau}}^{\eta_l\tilde{m}}\left(1-\left(\frac{\tilde{\tau}}{\eta_l\tilde{m}}\right)^\rho\right)dF(\tilde{\tau}) - \beta\eta_h^{\rho\tilde{\gamma}}\int_{\eta_h\tilde{m}}^\infty\left(\left(\frac{\tilde{\tau}}{\eta_h\tilde{m}}\right)^\rho - 1\right)dF(\tilde{\tau}) = 0. \tag{A55}$$

The lefthand side is the marginal cost of raising $\tilde{m}$ net of the marginal benefit, divided by a common factor. The factors in (A55) can be interpreted in terms of mismatch factors and mismatch factor weights in the same fashion as the factors in (A53). The overmatch factor weight $\beta\eta_l^{\rho\tilde{\gamma}}$ falls in ρ, and the undermatch factor weight $\beta\eta_h^{\rho\tilde{\gamma}}$ rises in ρ, lowering the net marginal cost. Let $X \equiv \int_0^\infty(1-(\tilde{\tau}/\tilde{m})^\rho)dF(\tilde{\tau})$, $Y \equiv \int_{\check{\tau}}^{\eta_l\tilde{m}}(1-(\tilde{\tau}/\eta_l\tilde{m})^\rho)dF(\tilde{\tau})$, and $Z \equiv \int_{\eta_h\tilde{m}}^\infty((\tilde{\tau}/\eta_h\tilde{m})^\rho - 1)dF(\tilde{\tau})$ denote the three mismatch factors. We have $\rho((\partial X/\partial\rho) + \beta\eta_l^{\rho\tilde{\gamma}}(\partial Y/\partial\rho) - \beta\eta_h^{\rho\tilde{\gamma}}(\partial Y/\partial\rho)) < X + \beta\eta_l^{\rho\tilde{\gamma}}Y - \beta\eta_h^{\rho\tilde{\gamma}}Y = 0$: As ρ rises, changes in mismatch factors lower the net marginal cost. Therefore, a rising ρ lowers the net marginal cost, raising $\tilde{m}$ in (A55). Following the same steps in (A52), we have $\tilde{m}$ rising in ρ. In Figure 3, a rising ρ shifts the $\tilde{m}$ reaction function up and shifts the $\check{\tau}$ reaction function to the right, raising the optimal $\tilde{m}$ and, in the case of a type 2 solution, raising the optimal $\check{\tau}$. In

(A54), a rising ρ can turn a type 2 solution to a type 1 solution in contrast with a rising $\tilde{\gamma}$.

Now consider the bounds of the task range given by a fixed ratio of talent productivity or production cost as in section 4: $\tilde{\eta}_h = \eta_h^\rho > 1$ and $\tilde{\eta}_l = \eta_l^\rho < 1$ are held constant. Then, η_h falls and η_l rises in ρ. In (A54), the righthand side rises in ρ, so a rising ρ can turn a type 1 solution to a type 2 solution but not the other way around. In (A52) and (A53), mismatch factor weights $\eta_l^{\rho\tilde{\alpha}}$ and $\eta_h^{\rho\tilde{\alpha}}$ are held constant while the undermatch factor $\int_{\eta_h\tilde{m}}^{\infty}(\tilde{\tau}^\rho - (\eta_h\tilde{m})^\rho)dF(\tilde{\tau})$ rises as η_h falls, and the overmatch factor $\int_{\check{\tau}}^{\eta_l\tilde{m}}((\eta_l\tilde{m})^\rho - \tilde{\tau}^\rho)dF(\tilde{\tau})$ rises as η_l rises. The latter effect cancels the negative effect of ρ on η_l^ρ that was present when η_l was held constant. If this latter effect is strong enough, the optimal $\tilde{m}$ falls. In (A55), holding $\tilde{m}$, the net marginal cost eventually becomes negative as ρ continues to rise. Therefore, even when the optimal $\tilde{m}$ falls, it eventually rises to a higher value as ρ continues to rise. Overall, a rising ρ raises the optimal task levels with qualifications as a rising $\tilde{\gamma}$ does. One difference is that, as can be seen in (A55), the optimal $\tilde{m}$ rises without a bound even when η_h^ρ and η_l^ρ are held constant.

Unlike a rising $\tilde{\gamma}$, a rising ρ raises the current-income maximizing task level $\tilde{m} = (E[\tilde{\tau}^\rho])^{1/\rho}$ and changes the associated maximum current income $\bar{y} = (E[\tilde{\tau}^\rho])^{\tilde{\gamma}}$. This implies that the responses of career-based task choices and incomes to a rising ρ are ambiguous. Formally, define career-based tasks as multiples of the current-income maximizing task: $\tilde{M} \equiv \tilde{m}/E[\tilde{\tau}^\rho]^{1/\rho}$, $\tilde{T} \equiv \tilde{\tau}/E[\tilde{\tau}^\rho]^{1/\rho}$, and $G(\tilde{T}) \equiv F(\tilde{T}E[\tilde{\tau}^\rho]^{1/\rho})$. With these changes of variables, (A51) to (A54) become

