EXERCISE SESSION JULY 11, 2007

ROMER'S MODEL WITH MODIFIED PROD. FN. IN R&D (FINAL EXAM 2006)

a) FINAL GOOD SECTOR

$$\max \; Y_i - wL_i - \sum_{j=1}^{A} P_j X_{ij} \quad ; \quad Y_i = L_{Y,i}^{\beta} \sum_{j=1}^{A} X_{ij}^{1-\beta}$$

FOC: $\frac{\partial Y_i}{\partial L_i} = w \Rightarrow \beta \frac{Y_i}{L_{Y,i}} = w$

$$\boxed{L_{Y,i} = \frac{\beta}{w} Y_i}$$

$$\frac{\partial Y_i}{\partial X_{ij}} = P_j \Rightarrow L_{Y,i}^{\beta} (1-\beta) X_{ij}^{-\beta} = P_j$$

$$X_{ij}^{\beta} = \frac{1-\beta}{P_j} L_{Y,i}^{\beta}$$

$$X_{ij} = \left(\frac{1-\beta}{P_j}\right)^{\frac{1}{\beta}} L_{Y,i} \quad \forall j, i$$

$$\boxed{X_{ij} = \left(\frac{1-\beta}{P_j}\right)^{\frac{1}{\beta}} \frac{\beta}{w} Y_i} \quad \forall j, i$$

b) INTERMEDIATE GOOD SECTOR

$$\max \; P_j(X_j) X_j - X_j$$

FOC: $\frac{\partial P_j}{\partial X_j} X_j + P_j - 1 = 0$ $\rightsquigarrow$ we need to express $P_j(X_j)$ from the demand function from F-good sector

$$X_j = \sum_{i=1}^{A} X_{ij} = \sum_{i=1}^{A} \left(\frac{1-\beta}{P_j}\right)^{\frac{1}{\beta}} L_{Y,i} = \left(\frac{1-\beta}{P_j}\right)^{\frac{1}{\beta}} \sum_{i=1}^{A} L_{Y,i} = \left(\frac{1-\beta}{P_j}\right)^{\frac{1}{\beta}} L_Y$$

$$\Rightarrow X_j = \left(\frac{1-\beta}{P_j}\right)^{\frac{1}{\beta}} L_Y \quad \Rightarrow \quad P_j = (1-\beta) L_Y^{\beta} X_j^{-\beta}$$

$$\frac{\partial P_j}{\partial X_j} = -\beta(1-\beta) L_Y^{\beta} X_j^{-\beta-1}$$

↳ plug this into FOC

$$\underbrace{(1-\beta)L_Y^{\beta}(-\beta)\,x_j^{-\beta-1}}_{\frac{\partial P_j}{\partial x_j}}\,x_j + \underbrace{(1-\beta)L_Y^{\beta}x_j^{-\beta}}_{P_j} = 1$$

$$(1-\beta)\,L_Y^{\beta}(-\beta)\,x_j^{-\beta} + x_j^{-\beta}(1-\beta)\,L_Y^{\beta} = 1$$

$$x_j^{-\beta}\left[(1-\beta)L_Y^{\beta} - \beta(1-\beta)L_Y^{\beta}\right] = 1 \quad \Rightarrow \quad x_j^{\beta} = (1-\beta)^2 L_Y^{\beta}$$

$$\boxed{x_j = (1-\beta)^{\frac{2}{\beta}} L_Y}$$

plug into P_j

$$P_j = (1-\beta)\,L_Y^{\beta}\,x_j^{-\beta} = (1-\beta)\,\frac{L_Y^{\beta}}{(1-\beta)^2 L_Y^{\beta}} = \frac{1}{1-\beta} \quad \forall j$$

$$\Rightarrow \boxed{P_j = \frac{1}{1-\beta} \quad \forall j}$$

$\Rightarrow$ each I- good is sold for the same price

PROFIT :
$$\pi_j = (P_j - 1)\,x_j = \left(\frac{1}{1-\beta} - 1\right)(1-\beta)^{\frac{2}{\beta}} L_Y =$$

$$= \frac{1-(1-\beta)}{1-\beta}\,(1-\beta)^{\frac{2}{\beta}} L_Y =$$

$$= \beta\,(1-\beta)^{\frac{2}{\beta}-1}\,L_Y$$

$$\boxed{\pi_j = \beta(1-\beta)^{\frac{2-\beta}{\beta}} L_Y \quad \forall j}$$

c) R&D sector

$$\hat{\eta} = \frac{\eta}{L_R^{\lambda-1}A^{\varepsilon}} \quad , \quad \frac{\dot{L}}{L} = n$$

(i) $\dot{A} = \dfrac{L_R}{\hat{\eta}}$ $\rightarrow$ this is a production function of R&D

L_R - # labor working in R&D sector

$\hat{\eta}$ - # of labor necessary to invent new good

$\Rightarrow \dfrac{L_R}{\hat{\eta}}$ = how many new products can be invented

~~→ interpretation of $\hat{\xi} = \frac{\xi}{L_R^{\lambda-1} A^{\varepsilon}}$~~

