R-C-K Model with GNT purchases in util. fn.

a) HH max. problem:

$$\max_{c_t} \int_0^\infty \frac{(c_t+g_t)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} dt \qquad (1)$$

s.t. BC: $\dot{a}_t = r_t a_t + w_t - n a_t - (1+\tau_c) c_t$ (2)

NPG condition: $\lim_{t\to\infty} a_t e^{-\int_0^t (r(s)-n)ds} \geq 0$ (3)

$a_0 > 0$; $\theta, \rho > 0$; $\rho > n$; $\tau_c \in (0;1)$

$a_t = k_t + b_t$ (4)

loans to finance cons.

- constraint on the amount of borrowing
- PV of assets must be asymptotically nonnegative
- in the long-run, HH's debt per person (negative values of a_t) cannot grow as fast as $r(t)-n$
Level of debt A_t $r(t)$

b) Hamiltonian:

$$H = \frac{(c_t+g_t)^{1-\theta}}{1-\theta} e^{-(\rho-n)t} + \lambda_t \left[w_t + (r_t-n)a_t - (1+\tau_c)c_t \right] \qquad (5)$$

a_t - state
c_t - control

FOC.:

$$\frac{\partial H}{\partial c_t} = 0 \quad : \quad (c_t+g_t)^{-\theta} e^{-(\rho-n)t} = \lambda_t (1+\tau_c) \qquad (6)$$

$$\frac{\partial H}{\partial a_t} = -\dot{\lambda}_t \quad : \quad (r_t - n)\lambda_t = -\dot{\lambda}_t \qquad (7)$$

TVC: $\lim_{t\to\infty} \lambda_t a_t = 0$ (8)

c) GNT's flow BC: $g_t = \tau_c c_t \quad \forall t$ (9)

No need to specify NPG condition for GNT since GNT needs to maintain balanced budget every period. Moreover, GNT purchases are solely financed by consumption tax, GNT does NOT need to borrow money to finance expenditures.

d) Because $g_t = \tau_c c_t \quad \forall t \Rightarrow$ plug to FOC (6) :

$$\left[(1+\tau_c)c_t\right]^{-\theta} e^{-(\rho-n)t} = \lambda_t (1+\tau_c)$$

$$(1+\tau_c)^{-\theta-1} c_t^{-\theta} e^{-(\rho-n)t} = \lambda_t$$

$$\ln \lambda_t = (-\theta-1)\ln(1+\tau_c) - \theta \ln c_t - (\rho-n)t \qquad / \frac{d}{dt}$$

$$\frac{\dot{\lambda}_t}{\lambda_t} = -\theta \frac{\dot{c}_t}{c_t} - (\rho-n) \tag{10}$$

(10) + (7) : $\quad -(r_t - n) = -\theta \frac{\dot{c}_t}{c_t} - (\rho-n)$

Euler eq. $$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[r_t - \rho\right] \tag{11}$$

$$r_t = \rho + \theta \cdot \frac{\dot{c}_t}{c_t}$$

rate of return to saving; rate of time preference; rate of return "to consumption"

- In optimizing environment EULER EQ. says that HHs equate rates of return (return on assets, rate of time preference , return on shifting future consumption to present period) so HHs are indifferent at the margin between consuming and saving.
- Higher the rate of return to saving relative to the rate of time preference, the more it pays to depress the current level of c to enjoy ↑c later.

e) Firm's profit max. problem

$y_t = Ak_t^{\alpha} \qquad 0<\alpha<1$

Capital depreciates at $\delta>0$

$$\max \Pi_i = \max_{k_{t,i}} Ak_{t,i}^{\alpha} - (r_t+\delta)k_{t,i} - w_{t,i}$$

Representative firm - omit index i

FOC. $[k_t]$: $\quad r_t + \delta = \alpha A k_t^{\alpha-1}$

$$r_t = \alpha A k_t^{\alpha-1} - \delta \tag{12}$$

In competitive market : w set such that $\Pi = 0 \Rightarrow w_t = Ak_t^{\alpha} - \underbrace{(r_t+\delta)}_{\alpha A k_t^{\alpha-1}} k_t = (1-\alpha)Ak_t^{\alpha}$ (13)

