R-C-K Model with CARA util. fn. (midterm 2007)

a) $u(c) = -\frac{1}{\alpha} e^{-\alpha c}$, $\alpha > 0$

- increasing in c : $u'(c) = -\frac{1}{\alpha} e^{-\alpha c} \cdot (-\alpha) = e^{-\alpha c} > 0$
- concave $u''(c) = -\alpha e^{-\alpha c} < 0$

$$\theta = -\frac{\frac{\partial u'(c)}{u'(c)}}{\frac{\partial c}{c}} = -\frac{\partial u'(c)}{\partial c} \cdot \frac{c}{u'(c)} = -\frac{u''(c) \cdot c}{u'(c)}$$

$$\theta_t = -\frac{-\alpha e^{-\alpha c_t} \cdot c_t}{e^{-\alpha c_t}} = \underline{\underline{\alpha c_t}} \qquad (1)$$

b) HH max. problem

$$\max_{c_t} \int_0^\infty -\frac{1}{\alpha} e^{-\alpha c_t} \cdot e^{-\rho t} \, dt \qquad (2)$$

s.t. BC : $\dot{a}_t = r_t(1-\tau) a_t + w_t - c_t$ (3)

+ NPG : $\lim_{t\to\infty} a_t e^{-\int_0^t r(s) ds} \geq 0$

$a_0 > 0$; $\alpha, \rho > 0$; $\tau \in (0;1)$

$a_t = k_t + b_t$

a_t - state

c_t - control

c) Hamiltonian : $H = -\frac{1}{\alpha} e^{-\alpha c_t - \rho t} + \lambda_t \left[r_t(1-\tau) a_t + w_t - c_t \right]$ (4)

FOC's : $\frac{\partial H}{\partial c_t} = 0$ $\quad e^{-\alpha c_t - \rho t} = \lambda_t$ (5)

$\frac{\partial H}{\partial a_t} = -\dot{\lambda}_t$ $\quad r_t(1-\tau) \cdot \lambda_t = -\dot{\lambda}_t$ (6)

TVC : $\lim_{t\to\infty} \lambda_t a_t = 0$

d) log (5) : $-\alpha c_t - \rho t = \ln \lambda_t \quad / \frac{\partial}{\partial t}$

$$-\alpha \dot{c}_t - \rho = \underbrace{\frac{\dot{\lambda}_t}{\lambda_t} = -r_t(1-\tau)}_{(6)}$$

EE : $$\frac{\dot{c}_t}{c_t} = \frac{1}{\alpha c_t}\left[r_t(1-\tau) - \rho \right] = \frac{1}{\theta_t}\left[r_t(1-\tau) - \rho \right] \qquad (7)$$

or $$\underbrace{r_t(1-\tau)}_{\text{after-tax rate of return}} = \underbrace{\rho}_{\text{rate of time preference}} + \underbrace{\theta_t \frac{\dot{c}_t}{c_t}}_{\text{rate of return "to consumption"}}$$

- ↑ after-tax rate of return relative to the rate of time preference, the more it pays to depress the current level of c to enjoy ↑ c later

e) Firm's profit max. problem

C-D prod. fn: $Y_t = A \cdot K_t^{\beta} L_t^{1-\beta}$; $\beta \in (0;1)$

$$y_t = A k_t^{\beta} \qquad (8)$$

$$\max \pi_i = \max_{K_{t,i}} A k_{t,i}^{\beta} - (r_t + \delta) k_{t,i} - w_{t,i}$$

FOC : $r_t + \delta = \beta A k_t^{\beta - 1}$

$$\underline{r_t = \beta A k_t^{\beta-1} - \delta} \qquad (9)$$

$$\underline{w_t} = A k_t^{\beta} - (\beta A k_t^{\beta-1} - \delta + \delta) k_t = \underline{(1-\beta) A k_t^{\beta}} \qquad (10)$$

Plug in EE: $$g_c = \frac{\dot{c}_t}{c_t} = \frac{1}{\theta_t}\left[(1-\tau)(\beta A k_t^{\beta-1} - \delta) - \rho \right] \qquad (11)$$

BC: $$\dot{k}_t = (1-\tau)(\beta A k_t^{\beta-1} - \delta) k_t + (1-\beta) A k_t^{\beta} - c_t = (1 - \tau\beta) A k_t^{\beta} - (1-\tau)\delta k_t - c_t \qquad (12)$$