$$\check{T}^\rho = \frac{1}{\tilde{\gamma}(\eta_l\tilde{M})^{\rho\tilde{\alpha}}} + \frac{\tilde{\alpha}(\eta_l\tilde{M})^\rho}{\tilde{\gamma}}, \tag{A56}$$

$$\tilde{M}^\rho - 1 < \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h\tilde{M}}^{\infty} (\tilde{T}^\rho - (\eta_h\tilde{M})^\rho)dG(\tilde{T}), \tag{A57}$$

$$\tilde{M}^\rho - 1 + \beta\eta_l^{\rho\tilde{\alpha}} \int_{\check{T}}^{\eta_l\tilde{M}} ((\eta_l\tilde{M})^\rho - \tilde{T}^\rho)dG(\tilde{T}) < \beta\eta_h^{\rho\tilde{\alpha}} \int_{\eta_h\tilde{M}}^{\infty} (\tilde{T}^\rho - (\eta_h\tilde{M})^\rho)dG(\tilde{T}), \tag{A58}$$

and

$$\frac{1}{\eta_l^\rho} - 1 \geq \beta \eta_h^{\rho\tilde{\alpha}} \int_{\eta_h/\eta_l}^{\infty} \left(\tilde{T}^\rho - \frac{\eta_h^\rho}{\eta_l^\rho} \right) dG(\tilde{T}). \tag{A59}$$

As ρ rises, G changes possibly countering the positive effect of a rising ρ on $\tilde{M}$ discussed above. Therefore, the overall effect of a rising ρ on $\tilde{M}$ is ambiguous and the effects on incomes as multiples of the maximum current income, $y/\bar{y} = \tilde{\gamma}\tilde{M}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{M}^{\tilde{\gamma}}$ and $y'/\bar{y} = \tilde{T} \cdot \tilde{\gamma}(\tilde{M}')^{\tilde{\alpha}} - \tilde{\alpha}(\tilde{M}')^{\tilde{\gamma}}$, are also ambiguous.

Appendix A12: Conditions for a Negative First-period Optimal Income y in Section 5

We have $y = \tilde{\gamma}\tilde{m}^{\tilde{\alpha}} - \tilde{\alpha}\tilde{m}^{\tilde{\gamma}} < 0$ iff $\tilde{m} > \tilde{\gamma}/\tilde{\alpha}$. As $\tilde{\gamma} \to \infty$, $\tilde{\gamma}/\tilde{\alpha} \to 1$, so y becomes negative eventually as $\tilde{\gamma}$ rises as long as $\lim_{\tilde{\gamma}\to\infty} \tilde{m} > 1$, which holds in both sections 3 and 4.2. For the remainder, assume the Pareto distribution of talent (assumption 3-2). Suppose $\eta_l \leq \tilde{\alpha}/\tilde{\gamma}$. Then, $\eta_l \tilde{m} \leq 1$ and $\check{\tau} = 1$ when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$. Then, the optimal $\tilde{m} > \tilde{\gamma}/\tilde{\alpha}$ and $y < 0$ iff (A1) holds when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$, which, given $\bar{\tau} = n/(n-1) \cdot (k/n)^{1/n} = 1$, is equivalent to

$$\left(\frac{n}{\tilde{\gamma} - 1} \right)^n \left(\frac{\tilde{\gamma}}{n-1} \right)^{n-1} < \frac{\beta}{\eta_h^{n-\tilde{\gamma}}}. \tag{A60}$$

Now suppose $\eta_l > \tilde{\alpha}/\tilde{\gamma}$. Then, $\eta_l \tilde{m} > 1$ and $\check{\tau} > 1$ when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$. Then, the optimal $\tilde{m} > \tilde{\gamma}/\tilde{\alpha}$ and $y < 0$ iff (A2) holds when $\tilde{m} = \tilde{\gamma}/\tilde{\alpha}$ and $\check{\tau}$ is given by (1), which is equivalent to

$$\left(\frac{n}{\tilde{\gamma} - 1} \right)^n \left(\frac{\tilde{\gamma}}{n-1} \right)^{n-1} < \frac{\beta}{\eta_h^{n-\tilde{\gamma}}} - \frac{\beta}{\eta_l^{n-\tilde{\gamma}}} \left(1 - \frac{1 - n + n(1 - (1/\tilde{\gamma})(1 - (1/\eta_l)^{\tilde{\gamma}}(\tilde{\alpha}/\tilde{\gamma})^{\tilde{\gamma}}))}{(1 - (1/\tilde{\gamma})(1 - (1/\eta_l)^{\tilde{\gamma}}(\tilde{\alpha}/\tilde{\gamma})^{\tilde{\gamma}}))^n} \right). \tag{A61}$$

In order to assess the plausibility of (A60) and (A61), consider $n = 2$ and $\tilde{\gamma} = 2$ used in Appendix A7, and $\eta_l \leq 1/2$ so that (A60) is the relevant condition. Then, (A60) becomes $\beta > 8$; that is, the weight of the second period income must be more than eight times the weight of the first-period income. If $\tilde{\gamma}$ rises to 3 keeping $\eta_l \leq 1/2$, (A60) becomes $\beta > 3/\eta_h$. The threshold value of β is now less than three, and falls below one if $\eta_h > 3$.