$b_t = 0 \quad \forall t \Rightarrow a_t = k_t$ pluging to HH's BC (2):

$$\dot{k}_t = (\alpha A k_t^{\alpha-1} - \delta)k_t + (1-\alpha)Ak_t^{\alpha} - nk_t - (1+\tau_c)c_t = Ak_t^{\alpha} - (n+\delta)k_t - (1+\tau_c)c_t \tag{14}$$

(11): $$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\alpha A k_t^{\alpha-1} - \delta - \rho\right] \tag{15}$$

+ TVC : $$\lim_{t\to\infty} k_t e^{-\int_0^t (r(s)-n)ds} = 0 \quad ; \quad k_0 > 0 \tag{16}$$

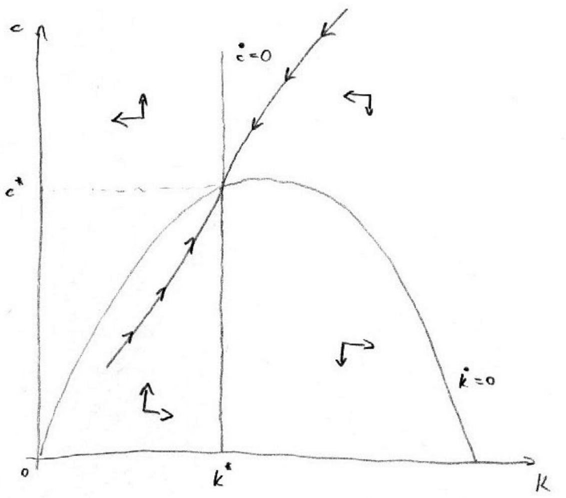
Competitive market EQ is characterized by (14) - (16).

f) at s.-s.: $\dot{c}_t = 0$
$\dot{k}_t = 0$

(15): $\alpha A k^{*\alpha-1} - \delta - \rho = 0 \quad \rightarrow \quad k^* = \left(\frac{\alpha A}{\delta+\rho}\right)^{\frac{1}{1-\alpha}}$ ← just fn. of parameters (17)

(14): $A k^{*\alpha} - (n+\delta)k^* - (1+\tau_c)c^* = 0$

$$c^*(\tau_c) = \frac{1}{1+\tau_c}\left[A k^{*\alpha} - (n+\delta)k^*\right] \qquad (18)$$



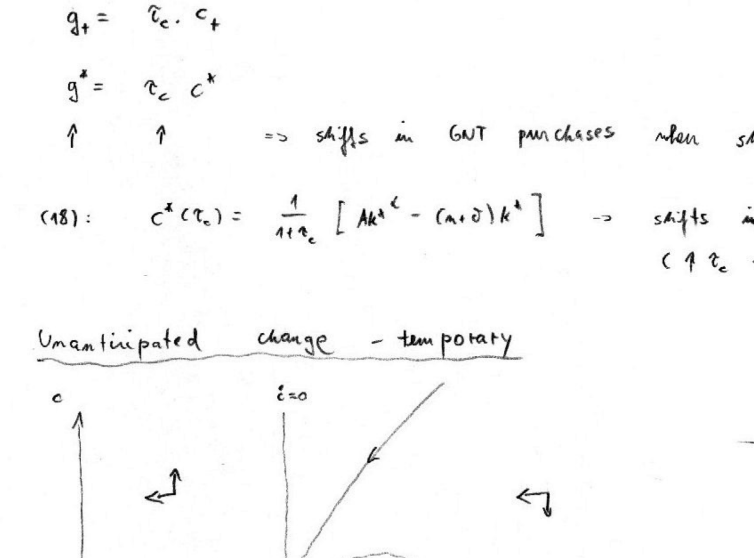
g) $g_t = \tau_c \cdot c_t$

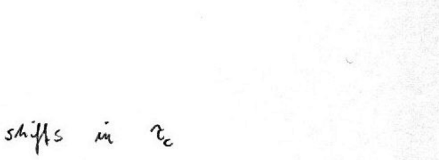
$g^* = \tau_c \; c^*$

↑ ↑ => shifts in GVT purchases when shifts in τ_c

(18): $c^*(\tau_c) = \frac{1}{1+\tau_c}\left[A k^{*\alpha} - (n+\delta)k^*\right]$ → shifts in τ_c just affect the "height" of $\dot{k}=0$ locus ($\uparrow \tau_c \rightarrow \downarrow c^*$)