A^{ε} → the more products are in the economy, the less labor is needed to invent something new, "standing on shoulders" - kind of using existing know-how

$L_R^{\lambda-1}$ → "stepping on toes" → after reaching certain ~~~~ level of productivity, we need less and less labor to work on one thing, thus it might happen that ~~~~ too many people are uselessly working on the same thing

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RATE OF TECHNOLOGICAL CHANGE:

$$\frac{\dot{A}}{A} = \frac{\frac{L_R}{\hat{\xi}}}{A} = \frac{L_R}{A\hat{\xi}} = \frac{L_R}{A \cdot \frac{\xi}{L_R^{\lambda-1} A^{\varepsilon}}} = \frac{L_R \cdot L_R^{\lambda-1} A^{\varepsilon}}{A \xi} = \frac{L_R^{\lambda} A^{\varepsilon-1}}{\xi}$$

along BGP, $\frac{\dot{A}}{A}$ = constant $\Rightarrow \frac{d}{dt}\left(\frac{\dot{A}}{A}\right) = 0$

$$\frac{d}{dt}\left(\frac{L_R^{\lambda} A^{\varepsilon-1}}{\xi}\right) = \lambda \underbrace{\left(\frac{\dot{L}_R}{L_R}\right)^*}_{= n} + (\varepsilon-1)\left(\frac{\dot{A}}{A}\right)^* = 0$$

$$\Rightarrow \left(\frac{\dot{A}}{A}\right)^* = \frac{\lambda n}{1-\varepsilon}$$

This is not a true endogenous growth model, because if there is no population growth, there will be endogenous growth.

(ii) $V(t) = \int_t^{\infty} \pi_j(s)\, e^{-\int_t^s r(v)\,dv}\, ds$ ... present discounted profit from blueprint

to derive no-arbitrage condition, we differentiate with respect to time

$$V(t) = \int_t^\infty \pi_j(s)\, e^{-\int_t^s r(r)dr}\, ds$$

$$\dot V(t) = \int_t^\infty \frac{d}{dt}\left(\pi_j(s)\, e^{-\int_t^s r(r)dr}\, ds\right) - 1 \cdot \pi_j(t)\, e^{-\int_t^t r(r)dr} =$$

$$= \int_t^\infty \pi_j(s) \underbrace{\frac{d}{dt}\left(e^{-\int_t^s r(r)dr}\right) ds}_{B} - \pi_j(t)$$

$$B = \frac{d}{dt}\left(e^{-\int_t^s r(r)dr}\right) = e^{-\int_t^s r(r)dr} \cdot \left[-\left(\int_t^s 0\, dr - 1\, r(t)\right)\right] =$$

$$= e^{-\int_t^s r(r)dr}\, r(t)$$

$\Rightarrow$

$$\dot V(t) = \int_t^\infty \pi_j(s)\, e^{-\int_t^s r(r)dr}\, r(t)\, ds - \pi_j(t) =$$

$$= r(t)\int_t^\infty \pi_j(s)\, e^{-\int_t^s r(r)dr}\, ds - \pi_j(t) =$$

$$= r(t)V(t) - \pi_j(t)$$

$$\Rightarrow \quad r(t) = \frac{\pi(t)}{V(t)} + \frac{\dot V(t)}{V(t)} \quad \Rightarrow \quad rV = \pi + \dot V$$

current profit

future capital gain

profit from not investing into R&D, but going to asset market

GROWTH RATE OF PROFITS:

from $*$): $\quad \pi_j = \beta(1-\beta)^{\frac{2-\beta}{\beta}} L_Y$

$$\frac{\dot\pi}{\pi} = \frac{d}{dt}\left(\ln \pi\right) = \frac{d}{dt}\left(\ln \beta(1-\beta)^{\frac{2-\beta}{\beta}} + \ln L_Y\right) = \frac{\dot L_Y}{L_Y} = n$$

because $\quad L = L_R + L_Y \quad \& \quad \frac{\dot L}{L} = n, \ \frac{\dot L_R}{L_R} = n \Rightarrow \frac{\dot L_Y}{L_Y} = n$

growth rate of V:

- from no-arbitrage condition : $\frac{\dot{V}(t)}{V(t)} = r(t) - \frac{\pi(t)}{V(t)}$

on BGP, growth rate of $V(t)$ is constant $\Rightarrow$ $r(t)$ is constant on BGP, and $\pi(t)$ & $V(t)$ grow at the same rate

$\Rightarrow \left(\frac{\dot{V}}{V}\right)^* = \left(\frac{\dot{\pi}}{\pi}\right)^* = n$

(iii) FREE - ENTRY CONDITION INTO R&D

means that any firm can pay the R&D sector costs ~~to secure the present value~~

free entry : $V(t) = \hat{\eta} \cdot w_R$

$V(t)$ ↓ PV of blueprint (in units of F-good)

$\hat{\eta}$ → quantity of labor needed to invent new product; thus we need to multiply it by w_R to have it in units of F-good