Appendix B: A Three-Period Model with a Numerical Exercise

There are three periods, denoted by $t = 0, 1, 2$. There are at least three trades, indexed by $s \geq 0$. The trades are ordered by talent rewards: $\tilde{\gamma}^s \geq \tilde{\gamma}^{s'}$ for $s < s'$. The bounds of the talent-update task range η_l^s and η_h^s may depend on $\tilde{\gamma}^s$ as will be discussed shortly. Trades are equivalent in all other aspects. In particular, the scale factor $\Upsilon^s = 1$ for all s so that given $\bar{\tau} = 1$, the unconditional maximum income $\bar{y}^s = \bar{\tau}^{\tilde{\gamma}^s} \cdot \Upsilon^s = 1$ for all s. The worker's talent across tasks evolves as follows. Let $\tau_t(\tilde{m}, s)$ denote the worker's (expected) talent for task $\tilde{m}$ in trade s at the beginning of t. Let $\tilde{m}_t$ and s_t denote the worker's task and trade at t. At the beginning of $t = 0$, $\tau_0(\tilde{m}, s) = \bar{\tau} = 1$ for all $(\tilde{m}, s)$. At the beginning of $t = 1$, the worker draws $\tau_1(\tilde{m}_0, s_0)$ from the distribution $F_0(\tau)$ with $E[\tau] = \bar{\tau} = 1$; $\tau_1(\tilde{m}, s_0) = \tau_1(\tilde{m}_0, s_0)$ if $\tilde{m} \in [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$; and $\tau_1(\tilde{m}, s) = \bar{\tau} = 1$ if $s \neq s_0$ or if $s = s_0$ and $\tilde{m} \notin [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$. At the beginning of $t = 2$, the worker draws $\tau_2(\tilde{m}_1, s_1)$ from the distribution $F_0(\tau)$ if $s_1 \neq s_0$ or if $s_1 = s_0$ and $\tilde{m}_1 \notin [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$, and from the distribution $F_1(\tau|\tau_1(\tilde{m}_1, s_1))$ with $E[\tau] = \tau_1(\tilde{m}_1, s_1)$ if $s_1 = s_0$ and $\tilde{m}_1 \in [\eta_l^{s_0}\tilde{m}_0, \eta_h^{s_0}\tilde{m}_0]$; $\tau_2(\tilde{m}, s_1) = \tau_2(\tilde{m}_1, s_1)$ if $\tilde{m} \in [\eta_l^{s_1}\tilde{m}_1, \eta_h^{s_1}\tilde{m}_1]$, and $\tau_2(\tilde{m}, s) = \bar{\tau} = 1$ if $s \neq s_1$ or if $s = s_1$ and $\tilde{m} \notin [\eta_l^{s_1}\tilde{m}_1, \eta_h^{s_1}\tilde{m}_1]$. In comparison with the two-period model, the (expected) talent for a task is subject to a further update, distributed by F_1, following the initial update, distributed by F_0.

The worker's trade is assumed to evolve as follows. The worker starts at trade 0 at $t = 0$ and, if he moves to another trade at $t = 1$, he moves to trade 1. If the worker moves to yet another trade at $t = 2$, he moves to any trade $s \neq s_1$ and obtains $\bar{y}^s = 1$. This pattern of the worker's trade would be without a loss of generality if the worker's expected lifetime income rises by raising $\tilde{\gamma}^{s_0}$ or $\tilde{\gamma}^{s_1}$ and, holding $\{s_0, s_1\}$, if the worker's expected lifetime income is higher when working in the trade with a higher $\tilde{\gamma}^s$ first. Since the upside income potential is higher with a higher $\tilde{\gamma}^s$, this property is plausible and appears to hold in the numerical exercise conducted. Note that the worker has three options at the beginning of $t = 1$. The worker can stay in trade 0 and attempt $\tilde{m} \in [\eta_l^0 \tilde{m}_0, \eta_h^0 \tilde{m}_0]$ (option 1). The worker can stay in trade 0 and attempt $\tilde{m} \notin [\eta_l^0 \tilde{m}_0, \eta_h^0 \tilde{m}_0]$ (option 2). The worker can move to trade 1 (option 3). In the two-period model, staying in the current trade and attempting a task outside the talent-update task range (option 2) was not viable since it was (weakly) dominated by moving to another trade and obtaining the unconditional maximum income $\bar{y} = 1$. In the three-period model, option 2 can dominate moving to trade 1 (option 3) as will become clear.