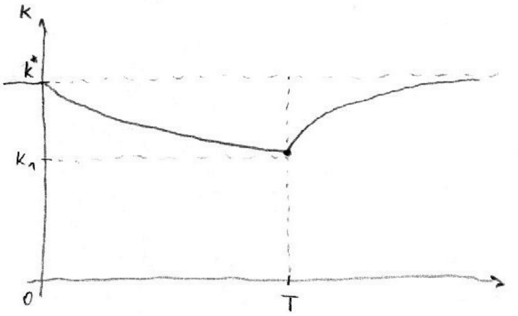
Unanticipated change - temporary

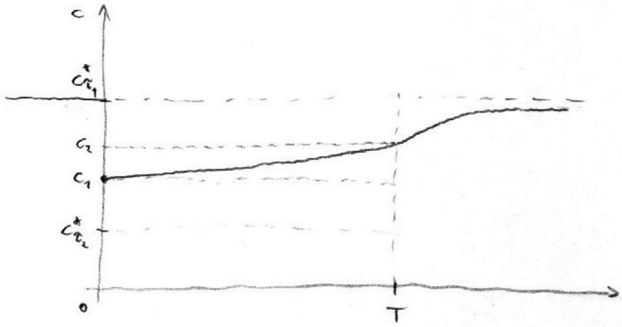




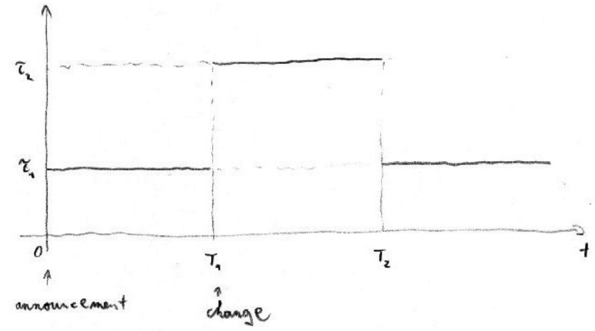
$\dot{c}=0$ unaffected
$\dot{k}_{\tau_2}=0$ lies below $\dot{k}_{\tau_1}=0$

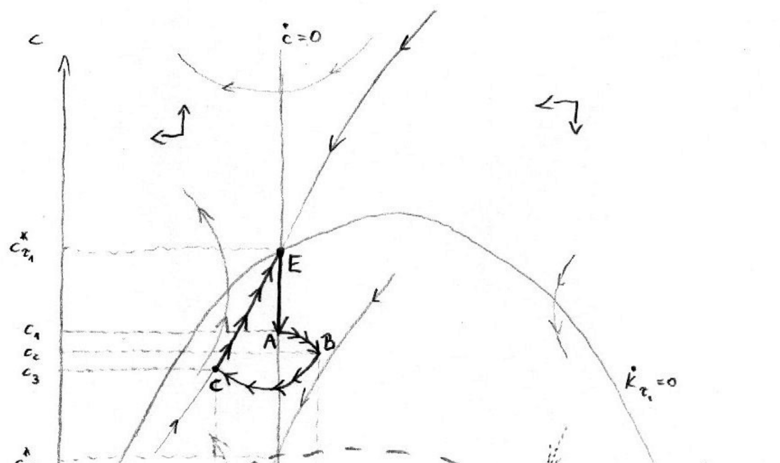
- c cannot jump at T. Agents know ahead of time that higher taxation will end (discontinuous jump in c would be inconsistent with cons.-smoothing behavior implied by HH's intertemporal optimization). Thus for economy to return to a balanced growth path, it must be somewhere on the original saddle path at T.
- at $t=0$ c jumps to point A
- between time $(0, T)$ paths under taxation τ_2 are valid, $c\uparrow$, $k\downarrow$
- at T returning back to original saddle path (point B), $c\uparrow$, $k\uparrow$; returning to E





