Macro 3, Final Examination 2007

1.| see lecture notes or Barro & Sala-i-Martin p. 146 The Romer model
p. 153 The Barro model with GNT

2.| Solow growth model

$\frac{\dot{A}}{A} = g \qquad \frac{\dot{L}}{L} = n$

$\dot{K} = s K^{\gamma} (AL)^{1-\gamma} - \delta K \quad , \quad 0<\gamma<1 \;,\; s \text{ const.} \;,\; \delta > 0$

a) $\hat{k} = \frac{K}{AL} \quad \rightarrow \quad K = \hat{k} AL$

$\dot{K} = \dot{\hat{k}} AL + \hat{k}\dot{A}L + \hat{k}A\dot{L} \qquad / \cdot \frac{1}{K}$

$$\left.\begin{array}{l} \frac{\dot{K}}{K} = \frac{\dot{\hat{k}}}{\hat{k}} + \underbrace{\frac{\dot{A}}{A}}_{g} + \underbrace{\frac{\dot{L}}{L}}_{n} \\ \frac{\dot{K}}{K} = s \cdot K^{\gamma-1}(AL)^{1-\gamma} - \delta = s\hat{k}^{\gamma-1} - \delta \end{array}\right\} \quad \underline{\gamma_{\hat{k}}} = \frac{\dot{\hat{k}}}{\hat{k}} = \underline{s\hat{k}^{\gamma-1} - (\delta+g+n)}$$

in ss $\dot{\hat{k}} = 0 \quad \rightarrow \quad \underline{\hat{k}^* = \left(\frac{\delta+g+n}{s}\right)^{\frac{1}{\gamma-1}}}$

$Y = K^{\gamma}(AL)^{1-\gamma} \qquad \hat{y} = \hat{k}^{\gamma} \quad \rightarrow \quad \underline{\hat{y}^* = \left(\frac{\delta+g+n}{s}\right)^{\frac{\gamma}{\gamma-1}}}$

$\ln Y = \gamma \ln K + (1-\gamma)(\ln A + \ln L)$

$$\underline{\gamma_Y} = \frac{\dot{Y}}{Y} = \gamma \cdot \frac{\dot{K}}{K} + (1-\gamma)\left(\frac{\dot{A}}{A} + \frac{\dot{L}}{L}\right) = \gamma\left(\underbrace{\frac{\dot{\hat{k}}}{\hat{k}}}_{0 \text{ in ss.}} + g + n\right) + (1-\gamma)(g+n) = \underline{g+n}$$

b) $\gamma_{\hat{k}} = \frac{d \ln \hat{k}}{dt} = s\hat{k}^{\gamma-1} - (\delta+g+n) = \underbrace{s e^{(\gamma-1)\ln \hat{k}} - (\delta+g+n)}_{F(x)}$

$x = \ln \hat{k} \;,\; x^* = \ln \hat{k}^*$

Taylor's expansion $\quad F(x) \approx F(x^*) + F'(x^*)(x - x^*)$

$F(x^*) = s e^{(\gamma-1)\ln \hat{k}^*} - (\delta+g+n) = 0$

$F'(x^*) = s e^{(\gamma-1)\ln \hat{k}^*} \cdot (\gamma-1)$

$$\gamma_{\hat{k}} \approx 0 + s(\gamma-1)e^{(\gamma-1)\ln \hat{k}^*}(\ln \hat{k} - \ln \hat{k}^*) = (\gamma-1)\underbrace{s(\hat{k}^*)^{\gamma-1}}_{= \delta+g+n} \cdot \ln \frac{\hat{k}}{\hat{k}^*} = -\underbrace{(1-\gamma)(\delta+g+n)}_{B} \ln \frac{\hat{k}}{\hat{k}^*}$$

c) $\underline{\sigma \text{ convergence}}$: dispersions of income (measured e.g. by standard deviation of the logarithm of income) across countries declines over time.

3. • HHs: $\max \int_0^\infty \ln c_t \, e^{-(\rho-n)t} \, dt$ (1)

s.t. $c_t + \dot{v}_t = w_t + (r_t - n) v_t$ (2)

$L_0 = 1, \quad v_0 > 0, \quad n > 0$

• final good $Y_i = L_i^{\beta} \sum_{j=1}^{A} x_{ij}^{1-\beta} \qquad 0<\beta<1$

• invention of intermediate nondurable good - fixed cost $\phi > 0$ of final goods

a) $H = \ln c_t + \lambda_t \left[w_t + (r_t - n) v_t - c_t \right]$

control c_t
state v_t
co-state λ_t

F.O.C. $[c_t]$: $\frac{\partial H}{\partial c} = 0 \qquad \frac{1}{c_t} - \lambda_t = 0 \qquad \lambda_t = \frac{1}{c_t}$ (3)