In the numerical exercise, I assume that $F_0(\tau)$ is a bounded Pareto distribution with the support $[1/\sigma_0, \sigma_0]$ where $\sigma_0 > 1$, and the shape parameter $n = 1/2$, which ensures that $E[\tau] = 1$. Similarly, $F_1(\tau|\tau')$ is a bounded Pareto distribution with the support $[\tau'/\sigma_1, \tau'\sigma_1]$ and the shape parameter $n = 1/2$, which ensures that $E[\tau|\tau'] = \tau'$. I set $\sigma_0 = 2$, which implies that the talent Gini at the beginning of $t = 1$ is $1 - 2/(\sigma_0 - 1) \cdot (\sigma_0/(\sigma_0 - 1) \cdot \log \sigma_0 - 1) = 0.23$ given $n = 1/2$. I also set $\sigma_1 = \sigma_0^\nu$ where $\nu \in [0, 1]$, which implies that the expected size of talent update becomes (weakly) smaller over time. Unlike in the two-period model, talent rewards at the second trade $\tilde{\gamma}^1$ affect the career paths. I set $\tilde{\gamma}^1 = \rho\tilde{\gamma}^0 + (1 - \rho) \cdot 1.5$ where $\rho \in [0, 1]$. This implies that $\tilde{\gamma}^1 = \tilde{\gamma}^0$ when $\tilde{\gamma}^0 = 1.5$ regardless of ρ. The value of $\tilde{\gamma} = 1.5$ implies that the fixed cost share of the gross output is $\tilde{\alpha}/\tilde{\gamma} = (\tilde{\gamma} - 1)/\tilde{\gamma} = 1/3$. I explore the changes of the worker's career paths when $\tilde{\gamma}^0$ rises from 1.5. If $\rho = 0$, $\tilde{\gamma}^1$ stays at 1.5 as $\tilde{\gamma}^0$ rises. If $\rho = 1$, as $\tilde{\gamma}^0$ rises, $\tilde{\gamma}^1$ rises as much as $\tilde{\gamma}^0$. A low value of ρ implies that rising talent rewards affect a narrow range of trades from the perspective of the worker so that when the worker fails in the initial trade, he moves to a trade with a significantly slow growth of talent rewards. A high value of ρ implies the opposite. I also parameterize the degree of rising task differentiation as $\tilde{\gamma}^s$ rises: $\eta_h^s = 1/\eta_l^s = 2^{\mu/(2\tilde{\alpha}^s)+1-\mu}$ for all s, where $\mu \in [0, 1]$. This implies that when $\tilde{\gamma}^1 = \tilde{\gamma}^0 = 1.5$, $\eta_h^s = 1/\eta_l^s = 2$. If $\mu = 0$, η_h^s stays at 2 as $\tilde{\gamma}_s$ rises as in section 3. If $\mu = 1$, $\eta_h^s = (2^{1/2})^{1/\tilde{\alpha}^s}$ falls as $\tilde{\gamma}_s$ rises as it does under the fixed ratio of talent productivity in section 4. I set the discount parameter $\beta = 1$: there are no discounts for the second and the third period incomes.

Tables 1 and 2 present the numerical results under $\rho = 0$ and $\rho = 1$, respectively, with $\nu = 1/2$ in both cases. In each table, rows 4 to 16 present results when the worker is not constrained while rows 17 to 29 present results when the worker must earn at least 85 percent of the unconditional maximum income in the first period ($y_0 \geq \omega = 0.85$) and a double of that amount over the first two periods ($y_0 + y_1 \geq 2\omega = 1.70$). This is equivalent to assuming that the worker holds no initial wealth, must spend at least 85 percent of the unconditional maximum income in each period, cannot borrow, but can save at the zero interest rate. The value of $\omega = 0.85$ was chosen to construct the narrative that as $\tilde{\gamma}^0$ rises, the income constraint is not binding for a moderate rise but is binding eventually as $\tilde{\gamma}^0$ continues to rise.

Benchmark: $\tilde{\gamma}^0 = 1.5$ and any values of μ and ρ

Column 2 presents results when $\tilde{\gamma}^0 = 1.5$, in which case the values of μ and ρ do not matter. Under $\tilde{\gamma}^0 = 1.5$, the interval $[\eta_l^0, \eta_h^0] = [0.5, 2]$ includes the intervals $[1/\sigma_0, \sigma_0] = [0.5, 2]$ and $[1/\sigma_1, \sigma_1] \approx [0.7, 1.4]$, eliminating the incentive to try a more talent-demanding task than the current-income maximizing task. Therefore, the optimal task m_0 is equal to one and the associated income y_0 is equal to one at $t = 0$. This outcome is constructed as a narrative benchmark in which the results mimic the uniform-talent case discussed in section 2.1. Nonetheless, the optimal talent threshold for staying in the initial trade $\check{\tau}_1$ is 1.07, greater than one. If the updated talent is moderately above $\bar{\tau} = 1$ at the beginning of $t = 1$, the worker moves to trade 1 since the upside income potential is higher when trying a task $\tilde{m} = 1$ in trade 1 (option 3) than trying a task within the talent-update task range $[\eta_l^0 \tilde{m}_0, \eta_h^0 \tilde{m}_0] = [0.5, 2]$ in trade 0 (option 1) given the larger sizes of talent updates in trade 1 (i.e., $\sigma_0 > \sigma_1$), or trying a task constrained to be outside the talent-update task range in trade 0 (option 2). Therefore, unlike in the two-period model, the trade-switch rate $F(\check{\tau}_1)$ is more than a 'failure' rate and includes workers drawn to another trade despite being able to earn more than the unconditional maximum income $\bar{y} = 1$ in the initial trade. The expected lifetime income $E[\sum_{t=0}^{2} y_t]$ is about 26% higher than 3.00 which would be obtained if the worker earned the unconditional maximum income $\bar{y} = 1$ in each period.