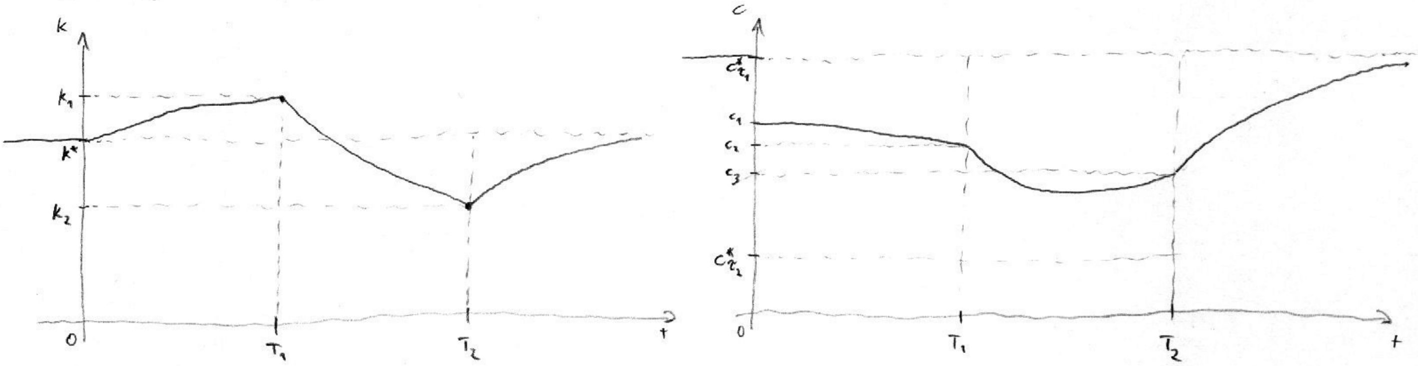
<u>Anticipated change - announced temporary change</u>





- c cannot jump in T_1, T_2. Everyone knows ahead of time that higher tax will be implemented at T_1 and lowered back at T_2. At T_2 economy must be somewhere on the original saddle path.
- at $t=0$ c jumps to a point like A

- between $\langle 0, T_1 \rangle$ old locus $\dot{k}_{\tau_1} = 0$ still holds, so we follow old right-down arrows to a point like B (above new saddle path; B cannot be on or below new saddle path because system wouldn't be able converge back to old saddle path at T_2); $c\downarrow, k\uparrow$
- between $\langle T_1, T_2 \rangle$ new locus $\dot{k}_{\tau_2} = 0$ holds, moving exactly to point C at T_2 which lies on old saddle path. $k\downarrow$; c is firstly $\downarrow$, when crossing $\dot{c} = 0$ starts to rise
- after T_2 converging back to point E; $c\uparrow, k\uparrow$



h) Social Planner problem

$$\max_{c_t, g_t} \int_0^\infty \frac{(c_t + g_t)^{1-\theta}}{1-\theta} e^{-(\rho - n)t} \, dt \qquad (19)$$

s.t. $Y_t = C_t + I_t + G_t \qquad [G_t = \tau_c \cdot C_t]$

$$I_t = \dot{K}_t + \delta K_t$$

$$Y_t = C_t + \dot{K}_t + \delta K_t + G_t \qquad / : N_t$$

$$y_t = c_t + \frac{\dot{K}_t}{N_t} + \delta k_t + g_t \qquad \dot{k}_t = \left(\frac{\dot{K_t}}{N_t}\right) = \frac{\dot{K}_t N_t - \dot{N}_t K_t}{N_t^2} = \frac{\dot{K}_t}{N_t} - \frac{\dot{N}_t}{N_t} \cdot \frac{K_t}{N_t} = \frac{\dot{K}_t}{N_t} - n k_t$$

$$y_t = c_t + \dot{k}_t + n k_t + \delta k_t + g_t$$

BC: $\dot{k}_t = y_t - (n+\delta) k_t - c_t - g_t$

$$\dot{k}_t = A k_t^\alpha - (n+\delta) k_t - c_t - g_t \qquad (20)$$

Hamiltonian: $H = \frac{(c_t + g_t)^{1-\theta}}{1-\theta} e^{-(\rho - n)t} + \lambda_t \left[A k_t^\alpha - (n+\delta) k_t - c_t - g_t \right]$