$[v_t]$: $-\frac{\partial H}{\partial v_t} + \lambda_t (\rho - n) = \dot{\lambda}_t \qquad \dot{\lambda}_t = -\lambda_t (r_t - n) + \lambda_t (\rho - n) = -\lambda_t (r_t - \rho)$ (4)

TVC: $\lim_{t \to \infty} e^{-(\rho-n)t} \lambda_t v_t = 0$

(3)+(4): $-\frac{\dot{c}_t}{c_t^2} = -\frac{1}{c_t}(r_t - \rho)$

$\underline{\frac{\dot{c}_t}{c_t} = r_t - \rho}$ Euler eq.

b) Final good's sector

$\max_{L_i, x_{ij}} \left(Y_i - wL_i - \sum_{j=1}^{A} p_j x_{ij} \right)$

$\max_{L_i, x_{ij}} \left(L_i^{\beta} \sum_{j=1}^{A} x_{ij}^{1-\beta} - wL_i - \sum_{j=1}^{A} p_j x_{ij} \right)$

FOC. $[L_i]$: $\beta L_i^{\beta-1} \sum_{j=1}^{A} x_{ij}^{1-\beta} - w = 0 \rightarrow w = \beta \cdot \frac{Y_i}{L_i} \rightarrow \underline{L_i = \frac{\beta}{w} Y_i}$

$[x_{ij}]$: $(1-\beta) L_i^{\beta} x_{ij}^{-\beta} - p_j = 0 \rightarrow p_j = (1-\beta) L_i^{\beta} x_{ij}^{-\beta} \rightarrow \underline{x_{ij} = \left(\frac{1-\beta}{p_j}\right)^{\frac{1}{\beta}} L_i} = \left(\frac{1-\beta}{p_j}\right)^{\frac{1}{\beta}} \frac{\beta}{w} Y_i$

c) Monopoly producer of intermediate good

$X_j = \sum_i x_{ij} = \sum_i \left(\frac{1-\beta}{p_j}\right)^{\frac{1}{\beta}} L_i = \left(\frac{1-\beta}{p_j}\right)^{\frac{1}{\beta}} \sum_i L_i = \underline{\left(\frac{1-\beta}{p_j}\right)^{\frac{1}{\beta}} L}$

$\downarrow$

$\log X_j = \frac{1}{\beta} \log \frac{1-\beta}{p_j} + \log L$

$\frac{d \log X_j}{d \log p_j} = -\frac{1}{\beta}$

$\max_{X_j} \left[X_j \, p_j(X_j) - X_j \right]$

FOC. $p_j + X_j \frac{dp_j}{dX_j} - 1 = 0$

$p_j \left(1 + \frac{X_j}{p_j} \cdot \frac{dp_j}{dX_j} \right) = 1$

$p_j \left(1 + \frac{1}{\frac{d \log X_j}{d \log p_j}} \right) = 1$

$$P_j(1-\beta) = 1$$

$$P_j = \frac{1}{1-\beta} \qquad x_j = \left(\frac{1-\beta}{\frac{1}{1-\beta}}\right)^{\frac{1}{\beta}} L = (1-\beta)^{\frac{2}{\beta}} L$$

d) $$\pi_j = P_j x_j - x_j = (P_j - 1)x_j = \left(\frac{1}{1-\beta} - 1\right)(1-\beta)^{\frac{2}{\beta}} L = \frac{\beta}{1-\beta}(1-\beta)^{\frac{2}{\beta}} L = \beta(1-\beta)^{\frac{2}{\beta}-1} L$$

Present Discounted Value V_t of returns:

$$V_t = \int_t^\infty \pi_j(s) e^{-\bar{r}(s-t)} ds = \int_t^\infty \pi_j(s) e^{-\int_t^s r_v dv} ds = \int_t^\infty \beta(1-\beta)^{\frac{2}{\beta}-1} L e^{-\int_t^s r_v dv} ds$$

$$\bar{r} = \frac{1}{s-t}\int_t^s r_v dv$$

Under free-entry: $V_t = \phi$

e) $v_t L_t = A_t V_t$

in EQ total asset holdings of population equal total value of the firms

f) $\boxed{n=0}$

i) $$\Phi = V_t = \int_t^\infty \beta(1-\beta)^{\frac{2}{\beta}-1} L e^{-\int_t^s r_v dv} ds \qquad (5)$$