At S-S. $\dot{c}_t = 0$, $\dot{k}_t = 0$ $\rightarrow$ $(1-\tau)(\beta A k^{*\,\beta-1} - \delta) - \rho = 0$

$$\underline{k^*(\tau) = \left[\frac{\beta A}{\frac{1}{1-\tau}\rho + \delta} \right]^{\frac{1}{1-\beta}}}$$

$\uparrow \tau \Rightarrow \downarrow k^*$

$\downarrow g_c$

f)

$\dot{k}_t$ if $\tau = 0$: $\dot{k}_t = A k_t^{\beta} - \delta k_t - c_t$

$\dot{k}_t$ in general : $\dot{k}_t = A k_t^{\beta} - \delta k_t - c_t \underbrace{- \tau\beta A k_t^{\beta} + \tau\delta k_t}_{?\ \ominus \text{ in SS}}$

$\Big\}$ $c^*(\tau) < \underset{\substack{\uparrow \\ \tau = 0}}{c^*}(\tau=0)$

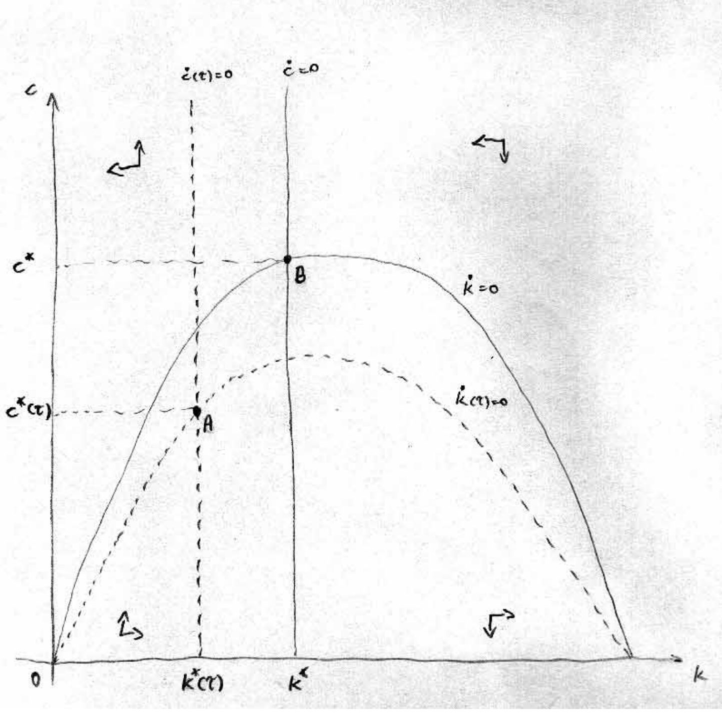
$$-\tau\beta A k_t^{\beta} + \tau\delta k_t < 0$$

$$\delta k_t < \beta A k_t^{\beta}$$

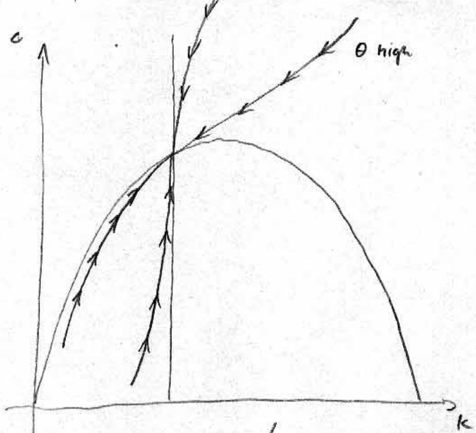
$$\frac{\delta}{\beta A} < k_t^{\beta-1}$$ pluging SS value

$$\frac{\delta}{\cancel{\beta A}} < \frac{\frac{1}{1-\tau}\rho + \delta}{\cancel{\beta A}}$$