The cross-sectional income distribution at $t = 1$ exhibits a flat profile up to the 63rd percentile filled by switchers ($s_1 \neq s_0$), and then a rising profile subsequently filled by stayers ($s_1 = s_0$). The cross-sectional income distribution at $t = 2$ exhibits a similar pattern but with a smaller fraction of switchers ($s_2 \neq s_1$), 42 percent, and higher incomes of stayers ($s_2 = s_1$), some of whom have moved up the task ladder twice following two positive talent updates. The 90th-percentile income rises to 2.45 at $t = 2$. For a loose reference, Baker et. al. (1994) show that among the management employees entering a US service-sector firm in 1975, the 95th percentile salary as a multiple of the starting salary rose to about three by 1987. The income constraints (rows 17 to 29) do not change any of the outcomes since the optimal task delivers an income larger than 85 percent of the unconditional maximum income in each period.

<u>Unconstrained Workers: $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$ and $\mu = 0$</u>

Columns 3 to 5 and Columns 6 to 8 in Tables 1 and 2 characterize the career tracks when $\tilde{\gamma}^0 = 3$ and $\tilde{\gamma}^0 = 5$, which imply that the fixed cost share of the gross output is 2/3 and 4/5, respectively. Consider the unconstrained worker (Columns 4 to 16). If $\mu = 0$, the interval $[\eta_l^0, \eta_h^0]$ stays constant as $\tilde{\gamma}^0$ rises, so the optimal first-period task $\tilde{m}_0$ remains at one. If $\mu = 0$ and $\rho = 1$, the optimal talent threshold for staying in the initial trade $\check{\tau}_1$ rises as $\tilde{\gamma}^0$ rises. As mentioned, given the larger sizes of talent updates in trade 1 (i.e., $\sigma_0 > \sigma_1$), the higher upside income potential in trade 1 draws workers whose $\tau_1(\tilde{m}_0, s_0)$ is moderately above one to trade 1, and this income advantage of trade 1 is amplified as both

$\tilde{\gamma}^0$ and $\tilde{\gamma}^1$ rise. If $\mu = 0$ and $\rho = 0$, $\check{\tau}_1$ instead falls as $\tilde{\gamma}^0$ rises. With $\rho = 0$, $\tilde{\gamma}^1$ stays at 1.5 while $\tilde{\gamma}^0$ rises, lowering the advantage of trade 1 instead. When $\tilde{\gamma}^0 = 5$, $\check{\tau}_1$ falls below one: Workers whose $\tau_1(\tilde{m}_0, s_0)$ are moderately below one tolerate a low current income (i.e., income below $\bar{y} = 1$) for the upside income potential at trade 0. Therefore, the fall of trade-switching disguises the low current-income stayers.

Unconstrained Workers: $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$ and $\mu = 0.5$ or $\mu = 1$

If $\mu = 0.5$ or $\mu = 1$, as $\tilde{\gamma}^0$ rises, the interval $[\eta_l^0, \eta_h^0]$ becomes smaller creating the talent-task gaps for high-enough talent updates as in the two-period model. Consequently, the optimal first-period task $\tilde{m}_0$ rises, lowering the first-period income y_0 as in the two-period model. This implies that the lower task bound $\eta_l^0 \tilde{m}_0^0$ (not shown in the table) rises as well, echoing Proposition 4. The upper task bound $\eta_h^0 \tilde{m}_0^0$ (not shown in the table) falls as $\tilde{\gamma}^0$ rises except when $\tilde{\gamma}^0$ rises from 3 to 5 under $\mu = 0.5$ and $\rho = 1$. The optimal task $\tilde{m}_0$ rises more for a higher μ, which shrinks the interval $[\eta_l^0, \eta_h^0]$ by more as $\tilde{\gamma}^0$ rises. This implies that the lower task bound $\eta_l^0 \tilde{m}_0^0$ rises as μ rises, echoing Proposition A2. On the other hand, the upper task bound $\eta_h^0 \tilde{m}_0^0$ falls as μ rises, echoing Proposition A1.