FOC:

$$\frac{\partial H}{\partial c_t} = 0 \qquad (c_t + g_t)^{-\theta} e^{-(\rho - n)t} = \lambda_t$$

$$\frac{\partial H}{\partial g_t} = 0 \qquad (c_t + g_t)^{-\theta} e^{-(\rho - n)t} = \lambda_t \qquad (21)$$

$$\frac{\partial H}{\partial k_t} = -\dot{\lambda}_t \qquad \left[\alpha A k_t^{\alpha - 1} - (n+\delta) \right] \lambda_t = -\dot{\lambda}_t$$

3 unknowns, but only 2 equations
-> indeterminacy between the choice of c_t & g_t

TVC $\lim_{t \to \infty} \lambda_t k_t = 0$

c_t, g_t -> perfect substitutes

when $g_t = 0 \ \forall t$ -> $\{c_t^o\}_{t=0}^\infty$ path of opt. consumption if $g_t = 0\ \forall t$ -> problem collapses to the standard problem without distortions (no tax, no GNT purchases)

- Any choice (c_t, g_t) that satisfy (21) and $g_t \leq c_t^\circ\ \forall t$ delivers the same utility as in case without taxes. => is a <u>social optimum</u>
- Competitive market EQ (decentralized version) is socially optimal.
- Lump-sum tax will not lead to change in social optimality of solution because the consumption tax already delivers socially optimal solution.
- Assumption of perfect substitutability between c_t & g_t - crucial role since HHs are indifferent between gaining utility from c_t or g_t

i) $$u(c,g) = \frac{(c^\gamma g^{1-\gamma})^{1-\theta}}{1-\theta}$$ elasticity of substitution between $c, g = 1$

$0 < \gamma < 1$

$\gamma(1-\theta) < 1$

$$H = \frac{(c_t^\gamma g_t^{1-\gamma})^{1-\theta}}{1-\theta} e^{-(\beta-n)t} + \lambda_t \left[w_t + (r_t - n)a_t - (1+\tau_c)c_t \right]$$

$$\frac{\partial H}{\partial c_t} = 0: \quad (c_t^\gamma g_t^{1-\gamma})^{-\theta} \cdot \gamma c_t^{\gamma-1} \cdot g_t^{1-\gamma} \cdot e^{-(\beta-n)t} = \lambda_t (1+\tau_c)$$

$g_t = \tau_c c_t$

$$(c_t^\gamma \tau_c^{1-\gamma} c_t^{1-\gamma})^{-\theta} \cdot \gamma\, \cancel{c_t^{\gamma-1}}\, \tau_c^{1-\gamma} \cancel{c_t^{1-\gamma}} \cdot e^{-(\beta-n)t} = \lambda_t (1+\tau_c)$$

$$c_t^{-\theta} \tau_c^{(1-\gamma)(1-\theta)} \cdot \gamma \cdot e^{-(\beta-n)t} = \lambda_t (1+\tau_c)$$

$$-\theta \ln c_t + (1-\gamma)(1-\theta) \ln \tau_c + \ln \gamma - (\beta-n)t = \ln \lambda_t + \ln(1+\tau_c)$$

$$-\theta \cdot \frac{\dot{c}_t}{c_t} - (\beta - n) = \frac{\dot{\lambda}_t}{\lambda_t}$$

$$\frac{\partial H}{\partial a_t} = -\dot{\lambda}_t \qquad (r_t - n)\lambda_t = -\dot{\lambda}_t$$

$$-\theta \frac{\dot{c}_t}{c_t} - (\beta - n) = -(r_t - n)$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}[r_t - \beta]$$ Euler eq. - no change