(iv) $l_R + l_Y = 1$ ... time allocation of one household

$L_R = l_R L$ .... aggregate labor in R&D and F-good sector

$L_Y = l_Y L$

household's problem :

$$\max_{c_t} \int_0^\infty e^{-(\rho-n)t} u(c_t)\, dt$$

s.t.

$$c_t + \dot{a}_t = w_R l_R + w_Y l_Y + r a_t - n a_t$$

In closed economy, the total assets must be equal to total value of firms:

$$aL = VA \quad \Rightarrow \quad a = \frac{VA}{L}$$

$$\Downarrow$$

$$\dot{a}L + \dot{L}a = \dot{V}A + \dot{A}V$$

$$\dot{a} = \frac{\dot{V}A}{L} + \frac{\dot{A}V}{L} - \frac{\dot{L}}{L}a$$

$$\dot{a} = \frac{\dot{V}}{V}\cdot\frac{VA}{L} + \dot{A}\frac{V}{L} - n\cdot\frac{VA}{L}$$

$\downarrow$

plug this into BC of HH

further we use that $w_R = w_Y$, otherwise people would work only in the sector with higher wage

BC:

$$c_t + \left[\frac{\dot{V}}{V}\cdot\frac{VA}{L} + \dot{A}\frac{V}{L} - n\frac{VA}{L}\right] = w(l_R + l_Y) + (r - n)\frac{VA}{L}$$

$$c_t + \dot{A}\frac{V}{L} = w + \frac{VA}{L}\left(r - n + n - \frac{\dot{V}}{V}\right)$$

$\uparrow$ use no-arbitrage con.

$$c_t + \dot{A}\frac{V}{L} = w + \frac{VA}{L}\cdot\frac{\pi}{V}$$

$$c_t + \underbrace{\dot{A}\frac{V}{L}}_{\text{investment into new firms}} = w + \underbrace{\frac{A\pi}{L}}_{\text{current profit of all firms divided among all people}}$$

(investment into 1 firm is V
number of new firms is $\dot{A}$)

(v) MARKET FAILURES

- I-good sector is monopolized, monopol works inefficiently, it produces less goods for higher price

- in R&D sector, too many workers work on the same thing which increases the cost of invention
  → NEGATIVE EXTERNALITY

- in R&D sector, the more I-goods exist, the lower are costs on inventing new ideas
  → POSITIVE EXTERNALITY

→ the decentralized economy considers L_R & A in ~~cost~~ cost function to be constant, while social planner takes into account future spillovers

(vi) SOCIAL PLANNER

$$\max \int_0^\infty u(c_t)\, e^{-(\rho-n)t}\, dt$$

s.t.

$$Y = C + AX$$

production of F-good; consumption; F-good needed to produce I-good

$$Y = L_Y^{\beta} AX^{1-\beta}$$

$$\dot{A} = \frac{L_R}{\hat{\eta}} = \frac{L_R}{\dfrac{\eta}{L_R^{\lambda-1} A^{\varepsilon}}} = \frac{L_R^{\lambda} A^{\varepsilon}}{\eta}$$

$$L_R + L_Y = L$$

controls : C, L_R, L_Y, X

states : A

NOTE: We can verify the SP's budget constraint by eliminating w from BC of household.

from zero profit condition in F-good sector:

$$Y_i - wL_{Y,i} - \sum_j x_{ij} P_j = 0 \qquad / \sum_i$$

$$\underbrace{\sum Y_i}_{Y} - w \underbrace{\sum_i L_{Y,i}}_{L_Y} - \sum_j \sum_i x_{ij} P_j = 0$$

$$wL_Y = Y - \sum_j P_j \underbrace{\sum_i x_{ij}}_{x_j} = Y - \sum_j P_j x_j \qquad (1)$$

from I-good sector

$$\pi_j = P_j x_j - x_j \qquad / \sum_j$$

$$\sum_j \pi_j = \sum P_j x_j - \sum_j x_j \qquad \text{since } \begin{array}{l} x_j \equiv x \quad \forall j \\ \pi_j \equiv \pi \quad \forall j \end{array}$$

$$A\pi = \sum_j P_j x_j - Ax$$

↳ plug into (1)

$$wL_Y = Y - (A\pi + Ax)$$

↳ plug into BC of HH

$$c + \dot{A}\frac{V}{L} = w + \frac{A\pi}{L} \qquad /\cdot L$$

$$C + \dot{A}V = wL + A\pi \qquad (wL = wL_Y + wL_R)$$

$$C + \dot{A}V = (Y - A\pi - Ax) + wL_R + A\pi$$

$$C + \dot{A}V = Y - Ax + wL_R$$

$$C + Ax + \underbrace{(\dot{A}V - wL_R)}_{=0 \text{ (from zero profit in R\&D)}} = Y \quad \Rightarrow \quad \boxed{C + Ax = Y}$$

because $\dot{A}V - wL_R = \frac{L_R}{\eta} \cdot V - wL_R = L_R\left(\frac{V}{\eta} - w\right) = 0$