If $\mu = 0.5$ or $\mu = 1$ and if $\rho = 1$, as $\tilde{\gamma}^0$ rises, the expected income from moving to trade 1 (option 3) rises along with the expected income from staying in trade 0. In particular, choosing a task without a task-range constraint in trade 1 (option 3) weakly dominates attempting a task outside the talent-update task range in trade 0 (option 2). Under the parameters chosen, the task range constraint is binding for option 2, so option 3 strictly dominates option 2. If $\mu = 0.5$, the income advantage of trade 1 due to the larger sizes of talent updates in trade 1 (i.e., $\sigma_0 > \sigma_1$) is amplified as $\tilde{\gamma}^0$ rises, raising $\check{\tau}_1$, as discussed for the case of $\mu = 0$. However, this effect is weaker than in the case of $\mu = 0$ because a higher μ narrows the range of the talent-update task range, which lowers the benefit of the larger sizes of talent updates. If $\mu = 1$, the effect nearly disappears and $\check{\tau}_1$ changes little as $\tilde{\gamma}^0$ rises.

If $\mu = 0.5$ or $\mu = 1$ and if $\rho = 0$, as $\tilde{\gamma}^0$ rises, the expected income from moving to trade 1 (option 3) stays constant while the upside income potential from staying in trade 0 rises, motivating the worker to stay in trade 0 and attempt a task outside the talent-update task range (option 2) if his updated talent is below a threshold, denoted by $\acute{\tau}_1$. Consequently, the probability of moving to trade 1 ($F_0(\check{\tau}_1)$) falls to zero while the probability of moving off the talent-update task range in trade 0 ($F_0(\acute{\tau}_1)$) is positive. The worker's current income y_1 from taking option 2 can be as low as 0.69 (when $\mu = 0.5$ and $\tilde{\gamma}^0 = 5$). The worker tolerates a low current income for the upside income potential at $t = 1$ as he does at $t = 0$. Therefore, the absence of trade-switchers disguises the low current-income stayers, which is qualitatively similar to the case of $\tilde{\gamma}^0 = 5$ under $\mu = 0$ and $\rho = 0$.

Unconstrained Workers: Income Changes

The unconstrained expected lifetime income $E[\sum_{t=0}^{2} y_t]$ grows substantially as $\tilde{\gamma}^0$ rises. The cross-sectional income distributions worsen as $\tilde{\gamma}^0$ rises. At $t = 1$, the 50-10 income ratio stays constant but the 90-50 ratio rises significantly, reflecting both positive talent updates delivering larger income gains and the negative talent updates leading to prolonged low incomes (options 2 or 3). At $t = 2$, the 50-10 income ratio rises but the 90-50 ratio rises much more. For a loose reference, the average CEO compensation of the top 350 US firms as a multiple of the average production worker compensation rose from 20 in 1965 to 278 in 2018 (Mishel and Wolfe 2019). If $\rho = 1$ or if $\mu = 0.5$ or $\mu = 1$, $\tilde{\gamma}^{s_1} = \tilde{\gamma}^0$ along any career paths (either $\rho = 1$ or $\rho = s_1 = 0$), as discussed above. As a consequence, the median income at $t = 2$ is (nearly) constant at 1.39 for $\tilde{\gamma}^0 = 3$ and 1.74 for $\tilde{\gamma}^0 = 5$, which reflects that the median income earners are workers who drew the final (expected) talent τ_2 of about 1.12 in trade s_1. If $\rho = 0$ and $\mu = 0$, workers who draw a low enough talent at the beginning of $t = 1$ move to trade 1 and face lower talent rewards (i.e., $\tilde{\gamma}^{s_1} = \tilde{\gamma}^1 < \tilde{\gamma}^0$) when $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$, so their income growth is dampened and the median income at $t = 2$ is lower at 1.26 for $\tilde{\gamma}^0 = 3$ and 1.32 for $\tilde{\gamma}^0 = 5$.

One half or more of the workers earning about the median income at $t = 2$ are workers who moved to trade 1 (option 3) or moved off the talent-update task range in trade 0 (option 2) at $t = 1$, and the rest are workers who stayed within the talent-update task range in trade 0 (option 1) at $t = 1$. Thus, many of the final-period median income earners have gone through a prolonged period of low income, and the life-time income of the final-period medium income earner can fall as $\tilde{\gamma}^0$ rises. For example, if $\mu = 1$ and $\rho = 0$, the minimum life-time income of the final-period medium income earner is 3.18 ($y_0 = 1$, $y_1 = 1$, $y_2 = 1.18$) when $\tilde{\gamma}^0 = 1.5$. The life-time income of the final-period medium income earners who took option 2 at $t = 1$ is 3.26 ($y_0 = 0.87$, $y_1 = 1$, $y_2 = 1.39$) when $\tilde{\gamma}^0 = 3$, and 2.74 ($y_0 = 0.17$, $y_1 = 0.83$, $y_2 = 1.74$) when $\tilde{\gamma}^0 = 5$. Therefore, the rise of the final-period median income disguises a possibly large income loss along the career paths leading to the median income.