$$\underset{\text{const.}}{\phi} = \underbrace{\beta(1-\beta)^{\frac{2}{\beta}-1}}_{\text{const.}} \underbrace{\int_t^\infty L(s) e^{-\int_t^s r_v dv} ds}_{\Downarrow \text{ const.}} \qquad / \frac{d}{dt}$$

$$0 = \int_t^\infty \frac{d}{dt}\left[L(s) e^{-\int_t^s r_v dv} ds\right] - 1 \cdot L_t e^{-\int_t^t r_v dv}$$

$$0 = \int_t^\infty L(s) \underbrace{\frac{d}{dt}\left(e^{-\int_t^s r_v dv}\right)}_{} ds - L_t$$

$$e^{-\int_t^s r_v dv} \frac{d}{dt}\left(-\int_t^s r_v dv\right) = e^{-\int_t^s r_v dv}\left(-\int_t^s 0\, dv + 1 \cdot r_t\right)$$

$$0 = r_t \int_t^\infty L(s) e^{-\int_t^s r_v dv} ds - L_t$$

if $n = 0$ $L_t = L = \text{const.}$

$$1 = r_t \int_t^\infty e^{-\int_t^s r_v dv} ds$$

$$\int_t^\infty e^{-\int_t^s r_v dv} ds = \frac{1}{r_t} \quad \rightarrow \text{plug to (5)}: \quad \Phi = \beta(1-\beta)^{\frac{2}{\beta}-1} L \cdot \frac{1}{r_t}$$

$$r_t = \frac{1}{\phi}\beta(1-\beta)^{\frac{2}{\beta}-1} L = r$$

ii) EE: $$\frac{\dot{c}_t}{c_t} = r_t - \rho = \frac{1}{\phi}\beta(1-\beta)^{\frac{2}{\beta}-1} L - \rho$$

$$Y = \sum_i Y_i = \sum_i L_i^\beta \sum_{j=1}^{A} x_{ij}^{1-\beta}$$

Intermediate firms charge same price $P_j = \frac{1}{1-\beta}$ $\rightarrow$ symmetrical $x_i \equiv x_{ij}$ $\rightarrow$ $Y_i = L_i^{\beta} A x_i^{1-\beta}$

Final good's firms identical too $\Rightarrow$ $\underline{Y = L^{\beta} A X^{1-\beta}}$, $/\ln, \frac{d}{dt}$

$$\frac{\dot{Y}}{Y} = \beta \cdot \frac{\dot{L}}{L} + \frac{\dot{A}}{A} + (1-\beta)\frac{\dot{X}}{X}$$

$$g_Y = \beta \cdot n + g_A + (1-\beta) g_X$$

Since $x_j = (1-\beta)^{\frac{2}{\beta}} L$ $\Rightarrow$ $g_X = n = 0$ $\Big\}$ $g_Y = g_A$

$$v_t L_t = A_t V_t \rightarrow \frac{\dot{v}_t}{v_t} + \underbrace{\frac{\dot{L}_t}{L_t}}_{=0} = \frac{\dot{A}_t}{A_t} + \underbrace{\frac{\dot{V}_t}{V_t}}_{0 \text{ because } V_t = \phi} \rightarrow g_v = g_A$$

From HHs BC: $\frac{\dot{v}_t}{v_t} = \frac{w_t}{v_t} + \underbrace{(r_t - n)}_{\text{const.}} - \frac{c_t}{v_t}$ $\rightarrow$ $g_c = g_w = g_v$

$$\underline{g_Y = g_c = \frac{1}{\phi}\beta(1-n)^{\frac{2}{\beta}-1} - \rho}$$

iii, <u>Market failures</u>

In monopolistic competition intermediate good's firms charge a price that is a markup over marginal costs $\rightarrow$ demand is $\downarrow$, less intermediate goods are produced $\rightarrow$ static inefficiency (lower output). This translates into the dynamic inefficiency: v is lower in the decentralized economy, so the growth is lower too.

iv, <u>SP problem</u>

$$\max \int_0^{\infty} L_0 \ln c_t \, e^{-(\rho - \overset{0}{n})t} \, dt$$

$$\text{s.t.} \quad C_t + \phi \dot{A}_t + A X = Y = L^{\beta} A_t X^{1-\beta} \rightarrow \dot{A}_t = \frac{1}{\phi}(L^{\beta} A_t X^{1-\beta} - A X - c_t L)$$

state: A_t
controls: C_t, X
co-state: μ

$$H = \ln c_t + \mu_t \frac{1}{\phi}(L^{\beta} A X^{1-\beta} - A X - c_t L)$$

FOC. $[c_t]$: $\frac{1}{c_t} - \mu_t \frac{1}{\phi} L = 0$ (6)