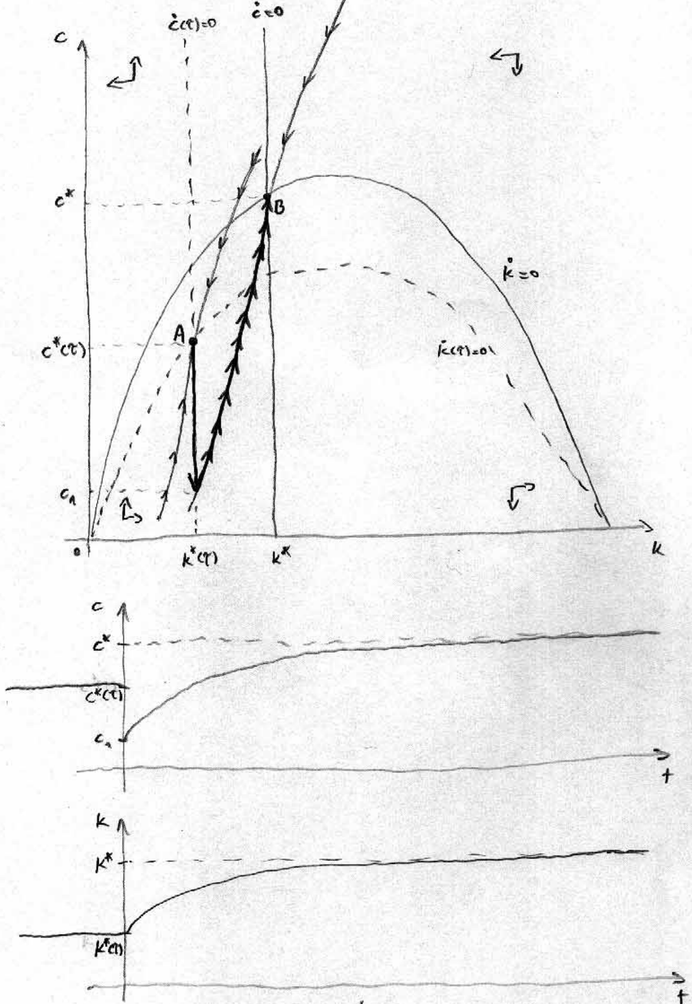
$$0 < \frac{\rho}{1-\tau} \qquad \uparrow$$