Constrained Workers: $\tilde{\gamma}^0 = 3$ or $\tilde{\gamma}^0 = 5$

Now consider constrained workers (rows 17 to 29). Under any values of μ and ρ considered, the income constraint of $\omega = 0.85$ is not binding when $\tilde{\gamma}^0 = 3$, so the outcome is identical to the unconstrained outcome. As mentioned, this is a narrative construction. When $\tilde{\gamma}^0 = 5$ and $\mu = 0$, the optimal first-period income y_0 is one, so the first-period task $\tilde{m}_0$ remains at one. When $\rho = 0$ in addition, the optimal second-period income y_1 of a worker who receives a talent update τ_1 above the threshold $\check{\tau}_1 = 0.96$ but below one, is less than one as was discussed. Nonetheless, income constraints are not binding as the sum of the optimal incomes over the first two periods remains above the constraint threshold: $y_0 + y_1 > 2\omega = 1.7$.

When $\tilde{\gamma}^0 = 5$ and $\mu = 0.5$ or $\mu = 1$, the income constraint in the first-period is binding and pulls down the first-period task $\tilde{m}_0$ in order to generate the first-period income $y_0 \geq 0.85$. In the table, $y_0 = 0.87$, which is the lowest value of $y_0 \geq 0.85$ when $\tilde{\gamma}^0 = 5$ and $\tau_0 = \bar{\tau} = 1$ due to the grid-based computation (151 possible values of τ or $\tilde{m}$ with the ratio of adjacent values equal to 1.014). This generates a rising and then falling level of the first-period task $\tilde{m}_0$ as $\tilde{\gamma}^0$ rises from 1.5 to 3 and then from 3 to 5, echoing Proposition 2. The constrained career paths lead to a smaller growth of the expected lifetime income and of some segments of the cross-sectional income profiles, as $\tilde{\gamma}^0$ rises. If $\mu = 0.5$ and $\rho = 0$, the optimal second-period income y_1 of a worker who stays off the talent-update task range in trade 0 (option 2) following a talent update τ_1 below the threshold $\acute{\tau}_1 = 1.03$, falls below $\omega = 0.85$. Unlike the case of $\mu = 0$ and $\rho = 0$ discussed above, the income constraint is binding in the second period as well as in the first period, forcing workers with τ_1 below $\check{\tau}_1 = 0.97$ to move to trade 1 (option 3). Note that the constrained career paths lead to a fall of the final-period median income below that under $\tilde{\gamma}^0 = 3$ (from 1.39 to 1.32). Therefore, the final-period median income can first rise and then fall as $\tilde{\gamma}^0$ rises from 1.5 to 3 and then from 3 to 5.

References for Appendix B

Baker, G., Gibbs, M., and Holmstrom, B. (1994), "The Wage Policy of a Firm," *Quarterly Journal of Economics* 109:921-955.

Mishel, L., and Wolfe, J. (2019), *CEO Compensation Has Grown 940% since 1978*, Economic Policy Institute Report (epi.org/171191).

Table 1: Numerical Exercise under $\rho = 1$ and $\nu = 0.5$

$\tilde{\gamma}^0$	1.5	3.0	3.0	3.0	5.0	5.0	5.0
μ	Any	0.0	0.5	1.0	0.0	0.5	1.0
ρ	Any	1.0	1.0	1.0	1.0	1.0	1.0
$\tilde{m}_0\vert_{\text{optimal}}$	1.00	1.00	1.06	1.16	1.00	1.12	1.21
$y_0\vert_{\text{optimal}}$	1.00	1.00	0.99	0.91	1.00	0.83	0.31
$\check{\tau}_1\vert_{\text{optimal}}$	1.07	1.10	1.10	1.07	1.15	1.13	1.07
$F_0(\check{\tau}_1)\vert_{\text{optimal}}$	0.63	0.65	0.65	0.63	0.68	0.67	0.63
$\acute{\tau}_1\vert_{\text{optimal}}$	-	-	-	-	-	-	-
$F_0(\acute{\tau}_1)\vert_{\text{optimal}}$	-	-	-	-	-	-	-
$E[\sum_{t=0}^{2} y_t]\vert_{\text{optimal}}$	3.79	5.58	5.52	5.14	12.70	12.00	7.78
$y_1(10\text{th})\vert_{\text{optimal}}$	1.00	1.00	0.99	0.97	1.00	0.87	0.77
$y_1(50\text{th})\vert_{\text{optimal}}$	1.00	1.00	0.99	0.97	1.00	0.87	0.77
$y_1(90\text{th})\vert_{\text{optimal}}$	2.16	4.66	4.65	4.28	13.00	12.97	9.32
$y_2(10\text{th})\vert_{\text{optimal}}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\vert_{\text{optimal}}$	1.18	1.39	1.39	1.39	1.74	1.74	1.74
$y_2(90\text{th})\vert_{\text{optimal}}$	2.45	6.23	6.00	5.33	21.11	19.70	11.55
$\tilde{m}_0\vert_{\omega=0.85}$	1.00	1.00	1.06	1.16	1.00	1.10	1.10
$y_0\vert_{\omega=0.85}$	1.00	1.00	0.99	0.91	1.00	0.87	0.87
$\check{\tau}_1\vert_{\omega=0.85}$	1.07	1.10	1.10	1.07	1.15	1.13	1.07
$F_0(\check{\tau}_1)\vert_{\omega=0.85}$	0.63	0.65	0.65	0.63	0.68	0.67	0.63
$\acute{\tau}_1\vert_{\omega=0.85}$	-	-	-	-	-	-	-
$F_0(\acute{\tau}_1)\vert_{\omega=0.85}$	-	-	-	-	-	-	-
$E[\sum_{t=0}^{2} y_t]\vert_{\omega=0.85}$	3.79	5.58	5.52	5.14	12.70	11.99	7.40
$y_1(10\text{th})\vert_{\omega=0.85}$	1.00	1.00	0.99	0.97	1.00	0.87	0.87
$y_1(50\text{th})\vert_{\omega=0.85}$	1.00	1.00	0.99	0.97	1.00	0.87	0.87
$y_1(90\text{th})\vert_{\omega=0.85}$	2.16	4.66	4.65	4.28	13.00	12.90	7.32
$y_2(10\text{th})\vert_{\omega=0.85}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\vert_{\omega=0.85}$	1.18	1.39	1.39	1.39	1.74	1.74	1.74
$y_2(90\text{th})\vert_{\omega=0.85}$	2.45	6.23	6.00	5.33	21.11	19.70	10.04