$[X]$: $(1-\beta) L^{\beta} A X^{-\beta} - A_t = 0$ $\rightarrow$ $X = (1-\beta)^{\frac{1}{\beta}} L$ (7)

$[A_t]$: $-\frac{\partial H}{\partial A_t} + \mu_t(\rho - \underset{0}{n}) = \dot{\mu}_t$ $\quad \dot{\mu}_t = -\mu_t \frac{1}{\phi}(L^{\beta} X^{1-\beta} - X) + \mu_t \cdot \rho$ (8)

TVC: $\lim_{t\to\infty} e^{-\rho t} \mu_t A_t = 0$

(6): $1 = \frac{1}{\phi} \mu_t c_t L$ $/\ln; \frac{d}{dt}$

$$0 = \frac{\dot{\mu}_t}{\mu_t} + \frac{\dot{c}_t}{c_t}$$

(8): $\frac{\dot{\mu}_t}{\mu_t} = -\frac{1}{\phi}(L^{\beta} X^{1-\beta} - X) + \rho$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\phi}(L^{\beta} X^{1-\beta} - X) - \rho = \frac{1}{\phi}\left[L^{\beta}(1-\beta)^{\frac{1-\beta}{\beta}} L^{1-\beta} - (1-\beta)^{\frac{1}{\beta}} L\right] - \rho =$$

$$= \frac{1}{\phi}(1-\beta)^{\frac{1}{\beta}} L \cdot \left[(1-\beta)^{\frac{1-\beta}{\beta} - \frac{1}{\beta}} - 1\right] - \rho = \frac{L}{\phi}(1-\beta)^{\frac{1}{\beta}}\left[(1-\beta)^{-1} - 1\right] - \rho =$$

$$= \frac{L}{\phi}(1-\beta)^{\frac{1}{\beta}}\left(\frac{1-1+\beta}{1-\beta}\right) - \rho = \underbrace{\frac{L}{\phi}(1-\beta)^{\frac{1}{\beta}-1}\beta}_{r_s} - \rho$$

Social return on savings / investment $r_s = \frac{L}{\phi}(1-\beta)^{\frac{1}{\beta}-1}\beta$

In decentralized economy $r_{DE} = \frac{L}{\phi}(1-\beta)^{\frac{2}{\beta}-1}\beta$

$\Bigg\}$ $r_s \neq r_{DE}$, $r_{DE} < r_s$

v) To reach social optimum we need to remove the inefficiency from monopolistic competition by providing subsidies on the intermediate good. This would raise X in DE to the optimum level X_{so} so that instantenous profits of intermediate good sector are:

$$\pi_j = \left(\frac{1}{1-\beta}-1\right)X_{so} = \left(\frac{1}{1-\beta}-1\right)(1-\beta)^{\frac{1}{\beta}}L = \underline{\beta(1-\beta)^{\frac{1}{\beta}-1}L}.$$

Similarly as in f) part i) we obtain $r_t = \frac{L}{\phi}\beta(1-\beta)^{\frac{1}{\beta}-1}$ => subsidy removed dynamic inefficiency, interest rate is in its optimal value

To finance subsidies Government should impose lump-sum taxes on HHs (to avoid distortionary effects of other types of taxation)

<u>HHs BC:</u> $\underline{c_t + \dot{v}_t = w_t + (r_t - n)v_t - \tau}$

<u>GNT BC:</u> $\tau L = \underbrace{\sum_{j=1}^{A} s P_j x_j}_{\text{subsidy}}$ -> due to symmetry $\underline{\tau L = sAPX_{so}} = sA \cdot \frac{1}{1-\beta}(1-\beta)^{\frac{1}{\beta}}L$

$$\tau = s(1-\beta)^{\frac{1}{\beta}-1}A$$

g) $\boxed{n>0}$

i) from f) part i):

$$0 = r_t \int_t^{\infty} L(s)e^{-\int_t^s r_v dv} ds - L_t$$

$$\int_t^{\infty} L(s)e^{-\int_t^s r_v dv} ds = \frac{L_t}{r_t}$$ plug back to (5)

$$\phi = \beta(1-\beta)^{\frac{2}{\beta}-1}\frac{L_t}{r_t}$$

$$\underline{r_t = \frac{1}{\phi}\beta(1-\beta)^{\frac{2}{\beta}-1}L_t}$$

since population is no longer const. => r_t is not const.

ii) Recall EE $\frac{\dot{c}_t}{c_t} = \frac{1}{\phi}\beta(1-\beta)^{\frac{2}{\beta}-1}L_t - \rho$

Per capita growth is increasing with size of population so the economy is growing faster and faster and will reach infinite consumption in finite time.