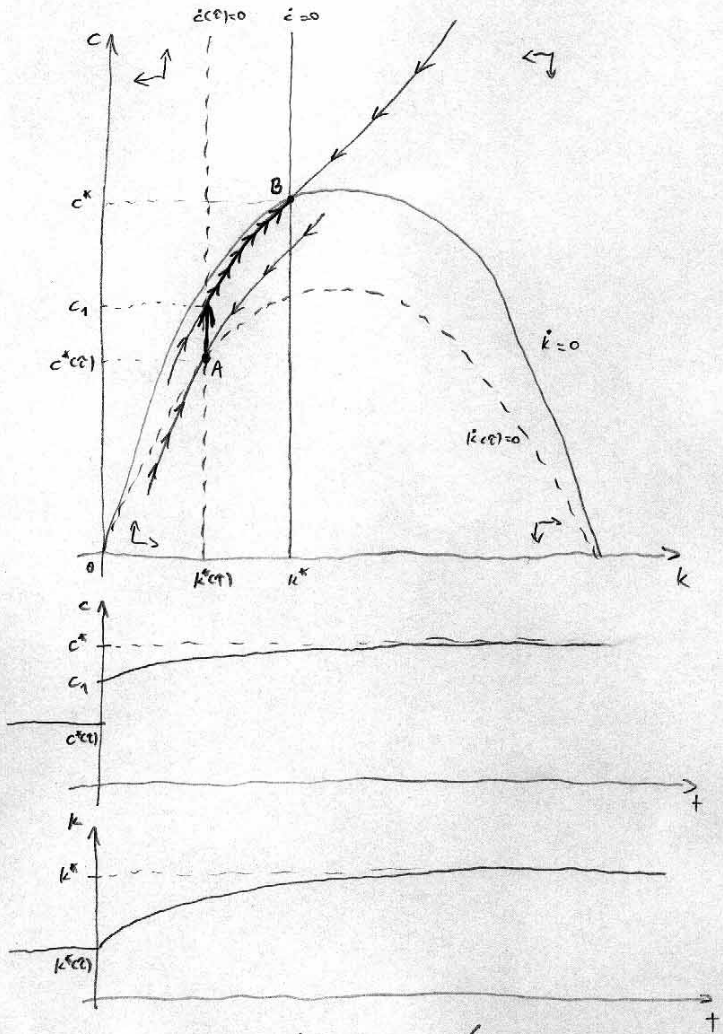


$\theta_t = \alpha c_t$

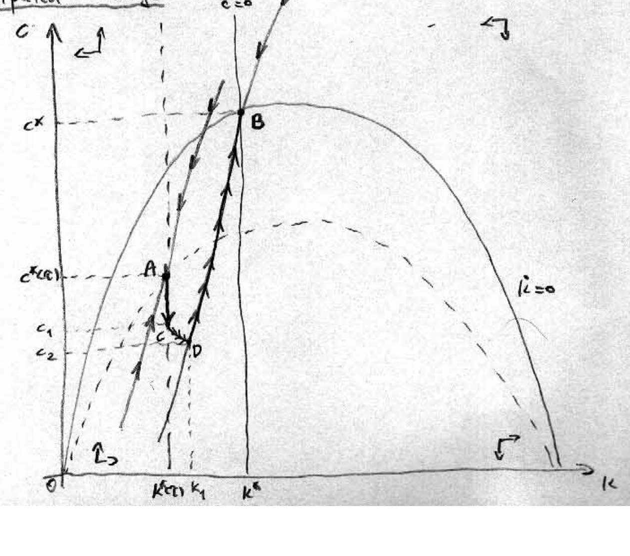


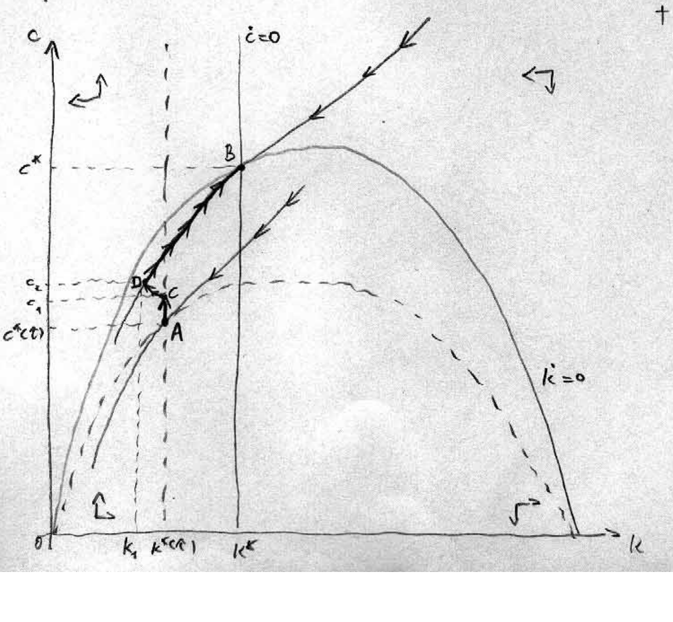
Unexpected change

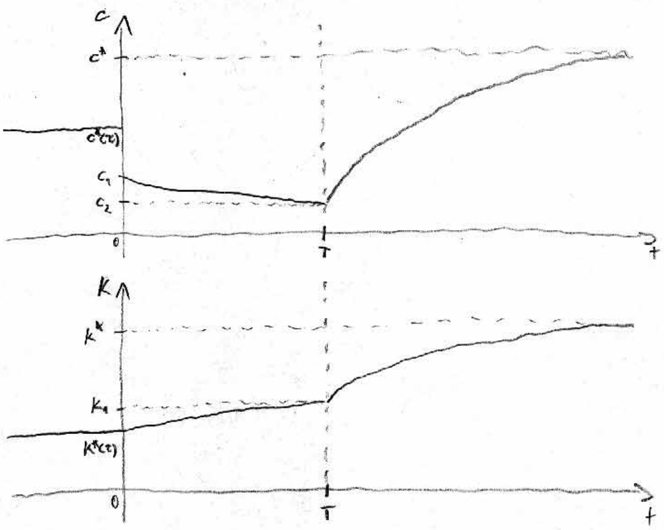


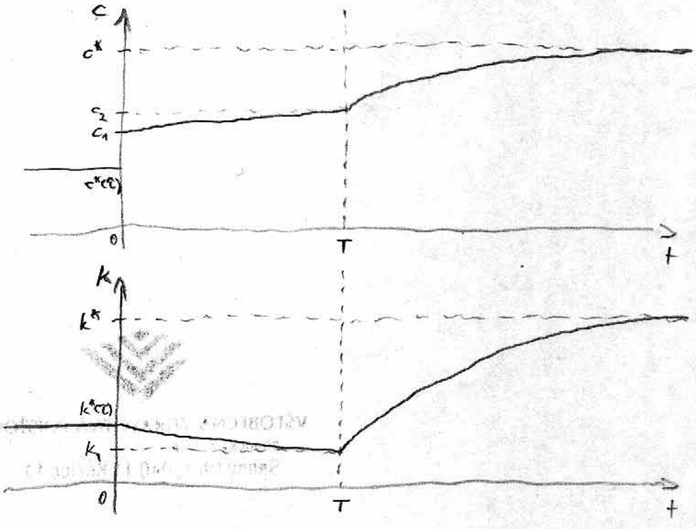


Anticipated change:









8) BC: $\dot{a}_t = r_t(1-\tau)a_t + w_t - c_t + f_t$

lump-sum transfer $\quad f_t = \tau r_t a_t$

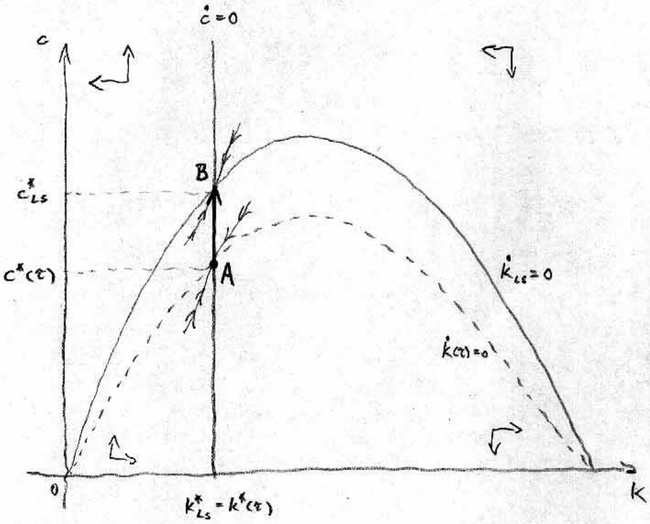
EE stays the same: $\dfrac{\dot{c}_t}{c_t} = \dfrac{1}{\theta_t}\left[r_t(1-\tau) - \rho \right]$

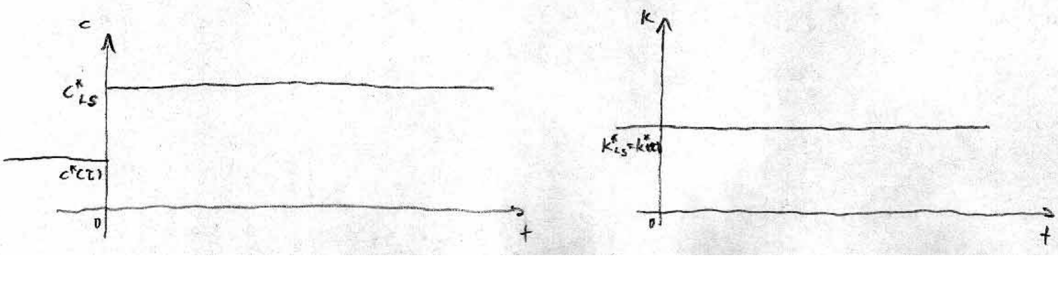
$$\dot{c}_t = 0 \quad \rightarrow \quad k^*_{LS} = \left[\frac{\beta A}{\frac{1}{1-\tau}\rho + \delta} \right]^{\frac{1}{1-\beta}} = k^*(\tau)$$

locus $\dot{k} = 0$ shifts:

$$\dot{k}_t = (1-\tau)(\beta A k_t^{\beta-1} - \delta)k_t + (1-\beta)A k_t^{\beta} - c_t + f_t = \underline{A k_t^{\beta} - \delta k_t - c_t}$$

$$f_t = \tau r_t k_t = \tau(\beta A k_t^{\beta-1} - \delta)k_t$$





AK Model with Physical & Human Capital

a) $Y = K^\alpha (hL)^{1-\alpha} \quad / : L$

$y = k^\alpha h^{1-\alpha}$

$\dot{K} = s_K Y - \delta K \quad / : L$

$\left(\frac{\dot{K}}{L}\right) = s_K \cdot y - \delta k \qquad \dot{k} = \left(\frac{\dot{K}}{L}\right) = \frac{\dot{K}L - K\dot{L}}{L^2} = \left(\frac{\dot{K}}{L}\right) - \underbrace{\frac{K}{L} \cdot \frac{\dot{L}}{L}}_{k \cdot n}$