Table 2: Numerical Exercise under $\rho = 0$ and $\nu = 0.5$

$\tilde{\gamma}^0$	1.5	3.0	3.0	3.0	5.0	5.0	5.0
μ	Any	0.0	0.5	1.0	0.0	0.5	1.0
ρ	Any	0.0	0.0	0.0	0.0	0.0	0.0
$\tilde{m}_0\|_{\text{optimal}}$	1.00	1.00	1.15	1.20	1.00	1.16	1.23
$y_0\|_{\text{optimal}}$	1.00	1.00	0.93	0.87	1.00	0.63	0.17
$\check{\tau}_1\|_{\text{optimal}}$	1.07	1.00	-	-	0.96	-	-
$F_0(\check{\tau}_1)\|_{\text{optimal}}$	0.63	0.58	-	-	0.55	-	-
$\acute{\tau}_1\|_{\text{optimal}}$	-	-	1.03	1.07	-	1.03	1.07
$F_0(\acute{\tau}_1)\|_{\text{optimal}}$	-	-	0.60	0.63	-	0.60	0.63
$E[\sum_{t=0}^{2} y_t]\|_{\text{optimal}}$	3.79	5.20	5.23	5.10	10.96	10.91	7.77
$y_1(10\text{th})\|_{\text{optimal}}$	1.00	1.00	0.83	1.00	1.00	0.69	0.83
$y_1(50\text{th})\|_{\text{optimal}}$	1.00	1.00	0.83	1.00	1.00	0.69	0.83
$y_1(90\text{th})\|_{\text{optimal}}$	2.16	4.66	4.66	4.36	13.00	13.00	9.61
$y_2(10\text{th})\|_{\text{optimal}}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\|_{\text{optimal}}$	1.18	1.26	1.39	1.39	1.32	1.74	1.74
$y_2(90\text{th})\|_{\text{optimal}}$	2.45	4.66	4.79	4.96	13.00	13.00	11.33
$\tilde{m}_0\|_{\omega=0.85}$	1.00	1.00	1.15	1.20	1.00	1.10	1.10
$y_0\|_{\omega=0.85}$	1.00	1.00	0.93	0.87	1.00	0.87	0.87
$\check{\tau}_1\|_{\omega=0.85}$	1.07	1.00	-	-	0.96	0.97	-
$F_0(\check{\tau}_1)\|_{\omega=0.85}$	0.63	0.58	-	-	0.55	0.56	-
$\acute{\tau}_1\|_{\omega=0.85}$	-	-	1.03	1.07	-	-	1.06
$F_0(\acute{\tau}_1)\|_{\omega=0.85}$	-	-	0.60	0.63	-	-	0.62
$E[\sum_{t=0}^{2} y_t]\|_{\omega=0.85}$	3.79	5.20	5.23	5.10	10.96	10.52	7.23
$y_1(10\text{th})\|_{\omega=0.85}$	1.00	1.00	0.83	1.00	1.00	1.00	1.00
$y_1(50\text{th})\|_{\omega=0.85}$	1.00	1.00	0.83	1.00	1.00	1.00	1.00
$y_1(90\text{th})\|_{\omega=0.85}$	2.16	4.66	4.66	4.36	13.00	12.90	7.32
$y_2(10\text{th})\|_{\omega=0.85}$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$y_2(50\text{th})\|_{\omega=0.85}$	1.18	1.26	1.39	1.39	1.32	1.32	1.74
$y_2(90\text{th})\|_{\omega=0.85}$	2.45	4.66	4.79	4.96	13.00	13.00	9.02