$\dot{k} = s_K y - (\delta + n) k$

$\dot{H} = s_H \cdot Y - \delta H$

$\dot{h} = s_H y - (\delta + n) h$

growth rates: $\gamma_k = \frac{\dot{k}}{k} = s_K \cdot \frac{y}{k} - (\delta + n)$

$\gamma_h = \frac{\dot{h}}{h} = s_H \cdot \frac{y}{h} - (\delta + n)$

Along balanced growth path (BGP) γ_k & γ_h are const. => ratios $\frac{y}{k}$ & $\frac{y}{h}$ are const.

$\gamma_k + \delta + n = s_K \cdot \frac{y}{k}$

$\log(\gamma_k + \delta + n) = \log s_K + \log y - \log k \quad / \frac{\partial}{\partial t}$

$0 = \frac{\dot{y}}{y} - \frac{\dot{k}}{k}$

$\gamma_y = \gamma_k$

similarly $\gamma_y = \gamma_h$ $\Big\}$ $\gamma_y = \gamma_k = \gamma_h$

$s_K \frac{y}{k} - \cancel{(\delta + n)} = s_H \cdot \frac{y}{h} - \cancel{(\delta + n)}$

$\frac{h}{k} = \frac{s_H}{s_K}$

$\gamma_y = s_K \cdot \frac{y}{k} - (\delta + n) = s_K \cdot \left(\frac{h}{k}\right)^{1-\alpha} - (\delta + n) = s_K \cdot \left(\frac{s_H}{s_K}\right)^{1-\alpha} - (\delta + n) = \underline{\underline{s_K^\alpha s_H^{1-\alpha} - (\delta + n)}}$

prod. fn

b) $y = k^\alpha h^{1-\alpha} = \left(\frac{h}{k}\right)^{1-\alpha} k = \left(\frac{s_H}{s_K}\right)^{1-\alpha} k = Ak$

$Y = AK \quad , \quad \underline{\underline{A = \left(\frac{s_H}{s_K}\right)^{1-\alpha}}}$

c) • in the decentralized economy we would have households which carry out production directly and maximize their utility w.r.t chosen level of consumption and investment into both types of capital

• need externalities to enforce exogenously given saving rates

• if we endogenize the determination of s_K & s_H, the decentralized equilibrium will no longer be socially optimal (saving rate will become dependent on the risk aversion coefficient of the agents and would be smaller than optimal).

d) see also Barro, 5.1.2 p. 175

• FOC's imply net MPK = net MPH $\Rightarrow \frac{K}{H} = \frac{\alpha}{1-\alpha}$

• If economy starts with such $K(0), H(0)$ that $\frac{K(0)}{H(0)} \neq \frac{\alpha}{1-\alpha}$ => need simultaneous ⊖ and ⊕ jumps in both types of stock. This is not possible when imposing inequality restrictions $I_H \geq 0$, $I_K \geq 0$.

• ex. $\frac{K(0)}{H(0)} < \frac{\alpha}{1-\alpha}$ -> H is abundant relative to K, we would like to ↓ H by discrete amount, but $I_H \geq 0$ is binding => $I_H = 0$ and $\gamma_H = \frac{\dot{H}}{H} = -\delta$ (excessive capital depreciate at exog. given rate δ)

• gradually $\frac{K}{H}$ increases until it reaches optimal value $\frac{\alpha}{1-\alpha}$

H is declining at the rate δ
K is increasing (at a rate that decreases toward $\gamma^* > 0$)
growth rate at S-S.

> Economy decreases monotonically from the start until the moment when $\frac{K}{H}$ is optimal, afterwards all variables (Y, K, H) grow at the same rate γ^*. Economy starts from higher growth rate than in EQ due to imbalance of production factors.

• imbalance effect: